BI-LIPSCHITZ RIGIDITY OF DISCRETE SUBGROUPS

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Abstract. We obtain a bi-Lipschitz rigidity theorem for a Zariski dense discrete subgroup of a connected simple real algebraic group. As an application, we show that any Zariski dense discrete subgroup of a higher rank semisimple algebraic group G cannot have a C^1-smooth slim limit set in G/P for any non-maximal parabolic subgroup P.

1. Introduction

For $i = 1, 2$, let G_i be a connected simple real algebraic group and Γ_i a Zariski dense discrete subgroup of G_i. Let

$$\rho : \Gamma_1 \to \Gamma_2$$

be an isomorphism. The classical rigidity problem searches for a condition on ρ which guarantees that ρ is algebraic, that is, it extends to a Lie group isomorphism $G_1 \to G_2$.

If Γ_1 is a lattice in G_1 and either

- $G_1 = G_2$ has rank one and is not locally isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, or
- G_1 has higher rank,

then *any* isomorphism $\rho : \Gamma_1 \to \Gamma_2$ is algebraic by celebrated theorems of Mostow, Prasad, and Margulis ([16], [17], [15]). On the other hand, there are very few rigidity theorems for non-lattice discrete subgroups, especially in higher rank. In this article, we provide a rigidity criterion on $\rho : \Gamma_1 \to \Gamma_2$ in terms of its boundary map between the limit sets of Γ_1 and Γ_2.

Since Γ_i is Zariski dense, for a parabolic subgroup P_i of G_i, there exists a unique Γ_i-minimal subset Λ_i in $\mathcal{F}_i = G_i/P_i$, called the limit set. When both parabolic subgroups P_1 and P_2 are maximal, our result takes the following simple form:

Theorem 1.1 (Bi-Lipschitz rigidity theorem I)**.** *Assume that P_1 and P_2 are maximal parabolic subgroups. Let $\rho : \Gamma_1 \to \Gamma_2$ be an isomorphism. If there exists a bi-Lipschitz ρ-equivariant map $f : \Lambda_1 \to \Lambda_2$, then ρ extends to a Lie group isomorphism*

$$\bar{\rho} : G_1 \to G_2$$

which induces a diffeomorphism $\bar{f} : \mathcal{F}_1 \to \mathcal{F}_2$ such that $\bar{f}|_{\Lambda_1} = f$.

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Recall that $f : \Lambda_1 \to \Lambda_2$ is bi-Lipschitz if there exists $C \geq 1$ such that for all $\xi, \eta \in \Lambda_1$,

$$C^{-1} d_{\mathcal{F}_1}(\xi, \eta) \leq d_{\mathcal{F}_2}(f(\xi), f(\eta)) \leq C d_{\mathcal{F}_1}(\xi, \eta) \tag{1.1}$$

where $d_{\mathcal{F}_i}$ is a Riemannian metric on $\mathcal{F}_i$ for $i = 1, 2$. Since any two Riemannian metrics on $\mathcal{F}_i$ are bi-Lipschitz equivalent to each other, this notion is well-defined. We note that there can be at most one ρ-equivariant map $f : \Lambda_1 \to \Lambda_2$ [12, Lemma 4.5]. We emphasize that we do not require f to be defined on all of $\mathcal{F}_1$, but only on Λ_1. For $G_1 = G_2 = \mathrm{SO}(n, 1)^\circ$, $n \geq 2$, Theorem 1.1 was proved by Tukia [29, Theorem D].

Remark 1.2.

1. The hypothesis that G_1 and G_2 are simple is necessary; see Remark 4.8.
2. The global bi-Lipschitz hypothesis on f can be replaced by the condition that f is bi-Lipschitz on some non-empty open subset of Λ_1; see Lemma 4.9.

We now state a general version of Theorem 1.1 where P_1 and P_2 are arbitrary parabolic subgroups.

Theorem 1.3 (Bi-Lipschitz rigidity theorem II)**.** *Let $\rho : \Gamma_1 \to \Gamma_2$ be an isomorphism. If there exists a bi-Lipschitz ρ-equivariant map $f : \Lambda_1 \to \Lambda_2$, then ρ extends to a Lie group isomorphism*

$$\bar{\rho} : G_1 \to G_2.$$

Moreover, there exists a parabolic subgroup P_2' of G_2 containing P_2 such that $\bar{\rho}(P_1) \subset P_2'$ up to a conjugation and the smooth submersion $G_1/P_1 \to G_2/P_2'$ induced by $\bar{\rho}$ coincides with the composition $\pi \circ f$ on Λ_1 where $\pi : G_2/P_2 \to G_2/P_2'$ is the canonical factor map.

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\ f\ } & \Lambda_2 \\ \big\downarrow & \circlearrowleft & \big\downarrow \pi \\ G_1/P_1 & \xrightarrow[\bar{\rho}]{} & G_2/P_2' \end{array}$$

See Theorem 4.7 for a stronger version which relaxes the bi-Lipschitz condition to a κ-bi-Hölder condition for $\kappa > 0$.

Remark 1.4. In general, P_2' is not the same as P_2. We use the theory of hyperconvex subgroups to construct a Zariski dense discrete subgroup of $\mathrm{SL}_8(\mathbb{R})$ which demonstrates this point in Proposition 6.1.

Theorem 1.3 also has consequences for the regularity of the limit set of Γ in G/P when G is a higher rank semisimple real algebraic group and P is a non-maximal parabolic subgroup.

Theorem 1.5 (Regularity of slim limit sets)**.** *Let G be a connected semisimple real algebraic group of rank at least 2 and P a non-maximal parabolic* *subgroup of G. Any Zariski dense discrete subgroup of G cannot have a slim limit set in G/P which is a C^1-submanifold.*

Note that any non-maximal parabolic subgroup P is contained in at least two non-conjugate maximal parabolic subgroups of G. We call a subset $S \subset G/P$ *slim* if there exists a pair of non-conjugate maximal parabolic subgroups P_1, P_2 containing P such that the canonical factor map $\pi_i : G/P \to G/P_i$ is injective on S for $i = 1, 2$.

$$\begin{array}{ccc} & G/P & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ G/P_1 & & G/P_2 \end{array}$$

In particular, the limit set of any subgroup of a P-Anosov or relatively P-Anosov subgroup is always slim. More generally, if any two points in the limit set are in general position, then the limit set is slim.

The non-maximal hypothesis on P in Theorem 1.5 is necessary, as there are many Zariski dense discrete subgroups of $\mathrm{PSL}_n(\mathbb{R})$, $n \geq 3$, whose limit sets are C^1-submanifolds of $\mathbb{P}(\mathbb{R}^n)$, e.g., images of Hitchin [14] and Benoist representations [2]. We remark that the limit sets of these examples are not C^2 as shown by Zimmer [32].

Remark 1.6.

(1) When G is of rank one, the limit set Λ of a Zariski dense subgroup of G is not a proper C^r-submanifold of G/P where $r = 1$ for $G = \mathrm{SO}(n, 1)^\circ$ and $r = 2$ for other rank one groups ([30, Proposition 3.12 and Corollary 3.13]). In higher rank, there exists $0 < r < \infty$, depending on G, such that Λ is not a proper C^r-submanifold of G/P for any parabolic subgroup P [5, Lemma 2.11].

(2) Theorem 1.5 was previously established for images of Hitchin representations [26, Corollary 6.1] and for images of $(1, 1, 2)$-hyperconvex representation of a surface group [18, Corollary 7.7]. We also mention [6], [31], and [20] for related work on the regularity of the limit set for certain classes of subgroups of $G = \mathrm{SO}(d, 2)$, $\mathrm{PSL}_d(\mathbb{R})$ and $\mathrm{SO}(p, q)$ respectively.

On the proofs. We deduce Theorem 1.3 and Theorem 1.5 from the following property of limit sets of a Zariski dense subgroup in higher rank:

Proposition 1.7. *Let G be a connected semisimple real algebraic group of rank at least 2. Let Q_1 and Q_2 be a pair of parabolic subgroups of G such that there is no parabolic subgroup of G containing Q_1 and a conjugate of Q_2 (e.g., a pair of non-conjugate maximal parabolic subgroups).*

If $\Gamma < G$ is a Zariski dense discrete subgroup, then there is no Γ-equivariant bi-Lipchitz map between the limit sets of Γ on G/Q_1 and G/Q_2.

Indeed, if ρ in Theorem 1.3 does not extend to a Lie group isomorphism $G_1 \to G_2$, then the following self-joining subgroup

$$\Gamma = (\mathrm{id} \times \rho)(\Gamma_1) = \{(g, \rho(g)) : g \in \Gamma_1\} \tag{1.2}$$

is a Zariski dense subgroup of the product $G = G_1 \times G_2$. On the other hand, a bi-Lipschitz map f as in Theorem 1.3 yields a bi-Lipschitz homeomorphism between the limit sets of the self-joining group Γ in $G/(P_1 \times G_2)$ and $G/(G_1 \times P_2)$, which then gives a desired contradiction by Proposition 1.7. We mention the recent work [10] and [11] on related rigidity theorems which use the idea of self-joinings.

If Γ has a C^1-slim limit set in G/P as in Theorem 1.5 and P_1 and P_2 are non-conjugate maximal parabolic subgroups containing P, we get a bi-Lipschitz map between the limit sets of Γ in G/P_1 and G/P_2 from the slimneess hypothesis. Therefore Proposition 1.7 implies Theorem 1.5.

For the proof of Proposition 1.7, we relate the exponential contraction rates of loxodromic elements $\gamma \in \Gamma$ on G/Q_i with the Jordan projections of the image of γ under Tits representations of G. This part of the argument is motivated by earlier work of Zimmer [32, Section 8]. We then show that the bi-Lipschitz equivalence of the limit sets gives an obstruction to Benoist's theorem [1] on the non-empty interior property of the limit cone of a Zariski dense subgroup (see the proof of Proposition 4.3).

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2. Preliminaries

Unless mentioned otherwise, let G be a connected semisimple *real* algebraic group throughout the paper. This means that G is the identity component $\mathbf{G}(\mathbb{R})^\circ$ for a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. A parabolic $\mathbb{R}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ is a proper algebraic subgroup defined over $\mathbb{R}$ such that the quotient $\mathbf{G}/\mathbf{P}$ is a projective algebraic variety. A parabolic subgroup P of G is of the form $\mathbf{P}(\mathbb{R})$ for a parabolic $\mathbb{R}$-subgroup $\mathbf{P}$ of $\mathbf{G}$; in this case, the quotient G/P is equal to $(\mathbf{G}/\mathbf{P})(\mathbb{R})$ and is a real projective variety, called a G-boundary [3]. Any parabolic subgroup P is conjugate to a unique standard parabolic subgroup of G, once we fix a root system associated to G.

To be precise, let A be a maximal real split torus of G. The rank of G is defined as the dimension of A. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebras of G and A. Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and set $A^+ = \exp \mathfrak{a}^+$, and a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. We denote by M the centralizer of A in K. For $g \in G$, we denote by $\mu(g)$ the Cartan projection of g, which is the unique element of $\mathfrak{a}^+$ such that $g \in K \exp \mu(g) K$.

Any $g \in G$ can be written as the commuting product $g = g_h g_e g_u$ where g_h is hyperbolic, g_e is elliptic and g_u is unipotent. The hyperbolic component

g_h is conjugate to a unique element $\exp \lambda(g) \in A^+$ and

$$\lambda(g) \in \mathfrak{a}^+ \tag{2.1}$$

is called the Jordan projection of g. When $\lambda(g) \in \operatorname{int} \mathfrak{a}^+$, $g \in G$ is called *loxodromic* in which case g_u is necessarily trivial and g_e is conjugate to an element $m \in M$.

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$ denote the set of all roots and Π the set of all simple roots given by the choice of $\mathfrak{a}^+$. The Weyl group $\mathcal{W}$ is given by $N_K(A)/M$ where $N_K(A)$ is the normalizer of A in K.

Consider the real vector space $\mathsf{E}^* = \mathsf{X}(A) \otimes_{\mathbb{Z}} \mathbb{R}$ where $\mathsf{X}(A)$ is the group of all real characters of A and let E be its dual. Denote by $(\cdot, \cdot)$ a $\mathcal{W}$-invariant inner product on E. We denote by $\{\omega_\alpha : \alpha \in \Pi\}$ the (restricted) fundamental weights of Φ defined by

$$2\frac{(\omega_\alpha, \beta)}{(\beta, \beta)} = c_\alpha \delta_{\alpha,\beta}$$

where $c_\alpha = 1$ if $2\alpha \notin \Phi$ and $c_\alpha = 2$ otherwise.

Fix an element $w_0 \in N_K(A)$ of order 2 representing the longest Weyl element so that $\operatorname{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The map

$$\mathrm{i} = -\operatorname{Ad}_{w_0} : \mathfrak{a} \to \mathfrak{a}$$

is called the opposition involution. It induces an involution of Φ preserving Π, for which we use the same notation i, so that $\mathrm{i}(\alpha) = \alpha \circ \mathrm{i}$ for all $\alpha \in \Phi$.

For a non-empty subset θ of Π, let $\mathfrak{a}_\theta = \cap_{\alpha \in \Pi - \theta} \ker \alpha$, and let P_θ denote a standard parabolic subgroup of G corresponding to θ; that is, $P_\theta = L_\theta N_\theta$ where L_θ is the centralizer of $\exp \mathfrak{a}_\theta$ and N_θ is the unipotent radical of P_θ which is generated by root subgroups associated to all positive roots which are not $\mathbb{Z}$-linear combinations of elements of $\Pi - \theta$. If $\theta = \Pi$, then $P = P_\Pi$ is a minimal parabolic subgroup. For a singleton $\theta = \{\alpha\}$, P_α is a maximal parabolic subgroup of G. Any parabolic subgroup P is conjugate to a unique standard parabolic subgroup P_θ for some non-empty subset $\theta \subset \Pi$.

We consider the θ-boundary:

$$\mathcal{F}_\theta = G/P_\theta.$$

We denote by $d_{\mathcal{F}_\theta}$ a Riemannian metric on $\mathcal{F}_\theta$. Let $P_\theta^+ = w_0 P_{\mathrm{i}(\theta)} w_0^{-1}$, which is the standard parabolic subgroup opposite to P_θ such that $P_\theta \cap P_\theta^+ = L_\theta$. Hence $\mathcal{F}_{\mathrm{i}(\theta)} = G/P_{\mathrm{i}(\theta)} = G/P_\theta^+$. The G-orbit $\mathcal{F}_\theta^{(2)} = \{(gP_\theta, gw_0P_{\mathrm{i}(\theta)}) : g \in G\}$ is the unique open G-orbit in $G/P_\theta \times G/P_\theta^+$ under the diagonal G-action. Two elements $\xi \in \mathcal{F}_\theta$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ are said to be in general position if $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$.

3. Contraction rates of loxodromic elements and Tits representations

The first part of the following theorem immediately follows as a special case of a theorem of Tits [25], and the second part is remarked in [1] and proved in [23].

Theorem 3.1 ([25, Theorem 7.2], [23, Lemma 2.13])**.** *For each $\alpha \in \Pi$, there exists an irreducible representation $\rho_\alpha : G \to \mathrm{GL}(V_\alpha)$ whose highest (restricted) weight χ_α is equal to $k_\alpha \omega_\alpha$ for some positive integer k_α and whose highest weight space is one-dimensional.*

Moreover, all weights of ρ_α are χ_α, $\chi_\alpha - \alpha$ and weights of the form $\chi_\alpha - \alpha - \sum_{\beta \in \Pi} n_\beta \beta$ with n_β non-negative integers.

These representations are called Tits representations of G. Fix $\alpha \in \Pi$ and, as before, set $\mathcal{F}_\alpha = G/P_\alpha$. We denote by V_1 and V_2 the weight spaces of ρ_α for the highest weight χ_α and the second highest weight $\chi_\alpha - \alpha$ respectively. We have $\dim V_1 = 1$ and $\dim V_2 \geq 1$. If we set $\xi_\alpha = [P_\alpha] \in \mathcal{F}_\alpha$, the map $g\xi_\alpha \mapsto gV_1$ gives an embedding

$$\mathcal{F}_\alpha \to \mathbb{P}(V_\alpha) \tag{3.1}$$

whose image is a closed subvariety. We may hence identify $\mathcal{F}_\alpha$ as a closed subvariety of $\mathbb{P}(V_\alpha)$. Let $\langle \cdot, \cdot \rangle_\alpha$ be a K-invariant inner product on V_α with respect to which A is symmetric and we have the orthogonal weight space decomposition of V_α. Using the norms on V_α and $\wedge^2 V_\alpha$ induced by this inner product, we get a K-invariant Riemannian metric d_α on $\mathbb{P}(V_\alpha)$:

$$d_\alpha([v],[w]) = \frac{\|v \wedge w\|}{\|v\|\|w\|} \quad \text{for } [v],[w] \in \mathbb{P}(V_\alpha).$$

Recall that an element $g \in G$ is *loxodromic* if there exist $a \in \operatorname{int} A^+$ and $m \in M$ such that $g = h_g a m h_g^{-1}$ for some $h_g \in G$. The element h_g is then uniquely determined modulo AM and $\lambda(g) = \log a \in \operatorname{int} \mathfrak{a}^+$.

Let $\pi_i = \pi_{\alpha,i} : V_\alpha \to V_i$ be the orthogonal projection for $i = 1, 2$. Recall the following standard lemma:

Lemma 3.2. *Let g be a loxodromic element of G. For $\xi \in \mathcal{F}_\alpha$, we have $\pi_1(h_g^{-1}\xi) \neq 0$ if and only if $g^n\xi$ converges to $h_g\xi_\alpha$ as $n \to \infty$.*

The point $y_\alpha^g := h_g\xi_\alpha \in \mathcal{F}_\alpha$ is called the attracting fixed point of g.

Lemma 3.3. *Let $g \in G$ be a loxodromic element and $\alpha \in \Pi$.*

1. *For all $\xi \in \mathcal{F}_\alpha$ with $\pi_1(h_g^{-1}\xi) \neq 0$, we have*
$$-\alpha(\lambda(g)) \geq \limsup_{n\to\infty} \frac{1}{n} \log d_\alpha(g^n\xi, y_\alpha^g).$$
2. *For all $\xi \in \mathcal{F}_\alpha$ with $\pi_1(h_g^{-1}\xi) \neq 0$ and $\pi_2(h_g^{-1}\xi) \neq 0$, we have*
$$-\alpha(\lambda(g)) = \lim_{n\to\infty} \frac{1}{n} \log d_\alpha(g^n\xi, y_\alpha^g).$$

Proof. It suffices to prove the claim when $h_g = e$, i.e., $g = am \in AM$ with $\log a \in \operatorname{int} \mathfrak{a}^+$. Considering $\xi \in \mathcal{F}_\alpha \subset \mathbb{P}(V_\alpha)$, choose a vector $v \in V_\alpha$ representing ξ. List all distinct weights of ρ_α given by Theorem 3.1 as follows: $\chi_1 = \chi_\alpha$, $\chi_2 = \chi_\alpha - \alpha$, and $\chi_i = \chi_\alpha - \alpha - \beta_i$, $3 \le i \le \ell$; in particular, $\beta_i \neq 0$ is a non-negative integral linear combinations of simple roots. Let V_i denote the weight space corresponding to χ_i and write $v = v_1 + v_2 + \cdots + v_\ell$ so that $v_i \in V_i$ for each $1 \le i \le \ell$. Suppose that $\pi_1(\xi) \neq 0$, that is $v_1 \neq 0$. We may then assume that v_1 is a unit vector relative to $\langle \cdot, \cdot \rangle_\alpha$. Since M commutes with A, M stabilizes each weight subspace, and in particular, $Mv_1 = \pm v_1$. Now

$$g^n v = e^{n\chi_\alpha(\log a)} m^n v_1 + e^{n(\chi_\alpha - \alpha)(\log a)} m^n v_2 + \sum_{i=3}^{\ell} e^{n(\chi_\alpha - \alpha - \beta_i)(\log a)} m^n v_i.$$

Hence the projection $p(g^n v)$ of $g^n v$ to the affine chart $\mathbb{A} = \{w \in V_\alpha : \pi_1(w) = v_1\}$ is

$$p(g^n v) = v_1 + e^{-n\alpha(\log a)} m^n v_2' + \sum_{i=3}^{\ell} e^{-n(\alpha+\beta_i)(\log a)} m^n v_i'$$

where $v_i' = \pm v_i$, depending on the sign of $m^n v_1$. Note that $\lim g^n \xi = V_1$, and that the metric d_α on a neighborhood on V_1 in $\mathbb{P}(V_\alpha)$ is bi-Lipschitz equivalent to the metric d on the affine chart $\mathbb{A}$, obtained by restricting the distance on V_α induced by $\langle \cdot, \cdot \rangle_\alpha$.

Since the weight spaces are orthogonal, we have

$$d(p(g^n v), v_1) = e^{-n\alpha(\log a)} \left(\|v_2\|^2 + \|w_n\|^2 \right)^{1/2}$$

where $w_n = \sum_{i=3}^{\ell} e^{-n\beta_i(\log a)} m^n v_i'$ and $\|\cdot\|$ is the norm induced by $\langle \cdot, \cdot \rangle_\alpha$. Since $\log a \in \operatorname{int} \mathfrak{a}^+$ and hence $\beta_i(\log a) > 0$ for all $3 \le i \le \ell$, we have

$$\lim_{n\to\infty} w_n = 0.$$

First consider the case when $\pi_2(\xi) = 0$, that is $v_2 = 0$. Since $\log \|w_n\| < 0$ for all large n, we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log d_\alpha(g^n \xi, y_\alpha^g) &= \limsup_{n\to\infty} \frac{1}{n} \log d(p(g^n v), v_1) \\ &= \limsup_{n\to\infty} \frac{1}{n} (-n\alpha(\log a) + \log \|w_n\|) \le -\alpha(\log a). \end{aligned}$$

Now suppose that $\pi_2(\xi) \neq 0$, that is $v_2 \neq 0$. Again since $w_n \to 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log d_\alpha(g^n \xi, y_\alpha^g) = \lim_{n\to\infty} \frac{1}{n} \log d(p(g^n v), v_1) = -\alpha(\log a).$$

This finishes the proof. ∎

4. Bi-Lipschitz rigidity of discrete subgroups

Let G be a connected semisimple real algebraic group and $X = G/K$ be the associated Riemannian symmetric space and fix $o = [K] \in X$.

We consider the following notion of convergence of a sequence in G to an element of $\mathcal{F}_\theta = G/P_\theta$ for a non-empty subset $\theta \subset \Pi$.

For a sequence $g_i o \in X$ and $\xi \in \mathcal{F}_\theta$, we write $\lim g_i o = \xi$ and say $g_i o \in X$ converges to ξ if

(1) $\min_{\alpha\in\theta} \alpha(\mu(g_i)) \to \infty$ as $i \to \infty$; and
(2) $\lim_{i\to\infty} \kappa_{g_i} P_\theta = \xi$ in $\mathcal{F}_\theta$ for some $\kappa_{g_i} \in K$ such that $g_i \in \kappa_{g_i} A^+ K$.

Definition 4.1. Let $\Gamma < G$ be a discrete subgroup and let $\mathcal{F} = G/P$ for a parabolic subgroup P. Let $\theta \subset \Pi$ be a unique subset such that P is conjugate to P_θ and hence $\mathcal{F} = \mathcal{F}_\theta$. The limit set of Γ in $\mathcal{F}_\theta$ is then defined as the set of all accumulation points of $\Gamma(o)$ in $\mathcal{F}_\theta$:

$$\Lambda_\theta = \Lambda_\theta(\Gamma) = \{\lim \gamma_i(o) \in \mathcal{F}_\theta : \gamma_i \in \Gamma\}.$$

It is a Γ-invariant closed subset of $\mathcal{F}_\theta$, which is non-empty provided Γ contains a sequence γ_i satisfying $\lim_{i\to\infty} \min_{\alpha\in\theta} \alpha(\mu(\gamma_i)) = \infty$. If Γ is Zariski dense, Λ_θ is the unique Γ-minimal subset of $\mathcal{F}_\theta$ and can also be described as the set of all $\xi \in \mathcal{F}_\theta$ such that the Dirac measure δ_ξ is the weak limit of $(\gamma_i)_* \operatorname{Leb}_\theta$ for some sequence $\gamma_i \in \Gamma$ where $\operatorname{Leb}_\theta$ denotes the unique K-invariant probability measure on $\mathcal{F}_\theta$ ([1], [21]). Moreover, if $\Theta \subset \theta$, then Λ_Θ is equal to the image of Λ_θ under the canonical projection $\mathcal{F}_\theta \to \mathcal{F}_\Theta$, by minimality.

The limit cone of Γ is defined as the smallest closed cone of $\mathfrak{a}^+$ containing all Jordan projections of loxodromic elements of Γ.

Theorem 4.2 (Benoist [1])**.** *If $\Gamma < G$ is Zariski dense, its limit cone has non-empty interior in $\mathfrak{a}$.*

For $\kappa > 0$ and $\theta_1, \theta_2 \subset \Pi$, a map $F : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$ is called κ-bi-Hölder if there exists $C > 0$ such that for all $x, y \in \Lambda_{\theta_1}$

$$C^{-1} d_{\mathcal{F}_{\theta_1}}(x,y)^\kappa \le d_{\mathcal{F}_{\theta_2}}(F(x),F(y)) \le C d_{\mathcal{F}_{\theta_1}}(x,y)^\kappa \tag{4.1}$$

where $d_{\mathcal{F}_{\theta_i}}$ is a Riemannian metric on $\mathcal{F}_{\theta_i}$ for $i = 1, 2$. Observe that if Γ is Zariski dense, any Γ-equivariant κ-bi-Hölder map $\Lambda_{\theta_1} \to \Lambda_{\theta_2}$ is a homeomorphism; the minimality of Λ_{θ_2} implies the surjectivity and the bi-Hölder property implies the injectivity. Therefore F is κ-bi-Hölder if and only if F is κ-Hölder and F^{-1} is κ^{-1}-Hölder.

Proposition 1.7 follows from the following for $\kappa = 1$:

Proposition 4.3. *Let $\Gamma < G$ be Zariski dense. Let θ_1 and θ_2 be disjoint non-empty subsets of Π. Then for any $\kappa > 0$, there exists no Γ-equivariant κ-bi-Hölder map $F : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$.*

Proof. For simplicity, we write $\Lambda_i = \Lambda_{\theta_i}$ and $d_{\theta_i} = d_{\mathcal{F}_{\theta_i}}$. Let $F : \Lambda_1 \to \Lambda_2$ be a Γ-equivariant homeomorphsim. Fix $\kappa > 0$. Since $\theta_1 \cap \theta_2 = \emptyset$, the union

$\bigcup_{\alpha_1\in\theta_1,\alpha_2\in\theta_2}\ker(\kappa\alpha_1-\alpha_2)$ is a finite union of hyperplanes of $\mathfrak{a}$. Therefore by Theorem 4.2, Γ contains a loxodromic element γ such that

$$\{\kappa\cdot\alpha(\lambda(\gamma)):\alpha\in\theta_1\}\cap\{\alpha(\lambda(\gamma)):\alpha\in\theta_2\}=\emptyset.$$

For each $i=1,2$, let $\alpha_i\in\theta_i$ be such that

$$\alpha_i(\lambda(\gamma))=\min\{\alpha(\lambda(\gamma)):\alpha\in\theta_i\}. \tag{4.2}$$

Note that

$$\kappa\cdot\alpha_1(\lambda(\gamma))\neq\alpha_2(\lambda(\gamma)). \tag{4.3}$$

Claim: If F^{-1} is κ^{-1}-Hölder, then

$$\alpha_2(\lambda(\gamma))\leq\kappa\cdot\alpha_1(\lambda(\gamma)). \tag{4.4}$$

By replacing Γ by a suitable conjugate, we may also assume that $\gamma=am\in\Gamma$ with $a\in\operatorname{int}A^+$ and $m\in M$. For each $i=1,2$, let $y_i=y^{\gamma}_{\alpha_i}$ denote the attracting fixed point of γ in $\mathcal{F}_i$; we have $y_i\in\Lambda_i$. As Γ is Zariski dense, Λ_i is Zariski dense in $\mathcal{F}_i$ for each $i=1,2$. Let $\pi_{\alpha,1}$ and $\pi_{\alpha,2}$ be as in Lemmas 3.2 and 3.3 for each $\alpha\in\Pi$. Since the set

$$\mathcal{O}=\{\xi\in\mathcal{F}_1:\pi_{\alpha,1}(\xi)\neq0,\pi_{\alpha,2}(\xi)\neq0\text{ for all }\alpha\in\theta_1\}$$

is a Zariski open subset of $\mathcal{F}_1$, the intersection $\mathcal{O}\cap\Lambda_1$ is a non-empty open subset of Λ_1. As F is a homeomorphism, the image $F(\mathcal{O}\cap\Lambda_1)$ is a non-empty open subset of Λ_2. Since $Z=\{\xi\in\mathcal{F}_2:\gamma^n\xi\not\to y_2\text{ as }n\to\infty\}$ is a proper Zariski closed subset of $\mathcal{F}_2$ by Lemma 3.2, $F(\mathcal{O}\cap\Lambda_1)$ cannot be contained in Z; otherwise it would imply that Λ_2 is contained in a proper Zariski closed subset by the Γ_2-minimality of Λ_2, which contradicts the Zariski density of Γ_2. Therefore there exists an element $\xi\in\mathcal{O}\cap\Lambda_1$ such that $\lim_{n\to\infty}\gamma^nF(\xi)=y_2$. By the equivariance and continuity of F, we have

$$F(y_1)=\lim F(\gamma^n\xi)=\lim\gamma^nF(\xi)=y_2. \tag{4.5}$$

Let $i=1,2$. Since $P_{\theta_i}=\bigcap_{\alpha\in\theta_i}P_\alpha$, we have a diagonal embedding

$$\mathcal{F}_i=G/P_{\theta_i}\to\prod_{\alpha\in\theta_i}\mathbb{P}(V_\alpha)$$

via the product of the maps in (3.1). Consider the metric d_i on $\mathcal{F}_i$ obtained as the restriction of $\sum_{\alpha\in\theta_i}d_\alpha$ to $\mathcal{F}_i$: for $\eta=gP_{\theta_1}$ and $\eta'=g'P_{\theta_2}$ with $g,g'\in G$,

$$d_i(\eta,\eta')=\sum_{\alpha\in\theta_i}d_\alpha(\eta,\eta')$$

where $d_\alpha(\eta,\eta'):=d_\alpha(gV_{\alpha,1},g'V_{\alpha,1})$ where $V_{\alpha,1}$ is the highest weight line of ρ_α as in (3.1). Since d_i is bi-Lipschitz equivalent to a Riemannian metric on $\mathcal{F}_i$, we have that $F^{-1}:(\Lambda_2,d_2)\to(\Lambda_1,d_1)$ is κ^{-1}-Hölder.

Since $\xi\in\mathcal{O}$ and $\lim\gamma^nF(\xi)=y_2$, we have by Lemma 3.3 that

$$-\alpha(\lambda(\gamma))=\lim\frac{1}{n}\log d_\alpha(\gamma^n\xi,y_1)\quad\text{for each }\alpha\in\theta_1$$

and

$$-\alpha(\lambda(\gamma)) \geq \limsup \frac{1}{n} \log d_\alpha(\gamma^n F(\xi), y_2) \text{ for each } \alpha \in \theta_2.$$

Since $d_{\alpha_1}(\eta,\eta') \leq d_1(\eta,\eta')$, $d_2(\eta,\eta') \leq \#\theta_2 \max_{\alpha\in\theta_2} d_\alpha(\eta,\eta')$, and F^{-1} is κ^{-1}-Hölder, we have

$$\begin{aligned}
-\alpha_1(\lambda(\gamma)) &= \lim \frac{1}{n} \log d_{\alpha_1}(\gamma^n \xi, y_1) \\
&\leq \lim \frac{1}{n} \log d_1(\gamma^n \xi, y_1) \\
&\leq \kappa^{-1} \limsup \frac{1}{n} \log d_2(F(\gamma^n\xi), F(y_1)) \\
&= \kappa^{-1} \limsup \frac{1}{n} \log d_2(\gamma^n F(\xi), y_2) \\
&= \kappa^{-1} \max_{\alpha\in\theta_2} \limsup \frac{1}{n} \log d_\alpha(\gamma^n F(\xi), y_2) \\
&\leq -\kappa^{-1} \min_{\alpha\in\theta_2} \alpha(\lambda(\gamma)) = -\kappa^{-1}\alpha_2(\lambda(\gamma)).
\end{aligned} \tag{4.6}$$

This implies that $\alpha_2(\lambda(\gamma)) \leq \kappa\alpha_1(\lambda(\gamma))$, proving the claim.

By switching the role of θ_1 and θ_2, this claim then implies that if F is κ-Hölder, then $\alpha_1(\lambda(\gamma)) \leq \kappa^{-1}\alpha_2(\lambda(\gamma))$. Therefore if F is κ-bi-Hölder, then $\kappa \cdot \alpha_1(\lambda(\gamma)) = \alpha_2(\lambda(\gamma))$, contradicting (4.3). This finishes the proof. □

The proof of Proposition 4.3 shows the following as well:

Proposition 4.4. *Let $\Gamma < G$ be Zariski dense and let $\theta_1, \theta_2 \subset \Pi$ be non-empty disjoint subsets. Suppose that Λ_{θ_1} and Λ_{θ_2} are C^1-submanifolds of $\mathcal{F}_{\theta_1}$ and $\mathcal{F}_{\theta_2}$ respectively. If $F : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$ is a Γ-equivariant homeomorphism, F cannot be C^1 with non-vanishing Jacobian at any $\xi \in \mathcal{A}$, where $\mathcal{A} \subset \Lambda_{\theta_1}$ is the set of all attracting fixed points of loxodromic elements $\gamma \in \Gamma$ such that $\{\alpha(\lambda(\gamma)) : \alpha \in \theta_1\} \cap \{\alpha(\lambda(\gamma)) : \alpha \in \theta_2\} = \emptyset$.*

Proof. Let $\gamma \in \Gamma$ be as above. For each $i = 1, 2$, let $y_i \in \Lambda_{\theta_i}$ be the attracting fixed point of γ. Then $F(y_1) = y_2$ by (4.5). Suppose that F is C^1 at y_1, and the Jacobian of F at y_1 is not zero. Then F^{-1} is also C^1 at y_2. Using the exponential maps and the Taylor series expansion of F, we get that there exist $c \geq 1$ and an open neighborhood U of y_1 in Λ_{θ_1} such that for all $y \in U$,

$$c^{-1} d_1(y, y_1) \leq d_2(F(y), F(y_1)) \leq c d_1(y, y_1). \tag{4.7}$$

Let $\alpha_i \in \theta_i$ be as in (4.2). Without loss of generality, we may assume $\alpha_1(\lambda(\gamma)) < \alpha_2(\lambda(\gamma))$ by switching the indexes if necessary. On the other hand, using (4.7), the computation (4.6) gives $\alpha_2(\lambda(\gamma)) \leq \alpha_1(\lambda(\gamma))$, which yields a contradiction. □

Remark 4.5. It would be interesting to know whether $\mathcal{A}$ can be replaced by the set of all *conical* limit points of Γ in Proposition 4.4. A point $\xi = gP_{\theta_1}$ is Γ-conical if $\limsup \Gamma g(K \cap P_{\theta_1})A^+ \neq \emptyset$, that is, there exists a sequence

$\gamma_i \in \Gamma$, $a_i \in A^+$ and $m_i \in K \cap P_{\theta_i}$ such that $\gamma_i g m_i a_i$ converges (see [13, Lemma 5.4] for an equivalent definition in terms of shadows).

This question is inspired by a related result for $G = \mathrm{SO}(n+1,1)^\circ$. Tukia [27] showed that if $f : \mathbb{S}^n \to \mathbb{S}^n$ is a homeomorphism which conjugates a discrete subgroup Γ_1 of G to another discrete group Γ_2 and has a non-vanishing Jacobian at a conical limit point of Γ_1, then Γ_1 is conjugate to Γ_2 (see also [7] for an extension of this result to other rank one groups). For a related result for $(1,1,2)$-hyperconvex groups, see [18, Corollary 7.5].

In the rest of this section, let G_i be a connected simple real algebraic group and θ_i be a non-empty set of simple roots of G_i for $i = 1, 2$. Let $\Gamma_i < G_i$ be a Zariski dense discrete subgroup and Λ_{θ_i} denote the limit set of Γ_i in $\mathcal{F}_i = G_i/P_{\theta_i}$.

Lemma 4.6. [12, Lemma 4.5] *For any isomorphism $\rho : \Gamma_1 \to \Gamma_2$, there exists at most one ρ-equivariant continuous map $f : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$.*

Indeed, f must send the attracting fixed point of any loxodromic element γ to that of $\rho(\gamma)$ whenever $\rho(\gamma)$ is loxodromic. Since the set of attracting fixed points of loxodromic elements is dense in Λ_{θ_1} by the Zariski density hypothesis on Γ_1 [1] and f is continuous, this determines the map f.

Theorem 1.3 is a special case of the following theorem for $\kappa = 1$:

Theorem 4.7. *Suppose that there exists a ρ-equivariant κ-bi-Hölder map $f : \Lambda_{\theta_1} \to \mathcal{F}_2$ for some $\kappa > 0$. Then ρ extends to a Lie group isomorphism $\bar{\rho} : G_1 \to G_2$. Moreover, there exists a non-empty subset $\Theta_2 \subset \theta_2$ such that $\bar{\rho}$ maps P_{θ_1} into a conjugate of P_{Θ_2} and the smooth submersion $G_1/P_{\theta_1} \to G_2/P_{\Theta_2}$ induced by $\bar{\rho}$ coincides with the composition $\pi \circ f$ on Λ_{θ_1} where $\pi : G_2/P_{\theta_2} \to G_2/P_{\Theta_2}$ is the canonical factor map.*

Proof. Let $G = G_1 \times G_2$. Define the following self-joining subgroup

$$\Gamma = (\mathrm{id} \times \rho)(\Gamma_1) = \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma_1\} < G.$$

Note that $P_1 := P_{\theta_1} \times G_2$ and $P_2 := G_1 \times P_{\theta_2}$ are parabolic subgroups of G. The maps $g_1 P_{\theta_1} \mapsto (g_1, e)P_1$ and $g_2 P_{\theta_2} \mapsto (e, g_2)P_2$ define diffeomorphisms between G_1/P_{θ_1} and G_2/P_{θ_2} with G/P_1 and G/P_2 respectively. Moreover, under this identification, the limit set Λ_{θ_i} of Γ_i in G_i/P_{θ_i} corresponds to the limit set Λ_i of the self-joining Γ in G/P_i for each $i = 1, 2$.

Since f is a ρ-equivariant continuous embedding of Λ_{θ_1} into G/P_{θ_2}, its image is a Γ_2-invariant compact subset. Since Λ_{θ_1} is a Γ_1-minimal subset, the image $f(\Lambda_{\theta_1})$ is also a Γ_2-minimal subset. Therefore $f(\Lambda_{\theta_1}) = \Lambda_{\theta_2}$ and hence we have a Γ-equivariant bijection $f : \Lambda_1 \to \Lambda_2$ which is κ-bi-Hölder.

Since P_1 and P_2 are parabolic subgroups corresponding to disjoint subsets of simple roots of G, Proposition 4.3 implies that Γ cannot be Zariski dense in G. Since both G_1 and G_2 are simple, the non-Zariski density of the self-joining group Γ implies that ρ extends to a Lie group isomorphism $\bar{\rho} : G_1 \to G_2$ (cf. [4]).

Since $\bar\rho(P_{\theta_1})$ must be a parabolic subgroup of G_2, there exists $g \in G_2$ such that $\bar\rho(P_{\theta_1}) = gP_{\theta_0}g^{-1}$ where θ_0 is a non-empty subset of some simple roots of G_2. We claim $\theta_0 \cap \theta_2 \neq \emptyset$. By replacing ρ by $\text{inn}(g) \circ \rho$ where $\text{inn}(g) : G_2 \to G_2$ is the conjugation by g, we may assume without loss of generality that $g = e$. The isomorphism $\bar\rho$ induces a diffeomorphism $\tilde\Phi : G_1/P_{\theta_1} \to G_2/P_{\theta_0}$ given by $\tilde\Phi(g_1 P_{\theta_1}) = \bar\rho(g_1)P_{\theta_0}$. Denote by Λ_{θ_0} the limit set of Γ_2 in G_2/P_{θ_0}. Since $\bar\rho|_{\Gamma_1} = \rho$ and hence $\tilde\Phi$ is ρ-equivariant, we have $\tilde\Phi(\Lambda_{\theta_1}) = \Lambda_{\theta_0}$. Then the composition $F := f \circ \tilde\Phi^{-1}$ restricted to Λ_{θ_0} yields a κ-bi-Hölder map between Λ_{θ_0} and Λ_{θ_2}. Since $\tilde\Phi^{-1}$ is ρ^{-1}-equivariant and f is ρ-equivariant, F is Γ_2-equivariant. So by applying Proposition 4.3 one more time, we obtain $\theta_0 \cap \theta_2 \neq \emptyset$. Setting $\Theta_2 = \theta_0 \cap \theta_2$, since P_{θ_0} and P_{θ_2} are subgroups of P_{Θ_2}, we get a map $\Phi := G_1/P_{\theta_1} \to G_2/P_{\Theta_2}$ by composing $\tilde\Phi$ with the canonical factor map $G_1/P_{\theta_0} \to G_2/P_{\Theta_2}$. The last claim $\Phi = \pi \circ f$ on Λ_{θ_1} follows from Lemma 4.6. This finishes the proof. □

Remark 4.8. The hypothesis that G_1 and G_2 are simple is necessary in Theorem 4.7. For example, consider a discrete Zariski dense subgroup Γ of a simple algebraic group G with a discrete faithful representation $\rho : \Gamma \to G$ which does not extend to G. Then $\Gamma_\rho = (\text{id} \times \rho)(\Gamma)$ is Zariski dense in G and the map $\gamma \to (\gamma, \rho(\gamma))$ gives an isomorphism $\Gamma \to \Gamma_\rho$. On the other hand, for any parabolic subgroup P of G, the isomorphism $G/P \simeq (G \times G)/(P \times G)$ provides an equivariant bi-Lipschitz bijection the limit set of Γ in G/P and the limit set of Γ_ρ in $(G \times G)/(P \times G)$.

We note that the global bi-Hölder condition in Proposition 4.3 and Theorem 4.7 can be relaxed to a local bi-Hölder condition by the following lemma.

Lemma 4.9. *Keep the notation as in Theorem 4.7 but assume G_1 and G_2 are semisimple, not just simple. Let $f : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$ be a ρ-equivariant homeomorphism which is κ-bi-Hölder on some non-empty open subset U of Λ_{θ_1} for some $\kappa > 0$. Then f is κ-bi-Hölder globally.*

Proof. Let $\Lambda_i = \Lambda_{\theta_i}$ for $i = 1, 2$. Since Λ_1 is Γ_1-minimal, $\Lambda_1 = \Gamma_1 U$ and hence, by compactness, we have Λ_1 is a finite union of $\gamma_k U$ for some $\gamma_1, \cdots, \gamma_n \in \Gamma_1$. If f is not κ-Hölder globally, by the compactness of Λ_1, we have a sequence $\xi_i \to \xi$ and $\eta_i \to \eta$ such that

$$\frac{d_{\mathcal{F}_2}(f(\xi_i), f(\eta_i))}{d_{\mathcal{F}_1}(\xi_i, \eta_i)^\kappa} \to \infty. \tag{4.8}$$

Since $\mathcal{F}_2$ is compact, we have $d_{\mathcal{F}_1}(\xi_i, \eta_i) \to 0$. Therefore, for some $1 \leq k \leq n$, $\xi_i, \eta_i \in \gamma_k U$ for all i. Noting that the action of each element of $g_i \in G_i$ on $\mathcal{F}_i$ is a diffeomorphism for $i = 1, 2$, we can let L be the maximum of the bi-Lipschitz constants of γ_k on $\mathcal{F}_1$ and of $\rho(\gamma_k)$ on $\mathcal{F}_2$. Now we have $d_{\mathcal{F}_2}(f(\xi_i), f(\eta_i)) \leq L d_{\mathcal{F}_2}(f(\gamma_k^{-1}\xi_i, \gamma_k^{-1}\eta_i))$ and $d_{\mathcal{F}_1}(\xi_i, \eta_i) \geq L^{-1} d_{\mathcal{F}_1}(\gamma_k^{-1}\xi_i, \gamma_k^{-1}\eta_i))$. Since f is κ-Hölder on U, it follows that the ratio in (4.8) is bounded, yielding a contradiction. This shows that f is κ-Hölder globally. Similarly by considering f^{-1}, we can show that f^{-1} is κ^{-1}-Hölder globally. □

Theorem 1.1 is now a special case of the following corollary of Theorem 4.7 together with Lemma 4.9:

Corollary 4.10. *Let α_i be a simple root of G_i for $i = 1, 2$. Suppose that there exists a ρ-equivariant bijection $f : \Lambda_{\alpha_1} \to \Lambda_{\alpha_2}$ which is κ-bi-Hölder on some non-empty open subset of Λ_{α_1} for some $\kappa > 0$. Then $\kappa = 1$ and ρ extends to a Lie group isomorphism $\bar{\rho} : G_1 \to G_2$ which induces a diffeomorphism $\bar{f} : G_1/P_{\alpha_1} \to G_2/P_{\alpha_2}$ such that $\bar{f}|_{\Lambda_1} = f$.*

Note that the conclusion $\kappa = 1$ follows since $\bar{f}$ is diffeomorphism and hence bi-Lipschitz.

Remark 4.11. In general, we cannot replace f bi-Lipschitz by Lipschitz in Theorem 1.1. For example, let Γ be a Schottky subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by two loxodromic elements a, b. Then for any $N \geq 2$, the representation ρ of Γ into $\mathrm{SL}_2(\mathbb{R})$ given by $a \mapsto a^N$ and $b \mapsto b^N$ induces an equivariant homeomorphism $\Lambda \to \Lambda$ which is Lipschitz, but not bi-Lipschitz. Clearly, ρ does not extend to $\mathrm{SL}_2(\mathbb{R})$.

On the other hand, we have the following corollary of the proof of Theorem 4.7 where f is required only to be Lipschitz under an extra hypothesis on the Hausdorff dimension of limit sets. In the statement below, a Möbius transformation is the extension of *any* isometry of $\mathbb{H}^{n+1}$ to its boundary $\mathbb{S}^n = \partial \mathbb{H}^{n+1}$.

Corollary 4.12. *For $i = 1, 2$, let Γ_i be a convex cocompact Zariski dense subgroup of $G_i = \mathrm{SO}^\circ(n_i + 1, 1)$, $n_i \geq 1$. Let $\Lambda_i \subset \mathbb{S}^{n_i}$ be the limit set of Γ_i. Suppose that the Hausdorff dimension of Λ_1 is equal to the Hausdorff dimension of Λ_2. Let $f : \Lambda_1 \to \Lambda_2$ be a ρ-equivariant homeomorphism which is Lipschitz on some non-empty open subset of Λ_1. Then ρ extends to a Lie group isomorphism of $G_1 \to G_2$ and f extends to a Möbius transformation of $\mathbb{S}^n$ for $n = n_1 = n_2$.*

Proof. By the proof of Lemma 4.9, f is Lipschitz on all of Λ_1. Let $\Gamma := (\mathrm{id} \times \rho)(\Gamma_1)$ be the self-joining subgroup of $G = G_1 \times G_2$. For $i = 1, 2$, let α_i be the simple root of $G = G_1 \times G_2$ from the i-th factor. Then for any loxodromic element $g = (\gamma, \rho(\gamma)) \in G$, $\alpha_1(\lambda(g))$ and $\alpha_2(\lambda(g))$ are equal to $\lambda(\gamma)$ and $\lambda(\rho(\gamma))$ respectively. Suppose that Γ is Zariski dense in G. The proof of Proposition 4.3 for Γ shows that if there exists a loxodromic element $g = (\gamma, \rho(\gamma)) \in \Gamma$ such that $\alpha_1(\lambda(g)) > \alpha_2(\lambda(g))$, then $f : \Lambda_1 \to \Lambda_2$ cannot be Lipschitz. On the other hand, if Λ_1 and Λ_2 have the same Hausdorff dimension, the middle direction $(1, 1) \in \mathfrak{a} \simeq \mathbb{R}^2$ is always contained in the interior of the limit cone of Γ by [9, Corollary 4.2]. Note that when Γ_i are cocompact lattices and $n_1 = n_2 = 2$, [9, Corollary 4.2] is due to Thurston [24]. Therefore, the desired element $g \in \Gamma$ can always be found. This implies that Γ cannot be Zariski dense in G. As before, this implies the conclusion. $\square$

5. Slim limit sets of G/P for P non-maximal

Let Γ be a Zariski dense subgroup of a connected semisimple real algebraic group G. Fix a subset $\theta \subset \Pi$ with $\#\theta \geq 2$. Recall from the introduction that a subset $S \subset \mathcal{F}_\theta$ is called *slim* if there exists a pair of distinct elements α_1 and α_2 of θ such that the limit set Λ_θ injects to G/P_{α_1} and G/P_{α_2} under the canonical projection map $\mathcal{F}_\theta \to G/P_{\alpha_i}$ for $i = 1, 2$.

In this section we prove the following theorem.

Theorem 5.1. *If $\#\theta \geq 2$ and Λ_θ is a slim subset of $\mathcal{F}_\theta$, then no non-empty open subset U of Λ_θ is contained in a proper C^1-submanifold of $\mathcal{F}_\theta$.*

We first prove the following lemma which connects Theorem 5.1 with Proposition 4.3.

Lemma 5.2. *Let $\theta_0 \subset \theta \subset \Pi$. Suppose that Λ_θ is a C^1-submanifold of $\mathcal{F}_\theta$ and that the canonical projection $\mathcal{F}_\theta \to \mathcal{F}_{\theta_0}$ is injective on Λ_θ. Then Λ_{θ_0} is a C^1-submanifold of $\mathcal{F}_{\theta_0}$ and $f_{\theta_0} : \Lambda_\theta \to \Lambda_{\theta_0}$ is a Γ-equivariant diffeomorphism.*

Proof. For simplicity, we write $\Lambda = \Lambda_\theta$. We suppose that Λ is a C^1-submanifold of $\mathcal{F}_\theta$. Since the projection $\mathcal{F}_\theta \to \mathcal{F}_{\theta_0}$ given by $f(gP_\theta) = gP_{\theta_0}$ is a smooth map, its restriction $f : \Lambda \to \mathcal{F}_{\theta_0}$ is a C^1 map which is also injective by hypothesis. We claim that there exists a point $x \in \Lambda$ where $df_x : T_x\Lambda \to T_{f(x)}\mathcal{F}_{\theta_0}$ is injective. Pick a point $x \in \Lambda$ which maximizes $\operatorname{rank} df_y$, $y \in \Lambda$. Then there exists a neighborhood of x in Λ where df has constant rank. Then if $r := \operatorname{rank} df_x$, there exist local coordinates near x where

$$f(x^1, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

Since f is injective, we must have $r = m$ and hence df_x is injective.

Now the set $\{x \in \Lambda : df_x \text{ is injective}\}$ is open and Γ-invariant. Since Γ acts minimally on Λ, this set must be all of Λ. Thus f is an immersion. Since f is an injective immersion and Λ is compact, f is a C^1-embedding. Hence f is a diffeomorphism onto its image, which is Λ_{θ_0}. In particular, Λ_{θ_0} is a C^1-submanifold of $\mathcal{F}_{\theta_0}$. $\square$

Proof of Theorem 5.1. By the hypothesis on the slimness of Λ_θ, there exists a pair of distinct elements α_1 and α_2 of θ such that Λ_θ injects to G/P_{α_1} and G/P_{α_2}.

Suppose on the contrary that some non-empty open subset U of Λ_θ is contained in some C^1-submanifold. Since Λ_θ is Γ-minimal, we have that for any $\xi \in \Lambda_\theta$, $\Gamma\xi$ is dense, so $\gamma\xi \in U$ for some $\gamma \in \Gamma$. Since $\xi \in \gamma^{-1}U$, it follows that Λ_θ is a C^1-submanifold of $\mathcal{F}_\theta$. By Lemma 5.2, we have Γ-equivariant diffeomorphisms $f_{\alpha_i} : \Lambda_\theta \to \Lambda_{\alpha_i}$ for each $i = 1, 2$. Hence

$f_{\alpha_2} \circ f_{\alpha_1}^{-1} : \Lambda_{\alpha_1} \to \Lambda_{\alpha_2}$ is a Γ-equivariant diffeomorphism, contradicting Proposition 4.3. This finishes the proof.

Remark 5.3. We remark that Proposition 4.3 implies that if Λ is a slim subset of G/P, then there exists a maximal parabolic subgroup Q containing P such that the projection $G/P \to G/Q$ restricted to Λ is not bi-Lipschitz.

Antipodal groups. Theorem 1.5 applies to the class of P-antipodal discrete subgroups of G, which contains any subgroup of a P-Anosov or a relatively P-Anosov subgroup. To define an antipodality, we recall that a parabolic subgroup P is called reflexive if its conjugacy class contains a parabolic subgroup P' opposite to P, that is, $P \cap P'$ is a common Levi subgroup of both P and P'. For example, a minimal parabolic subgroup of G is always reflexive. For a parabolic subgroup P, let $P_{\text{reflexive}}$ be the largest reflexive parabolic subgroup contained in P. If $P = P_\theta$, then $P_{\text{reflexive}} = P_{\theta \cup \mathrm{i}(\theta)}$.

Definition 5.4. A discrete subgroup Γ is called *P-antipodal* if its limit set in $G/P_{\text{reflexive}}$ is antipodal in the sense that any two distinct points are in general position.

If a discrete subgroup Γ is P-antipodal, then its limit set on G/P injects to G/P' for any P' containing P [13, Lemma 9.5]. Hence if Γ is P-antipodal for a non-maximal parabolic subgroup P, then its limit set is a slim subset of G/P. Therefore the following corollary is a special case of Theorem 1.5.

Corollary 5.5. *Let G be a connected semisimple real algebraic group of rank at least 2 and P a non-maximal parabolic subgroup of G. The limit set of a Zariski dense P-antipodal subgroup of G cannot be a C^1-submanifold of G/P.*

Note that there are many slim limit sets which are not antipodal (e.g., the limit set of a self-joining group defined in (1.2)).

6. An example

In this final section, we construct an example of a Zariski dense discrete subgroup of $\mathrm{SL}_8(\mathbb{R})$ which explains the necessity of introducing P_2' in the conclusion of Theorem 1.3 in the case when P_2 is not maximal. The examples we construct are Borel-Anosov and $(1,1,2)$-hyperconvex subgroups of $\mathrm{SL}_8(\mathbb{R})$.

We begin by setting up some notation. For any $d \geq 2$, let A be the diagonal subgroup of $\mathrm{SL}_d(\mathbb{R})$ consisting of diagonal elements with positive entries so that $\mathfrak{a}$ and $\mathfrak{a}^+$ can respectively be identified with $\mathfrak{a} = \{(u_1, \cdots, u_d) : \sum_{k=1}^d u_k = 0\}$ and $\mathfrak{a}^+ = \{(u_1, \cdots, u_d) \in \mathfrak{a} : u_1 \geq \cdots \geq u_d\}$. For $1 \leq k \leq d-1$, let

$$\alpha_k((u_1, \cdots, u_d)) = u_k - u_{k+1};$$

then $\Pi = \{\alpha_k : 1 \le k \le d-1\}$ is the set of all simple roots. For any $g \in \mathrm{SL}_d(\mathbb{R})$, its Jordan projection $\lambda(g) \in \mathfrak{a}^+$ satisfies

$$\alpha_k(\lambda(g)) = \log \frac{\lambda_k(g)}{\lambda_{k+1}(g)}$$

where $\lambda_1(g) \ge \cdots \ge \lambda_d(g)$ are the absolute values of the eigenvalues of g. Also, for $\theta \subset \Pi$, the boundary $\mathcal{F}_\theta = \mathrm{SL}_d(\mathbb{R})/P_\theta$ coincides with the partial flag manifold consisting of flags with subspaces of dimensions $\{k : \alpha_k \in \theta\}$.

Let Δ be a hyperbolic and denote by $\partial\Delta$ its Gromov boundary. For $\gamma \in \Delta$, denote by $|\gamma|$ the minimal word length of γ with respect to a fixed finite generating set of Δ. Recall from [8] that a representation $\rho : \Delta \to \mathrm{SL}_d(\mathbb{R})$ is $\{\alpha_k\}$-Anosov if there exist constants $c, C > 0$ so that for all $\gamma \in \Delta$,

$$\alpha_k(\lambda(\rho(\gamma)) \ge c \cdot \ell(\gamma) - C$$

where $\ell(\gamma)$ is the translation length of γ, i.e., $\ell(\gamma) = \inf_{\gamma_0 \in \Delta} |\gamma_0 \gamma \gamma_0^{-1}|$. If ρ is $\{\alpha_k\}$-Anosov, it admits a pair of unique continuous equivariant embeddings $\xi_\rho^k : \partial\Delta \to \mathrm{Gr}_k(\mathbb{R}^d)$ and $\xi_\rho^{d-k} : \partial\Delta \to \mathrm{Gr}_{d-k}(\mathbb{R}^d)$. Furthermore, the image of $(\xi_\rho^k, \xi_\rho^{d-k})$ coincides with the limit set of $\rho(\Delta)$ in $\mathcal{F}_{\{\alpha_k, \alpha_{d-k}\}}$. We say that ρ is Borel-Anosov if it is $\{\alpha_k\}$-Anosov for all $1 \le k \le d-1$. The image of a Borel-Anosov representation is called a Borel Anosov subgroup.

A representation $\rho : \Delta \to \mathrm{SL}_d(\mathbb{R})$ is $(1,1,2)$-hyperconvex if it is $\{\alpha_1, \alpha_2\}$-Anosov and for all distinct $x, y, z \in \partial\Delta$,

$$\xi_\rho^1(x) \oplus \xi_\rho^1(y) \oplus \xi_\rho^{d-2}(z) = \mathbb{R}^d.$$

Both being $\{\alpha_k\}$-Anosov and being $(1,1,2)$-hyperconvex are open conditions in the representation variety (see [19, Proposition 6.2]).

Proposition 6.1. *There exists a Zariski dense discrete subgroup $\Gamma < \mathrm{SL}_8(\mathbb{R})$ which admits an equivariant Lipschitz bijection $\Lambda_{\alpha_3} \to \Lambda_{\alpha_1}$. Moreover, Γ is Borel-Anosov, $(1,1,2)$-hyperconvex, and the projection map $p : \Lambda_{\{\alpha_1,\alpha_3\}} \to \Lambda_{\alpha_3}$ is a bi-Lipschitz bijection.*

Theorem 1.3 in this case applies with $f = p^{-1}$, $P_1 = P_{\alpha_3}$, $P_2 = P_{\{\alpha_1,\alpha_3\}}$ and $P_2' = P_{\alpha_3}$.

Let $\Delta = \langle a_1, a_2\rangle$ be the free group with two generators a_1, a_2. Let $N \ge 2$. Let $\tau_1 : \Delta \to \mathrm{SL}_2(\mathbb{R})$ be a convex cocompact representation and $\tau_2 : \Delta \to \mathrm{SL}_2(\mathbb{R})$ be defined so that $\tau_2(a_i) = \tau_1(a_i)^N$ for $i = 1, 2$. We may choose N large enough that

$$\frac{\alpha_1(\lambda(\tau_2(\gamma)))}{\alpha_1(\lambda(\tau_1(\gamma)))} \ge 4 \quad \text{for all non-trivial} \quad \gamma \in \Delta.$$

Let $\iota : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_4(\mathbb{R})$ be an irreducible representation, which is unique up to conjugations. Then each $\rho_i = \iota \circ \tau_i$ is a positive representation and hence Borel Anosov and $(1,1,2)$-hyperconvex [19, Corollary 6.13]. One easily checks that $\frac{\alpha_1(\lambda(\rho_2(\gamma)))}{\alpha_1(\lambda(\rho_1(\gamma)))} \ge 4$ for all non-trivial $\gamma \in \Delta$. Then a theorem of Tsouvalas [26, Theorem 1.9] implies that $f_{\rho_1,\rho_2} = \xi_{\rho_2}^1 \circ (\xi_{\rho_1}^1)^{-1}$ is 4-Hölder.

Let $\Phi_0 : \Delta \to \mathrm{SL}_8(\mathbb{R})$ denote the representation given by the direct sum $\Phi_0 = \rho_1 \oplus \rho_2$. One checks that

$$\lambda_1(\rho_2(\gamma)) > \lambda_2(\rho_2(\gamma)) > \lambda_1(\rho_1(\gamma)) > \cdots > \lambda_4(\rho_1(\gamma)) > \lambda_3(\rho_2(\gamma)) > \lambda_4(\rho_2(\gamma))$$

for all non-trivial $\gamma \in \Delta$ and that Φ_0 is Borel Anosov with limit maps given by

$$\zeta_0^k(x) = \begin{cases} \{0\} \oplus \xi_{\rho_2}^k(x) & \text{if } k = 1,2 \\ \xi_{\rho_1}^{k-2}(x) \oplus \xi_{\rho_2}^2(x) & \text{if } k = 3,4,5 \\ \mathbb{R}^4 \oplus \xi_{\rho_2}^{k-4}(x) & \text{if } k = 6,7. \end{cases}.$$

Then, the fact that f_{ρ_1,ρ_2} is 4-Hölder implies that $\zeta_0^1 \circ (\zeta_0^3)^{-1}$ is also 4-Hölder. In particular, $\zeta_0^1 \circ (\zeta_0^3)^{-1} : \Lambda_{\alpha_3}(\Phi_0(\Delta)) \to \Lambda_{\alpha_1}(\Phi_0(\Delta))$ is Lipschitz.

However, $\Phi_0(\Delta)$ is not Zariski dense. Since Δ is the free group on two generators, there exists an arbitrary small deformation $\Phi : \Delta \to \mathrm{SL}_8(\mathbb{R})$ of Φ_0 which is Borel Anosov with Zariski dense image. Arguing exactly as in [31, Section 9], one can show that Φ_0 and $\wedge^3\Phi_0$ are both $(1,1,2)$-hyperconvex. Therefore, we may assume that Φ and $\wedge^3\Phi$ are both $(1,1,2)$-hyperconvex.

One may then use standard techniques (cf. [31]) to show that if Φ is sufficiently close to Φ_0, then

$$\frac{2}{3} \le \frac{\alpha_1(\lambda(\Phi(\gamma)))}{\alpha_1(\lambda(\Phi_0(\gamma)))} \le \frac{3}{2} \quad \text{and} \quad \frac{2}{3} \le \frac{\alpha_1(\lambda(\wedge^3\Phi(\gamma)))}{\alpha_1(\lambda(\wedge^3\Phi_0(\gamma)))} \le \frac{3}{2}$$

for all non-trivial $\gamma \in \Delta$. Let $\zeta = (\zeta^k)$ be the limit map of $\Phi(\Delta)$ and $\hat\zeta_0^1 : \partial\Delta \to \Lambda_{\alpha_1}(\wedge^3\Phi_0(\Delta))$ and $\hat\zeta^1 : \partial\Delta \to \Lambda_{\alpha_1}(\wedge^3\Phi(\Delta))$ be limit maps of $\wedge^3\Phi_0$ and $\wedge^3\Phi$. One may again apply Tsouvalas's theorem [26, Theorem 1.9] to conclude that $\zeta^1 \circ (\zeta_0^1)^{-1}$ and $\hat\zeta_0^1 \circ (\hat\zeta^1)^{-1}$ are $\frac{2}{3}$-Hölder. There is a C^1-equivariant identification of $\Lambda_{\alpha_1}(\wedge^3\Phi_0(\Delta))$ with $\Lambda_{\alpha_3}(\Phi_0(\Delta))$ and an analogous identification for Φ, so we may conclude that $\zeta_0^3 \circ (\zeta^3)^{-1}$ is $\frac{2}{3}$-Hölder. Now set

$$\Gamma := \Phi(\Delta) < \mathrm{SL}_8(\mathbb{R}).$$

Then the limit map

$$\zeta^1 \circ (\zeta^3)^{-1} = \left(\zeta^1 \circ (\zeta_0^1)^{-1}\right) \circ \left(\zeta_0^1 \circ (\zeta_0^3)^{-1}\right) \circ \left(\zeta_0^3 \circ (\zeta^3)^{-1}\right)$$

is a $\frac{16}{9}$-Hölder and hence yields a Lipschitz map from Λ_{α_3} to Λ_{α_1}. Since Γ is Borel Anosov, the projection map $\Lambda_{\{\alpha_1,\alpha_3\}} \to \Lambda_{\alpha_3}$ is now a bi-Lipschitz homeomorphism. This proves Proposition 6.1.

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