

# Discrete Subgroups Generated by Lattices in Opposite Horospherical Subgroups

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## 0. INTRODUCTION

Let  $G$  be a center-free connected semisimple real algebraic group with no compact factors. The unipotent radical of a proper parabolic subgroup of  $G$  is called *horospherical*. Two horospherical subgroups are called *opposite* if they are the unipotent radicals of two opposite parabolic subgroups.

We recall the following theorem:

**THEOREM 0.1** ([12, Theorem 7.1.1], also see [18, Theorem 4.2]). *Suppose that the real rank of  $G$  is at least 2. Then for any irreducible non-uniform lattice  $\Gamma$  in  $G(\mathbb{R})$ , there exists a pair of opposite horospherical subgroups  $U_1$  and  $U_2$  defined over  $\mathbb{R}$  such that  $\Gamma \cap U_i(\mathbb{R})$  is a lattice in  $U_i(\mathbb{R})$  for  $i = 1, 2$ .*

This theorem was one of the main steps in proving the arithmeticity of a non-uniform lattice in such groups, without the use of the superrigidity theorem [13] which had settled the arithmeticity of both uniform and non-uniform lattices at once.

In this paper we study the converse problem, which may be stated as follows: suppose that one is given opposite horospherical real subgroups  $U_1$  and  $U_2$  and lattices  $F_1$  and  $F_2$  inside  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively. Then under what conditions is the group generated by  $F_1$  and  $F_2$  discrete? What discrete subgroups of  $G$  can arise in this way?

Our main result is that if  $G$  is absolutely simple, then (under some additional assumptions on  $U_1$  and  $U_2$ ) any discrete group generated by  $F_1$  and  $F_2$  is a non-uniform lattice in  $G(\mathbb{R})$ . In particular we prove the following:

**THEOREM 0.2.** *Let  $G$  be an adjoint absolutely simple  $\mathbb{R}$ -algebraic group with real rank at least 2,  $U_1, U_2$  a pair of opposite horospherical  $\mathbb{R}$ -subgroups of  $G$ . Suppose that  $G$  is split over  $\mathbb{R}$  and that  $U_1$  is not the unipotent radical of a Borel subgroup in a group of type  $A_2$ .*

*Let  $F_1$  and  $F_2$  be lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively. If  $F_1$  and  $F_2$  generate a discrete subgroup, then there exists a  $\mathbb{Q}$ -form of  $G$  with respect to which  $U_1$  and  $U_2$  are defined over  $\mathbb{Q}$  and  $F_i$  is commensurable to  $U_i(\mathbb{Z})$  for each  $i = 1, 2$ . Furthermore the discrete subgroup  $\Gamma_{F_1, F_2}$  generated by  $F_1$  and  $F_2$  is commensurable to  $G(\mathbb{Z})$ .*

Let us remark that Theorem 0.2 is not true in the group of real rank one; in fact there exist discrete subgroups of the form  $\Gamma_{F_1, F_2}$  which are not lattices. For example, the subgroup  $\Gamma_n$  of  $SL_2(\mathbb{R})$  generated by  $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$  for some nonzero  $n \in \mathbb{Z}$  is not a lattice if  $n > 2$ . To see this, note that the subgroup  $\Gamma_n$  is contained in the subgroup generated by the elements  $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , and the fundamental domain in the upper half plane for the latter subgroup is the set  $\{z \in \mathbb{H}^+ \mid |z| > 1, |\operatorname{Re}(z)| < n/2\}$  which has infinite volume when  $n > 2$ .

We can extend Theorem 0.2 in some cases by dropping the assumption that  $G$  is split over  $\mathbb{R}$ , giving the following more technical result.

**THEOREM 0.3.** *Let  $Z_i$  denote the center of  $U_i$  for each  $i = 1, 2$ . In Theorem 0.2 the assumption that  $G$  is split over  $\mathbb{R}$  may be replaced by one of*

the following:

- (1)  $U_1$  is commutative, and either  $\text{rank}_{\mathbb{R}}(\mathbf{G}) \geq 3$  or  $\mathbf{G}$  is not of type  $E_6$ .
- (2)  $U_1$  is Heisenberg, and either  $\text{rank}_{\mathbb{R}}(\mathbf{G}) \geq 3$  or  $\mathbf{G}$  is not of type  $A_2$ ,  $B_n$ , or  $D_n$ .
- (3)  $[U_1, U_1] = Z_1$ ,  $Z_1$  is not the root group of a highest real root, and either  $\text{rank}_{\mathbb{R}}(G_0) \geq 3$ , or  $G_0$  is not of type  $E_6$ , where  $G_0$  is the subgroup generated by  $Z_1$  and  $Z_2$ .
- (4) If  $[U_1, U_1] \neq Z_1$ ,  $Z_1$  is the root group of a highest real root, and either  $\text{rank}_{\mathbb{R}}(G'_0) \geq 3$  or  $G'_0$  is not of type  $E_6$ , where  $G'_0$  is the subgroup generated by the center of  $U'_1$  and of  $U'_2$ , and  $U'_i$  is the centralizer of  $\tilde{U}_i = \{g \in U_i \mid gug^{-1}u^{-1} \in Z_i \text{ for all } u \in U_i\}$  in  $U_i$ .

Since Theorem 0.2 follows from Theorem 0.3, we will refer to Theorem 0.3 as the main theorem hereafter.

As a corollary, we obtain a complete classification of discrete subgroups generated by lattices in opposite horospherical subgroups  $U_1$  and  $U_2$  considered in the main theorem. In particular, we note the following:

**COROLLARY 0.4.** *Let  $G$ ,  $U_1$ , and  $U_2$  be as in the main theorem. Then any discrete subgroup generated by two lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$  is an arithmetic subgroup.*

Therefore for the  $\mathbb{R}$ -split groups, combining Theorem 0.2 with Theorem 0.1, we obtain the following criterion for a discrete subgroup to be a non-uniform lattice:

**COROLLARY 0.5.** *Let  $G$  be an adjoint absolutely simple  $\mathbb{R}$ -split group with rank at least 2 and  $\Gamma$  a discrete subgroup. In addition, assume that  $G$  is not of type  $A_2$ . Then  $\Gamma$  is a non-uniform lattice if and only if there exists a pair of opposite horospherical subgroups  $U_1$  and  $U_2$  of  $G$  such that  $\Gamma \cap U_i$  is Zariski dense in  $U_i$  for each  $i = 1, 2$ .*

(Note that any arithmetic subgroup of  $G(\mathbb{R})$  which has a non-trivial unipotent element is a non-uniform lattice in  $G(\mathbb{R})$  (e.g., [16, Theorem 10.18]).)

*Remark.* We note that the main theorem presents a strong necessary condition for discreteness of a subgroup generated by lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ . We refer the readers to [9] for discreteness criteria of this kind in  $SL_2(\mathbb{R})$ .

We call a horospherical subgroup  $U$   $\mathbb{R}$ -Heisenberg if  $[U, U]$  is equal to the center  $Z(U)$  of  $U$ , i.e., 2-step nilpotent and  $Z(U)$  is the root group of a highest real root of  $G$ . If  $U$  is  $\mathbb{R}$ -Heisenberg and  $\dim Z(U) = 1$ , then  $U$  is Heisenberg. It should be noted that the main theorem would not cover the

cases when  $U_1$  is either  $\mathbb{R}$ -Heisenberg with  $\dim(Z(U)) > 1$  or Heisenberg in an  $\mathbb{R}$ -split group of type  $A_2$ , even if we were to drop the assumption on  $\mathbb{R}$ -anisotropic factors. On the other hand we can see that the main theorem (Theorem 0.3) implies Theorem 0.2 as follows: if  $G$  is split over  $\mathbb{R}$ , none of the subgroups  $G_0$ ,  $G'_0$ , and  $H$  has  $\mathbb{R}$ -anisotropic factors. Therefore the case when  $U_1$  is either commutative or Heisenberg follows from (1). When  $U_1$  is neither of those,  $U_1$  satisfies assumption (2) or (3) according to whether  $U_1$  is 2-step nilpotent or not, respectively.

The proof of the main theorem is given in three parts according to whether the horospherical subgroups involved are commutative (Theorem 4.1.1), Heisenberg (Theorem 4.2.11), or non- $\mathbb{R}$ -Heisenberg (Theorem 4.3.4). One of the main ideas for the first two cases is to use Raghunathan's conjecture proved by Ratner (Theorem 3.3.1) for the action of the group of real points of the commutator subgroup of  $N(U_1) \cap N(U_2)$  on the space of lattices in  $U_i(\mathbb{R})$  for each  $i = 1, 2$ . A theorem of Margulis on the construction of a representation (Theorem 2.4.2) enables us to reduce the non- $\mathbb{R}$ -Heisenberg horospherical subgroup cases to the commutative cases. In fact the assumptions on  $\mathbb{R}$ -anisotropic factors in (1)–(3) arise because of the dependence of our proof on Ratner's theorem.

The main theorem of this paper was announced in [14] together with a detailed sketch of the proof, in the case when  $G(\mathbb{R}) = SL_n(\mathbb{R})$ ,  $n \geq 3$ , and the horospherical subgroups involved are commutative.

## 1. PRELIMINARIES

### 1.1. Notation and Terminology

1.1.1. As usual,  $\mathbb{C}$ ,  $\mathbb{R}$ ,  $\mathbb{Q}$ ,  $\mathbb{Z}$ , and  $\mathbb{N}$  denote the complex numbers, the reals, the rationals, the integers, and the positive integers.

1.1.2. For a group  $H$ ,  $Z(H)$  denotes the center of  $H$ . For any subset  $F \subset H$ ,  $N(F)$  and  $C(F)$  denote the normalizer and the centralizer of  $F$  in  $H$ , respectively.

For a Lie algebra  $\mathfrak{h}$ , the commutator of two elements  $X, Y$  of  $\mathfrak{h}$  is denoted by  $[X, Y]$  and for any two subsets  $A$  and  $B$  of  $\mathfrak{h}$ ,  $[A, B]$  denotes the linear subspace generated by all the commutators  $[X, Y]$ ,  $X \in A$ ,  $Y \in B$ .  $Z(\mathfrak{h})$  denotes the center of  $\mathfrak{h}$ , that is,  $\{X \in \mathfrak{h} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{h}\}$ .

1.1.3. Two subgroups  $H_1$  and  $H_2$  are called *commensurable* if  $H_1 \cap H_2$  has a finite index in both of  $H_1$  and  $H_2$ .  $H_1 \times H_2$  denotes the direct product of  $H_1$  and  $H_2$  and  $L \rtimes N$  the semi-direct product of  $L$  and a normal subgroup  $N$ .

1.1.4. For an algebraic group  $H$ ,  $R(H)$  and  $R_u(H)$  denote the radical and the unipotent radical of  $H$ , respectively. For a Lie group  $H$ , we denote by  $H^0$  the connected component of the identity of  $H$ .

1.1.5. For a subfield  $k$  of  $\mathbb{R}$ , a linear algebraic  $\mathbb{R}$ -group  $G \subset GL_n(\mathbb{C})$  is defined over  $k$  if it consists of all matrices whose entries annihilate some set of polynomials on  $M_n(\mathbb{C})$  with coefficients in  $k$ . In this case, we denote by  $G(J)$  the subgroup  $\{g = (g_{ij}) \in G \mid g, g^{-1} \in GL_n(J), g_{ij} = \delta_{ij} \pmod{J}\}$  for any subring  $J$  of  $k$  and  $G(\mathbb{R}) = G \cap GL_n(\mathbb{R})$ .

1.1.6. A  $k$ -form of an algebraic  $\mathbb{R}$ -group  $G$  is a pair  $(\tilde{G}, f)$  where  $\tilde{G}$  is an algebraic group defined over  $k$  and  $f$  an isomorphism  $G \rightarrow \tilde{G}$  defined over  $\mathbb{R}$ . For a given  $k$ -form  $(\tilde{G}, f)$  of  $G$ , we denote  $f^{-1}(\tilde{G}(J))$  by  $G(J)$  for a subring  $J$  of  $k$ . If an algebraic  $\mathbb{R}$ -subgroup  $H$  of  $G$  is such that  $f(H)$  is a  $k$ -subgroup of  $\tilde{G}$ , we say that  $H$  is defined over  $k$  and denote  $H \cap G(J)$  by  $H(J)$ . A subgroup commensurable to  $G(\mathbb{Z})$  is called an arithmetic subgroup of  $G(\mathbb{R})$ .

1.1.7. We extend the definition of an arithmetic group to a semisimple Lie group with finite center as follows. Let  $G$  be a connected semisimple Lie group with finite center. A discrete subgroup  $\Gamma$  of  $G$  is called an arithmetic subgroup if there exist a connected adjoint semisimple algebraic  $\mathbb{Q}$ -group  $\mathbf{G}$  and an isogeny  $p: G \rightarrow \mathbf{G}(\mathbb{R})^0$  such that  $p(\Gamma)$  is commensurable to  $\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0$ .

1.1.8. A connected algebraic  $k$ -group  $G$  is called absolutely almost simple if it has no connected normal subgroup, and almost  $k$ -simple if it has no connected normal  $k$ -subgroup. A connected semisimple algebraic group  $G$  is called simply connected (resp. adjoint) if every central isogeny  $\phi: G' \rightarrow G$  (resp.  $\phi: G \rightarrow G'$ ), for  $G'$  connected, is an algebraic group isomorphism. In characteristic 0 case, a connected semisimple algebraic group is adjoint if and only if its center is trivial.

1.1.9. For a locally compact group  $G$ , a discrete subgroup  $\Gamma$  of  $G$  is called a lattice if the quotient  $G/\Gamma$  has a finite invariant measure. A lattice  $\Gamma$  in  $G$  is called *uniform* if  $G/\Gamma$  is compact, and *non-uniform*, otherwise. A lattice  $\Gamma$  in a connected semisimple Lie group  $G$  with finite center is called irreducible if for every normal subgroup  $N$  of positive dimension,  $\Gamma$  is dense when projected onto  $G/N$ .

1.1.10. A discrete subgroup  $F$  in a real vector space  $V$  is called a quasi-lattice and in particular, if  $F$  spans  $V$ , then we call it a lattice. The determinant of a lattice  $F$  in  $V$  is the volume of the quotient  $V/F$ .

1.1.11. A connected semisimple  $k$ -group  $G$  is *isotropic over  $k$*  if it contains a non-trivial  $k$ -split torus and is *anisotropic over  $k$*  otherwise.

For  $k = \mathbb{R}$ ,  $G$  is anisotropic over  $\mathbb{R}$  if and only if  $G(\mathbb{R})$  is compact.

1.1.12. Let  $G$  be a connected semisimple algebraic  $k$ -group. We call a subgroup  $U$  a horospherical  $k$ -subgroup if it is the unipotent radical of a parabolic  $k$ -subgroup  $P$  such that  $P \cap G'$  is a proper parabolic subgroup

of  $G'$  for each semisimple normal  $k$ -subgroup of  $G$ . It is known that the normalizer of a horospherical  $k$ -subgroup is a parabolic  $k$ -subgroup.

We note that, in our definition, the existence of a horospherical  $\mathbb{R}$ -subgroup of a semisimple  $\mathbb{R}$ -group  $G$  implies that  $G$  does not have any  $\mathbb{R}$ -anisotropic factors.

1.1.13. In a connected simple algebraic group  $G$ , a horospherical subgroup  $U$  is called *Heisenberg* if  $[U, U] = Z(U)$  and  $Z(U)$  has dimension one. We call a horospherical subgroup  $U$   $\mathbb{R}$ -Heisenberg if  $[U, U] = Z(U)$  and  $Z(U)$  is the root group of a highest real root of  $G$  (also see Subsection 1.3.1).

A parabolic (resp. horospherical) subgroup of a semisimple algebraic group is called *reflexive* if its conjugacy class contains an opposite parabolic (resp. horospherical) subgroup to it.

1.1.14. Let  $G$  be a connected semisimple algebraic  $k$ -group and  $P$  a parabolic  $k$ -subgroup. A Levi decomposition  $P = LR_u(P)$  of  $P$  is called a  $k$ -Levi decomposition if  $L$  is defined over  $k$ . We denote by  $\text{Ad}: L \rightarrow GL(\mathcal{Z})$  the representation of  $L$  in  $\mathcal{Z} = \text{Lie}(R_u(P))$  which is the restriction of the adjoint representation of  $P$ .

1.1.15. Let  $G$  be a connected semisimple algebraic  $\mathbb{R}$ -group and  $U_1$  and  $U_2$  a pair of opposite horospherical  $\mathbb{R}$ -subgroups. For lattices  $F_1$  and  $F_2$  in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively, we denote by  $\Gamma_{F_1, F_2}$  the subgroup of  $G(\mathbb{R})$  generated by  $F_1$  and  $F_2$ .

1.1.16. Let  $(G, U_1, U_2)$  be a triple where  $G$  is a connected semisimple adjoint algebraic  $\mathbb{R}$ -group with no  $\mathbb{R}$ -anisotropic factors and  $U_1$  and  $U_2$  a pair of opposite horospherical  $\mathbb{R}$ -subgroups. In order to avoid the lengthy repetition, we say the triple  $(G, U_1, U_2)$  has property (A) if every discrete subgroup of the form  $\Gamma_{F_1, F_2}$  is an arithmetic subgroup of  $G(\mathbb{R})^0$  where  $F_1$  and  $F_2$  are lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively. We use this terminology only in Section 4 where we prove the main theorem.

1.1.17. All algebraic groups are assumed to be connected and all fields, usually denoted by  $k$  or  $K$ , have characteristic 0. We freely use terminology from the theory of algebraic groups in [1, 5] and of Lie algebras in [8].

## 1.2. Some Known Algebraic Lemmas

1.2.1 (cf. [5, Chaps. 3–4]). Let  $G$  be a connected semisimple algebraic  $k$ -group,  $S$  a maximal  $k$ -split torus of  $G$ , and  $T$  a maximal  $k$ -torus containing  $S$ . Denote by  ${}_k\Phi = \Phi(S, G)$  (resp.  $\Phi = \Phi(T, G)$ ) the set of roots of  $G$  with respect to  $S$  (resp.  $T$ ). An element in  ${}_k\Phi$  is called a  $k$ -root. We choose compatible orderings on  $\Phi$  and  ${}_k\Phi$ , and let  ${}_k\Delta$  and  $\Delta$  be the simple roots for these orderings. Let  $j$  be the map  $\Phi \rightarrow {}_k\Phi \cup \{0\}$  induced by the restriction onto  $S$ .

For each  $b \in \Phi(T, G)$ , we denote by  $U_b$  the unique one parameter root subgroup associated with  $b$ . We observe that  $\text{Lie}(U_b) = \mathcal{Z}_b := \{v \in \text{Lie}(G) \mid \text{Ad } t(v) = b(t)v, t \in T\}$ ,  $\dim(\mathcal{Z}_b) = 1$ , and  $\text{Lie}(G) = \bigoplus_{b \in \Phi(T, G)} \mathcal{Z}_b \oplus \text{Lie}(T)$ .

A subset  $\Psi \subset {}_k\Phi$  is called closed if  $a, b \in \Psi$  and  $a + b \in {}_k\Phi$  imply  $a + b \in \Psi$ . If  $\Psi$  is closed, then we denote by  $G_\Psi$  the subgroup generated by  $T$  and the subgroups  $U_a$ ,  $a \in j^{-1}(\Psi \cup \{0\})$ , and by  $G_\Psi^*$  the subgroup generated by all the subgroups  $U_a$ ,  $a \in j^{-1}(\Psi)$ . The subgroups  $G_\Psi$  and  $G_\Psi^*$  are algebraic and do not depend on the choice of a maximal torus  $T$  containing  $S$ . The groups  $G_\Psi$  and  $G_\Psi^*$  are defined over  $k$  if and only if  $\Psi$  is invariant under  $\text{Gal}(K/k)$ ,  $K$  the separable closure of  $k$  [5, Proposition 3.14]. If  $G_\Psi^*$  is unipotent, then it will be also denoted by  $U_\Psi$  and the set  $\Psi$  in this case will be called unipotent. For a closed unipotent subset  $\Psi \subset {}_k\Psi$ , the unipotent subgroup  $U_\Psi$  is defined and split over  $k$ .

1.2.2. For  $\Theta \subset {}_k\Delta$ ,  $[\Theta]$  denotes the  $\mathbb{Z}$ -linear combinations of  $\Theta$  which are  $k$ -roots.

We define the following closed subsets of  ${}_k\Phi$ :

$$\begin{aligned} \pi_\theta &= [\Theta] \cup {}_k\Phi^+, & \pi_\theta^- &= [\Theta] \cup {}_k\Phi^-, \\ \beta_\theta &= {}_k\Phi^+ - [\Theta], & \beta_\theta^- &= {}_k\Phi^- - [\Theta]. \end{aligned}$$

For the sake of simplicity, we shall denote by  ${}_kP_\Theta, {}_kP_\Theta^-, {}_kV_\Theta, {}_kV_\Theta^-$ , the subgroups  $G_{\pi_\Theta}, G_{\pi_\Theta^-}, U_{\beta_\Theta}, U_{\beta_\Theta^-}$ . The subgroups  ${}_kP_\Theta, {}_kP_\Theta^-, {}_kV_\Theta$ , and  ${}_kV_\Theta^-$  are connected and defined over  $k$ .

The subgroups  ${}_kP_\Theta$  (resp.  ${}_kV_\Theta$ ),  $\Theta \subset {}_k\Delta$  are called standard parabolic (resp. horospherical)  $k$ -subgroups of  $G$  associated with  $S$  and  ${}_k\Phi^+$ . Every parabolic (resp. horospherical)  $k$ -subgroup of  $G$  is conjugate by an element of  $G(k)$  to a unique standard parabolic (resp. horospherical) subgroup.

LEMMA. (1)  $R_u({}_kP_\Theta) = {}_kV_\Theta, R_u({}_kP_\Theta^-) = {}_kV_\Theta^-$  for any  $\Theta \subset {}_k\Delta$ .

(2) Any pair of opposite parabolic (resp. horospherical)  $k$ -subgroups is conjugate by an element of  $G(k)$  to the pair  ${}_kP_\Theta, {}_kP_\Theta^-$  (resp.  ${}_kV_\Theta, {}_kV_\Theta^-$ ) for some  $\Theta \subset {}_k\Delta$ .

(3) Any two parabolic  $k$ -subgroups opposite to  $P$  are conjugate by a unique element of  $R_u(P)(k)$ .

1.2.3. The root system  $\Phi(S, G)$  (resp.  $\Phi(T, G)$ ) is irreducible if and only if  $G$  is almost  $k$ -simple (resp. absolutely almost simple). The type of an irreducible root system  $\Phi$  is by definition the type of its Dynkin diagram. For a  $k$ -simple  $k$ -group  $G$ , we refer the type of  $\Phi(T, G)$  by the absolute type, or simply the type of  $G$ , and the type of  $\Phi(S, G)$  by the  $k$ -type of  $G$ .

1.2.4. PROPOSITION [21, 3.1.2]. (1) *A connected simply connected (resp. adjoint) semisimple  $k$ -group decomposes uniquely into a direct product of simply connected (resp. adjoint) almost  $k$ -simple normal  $k$ -groups.*

(2) *A connected semisimple  $k$ -group decomposes into an almost direct product of almost  $k$ -simple normal  $k$ -groups.*

1.2.5. PROPOSITION. *For a connected semisimple  $k$ -group  $G$ , there exists a sequence  $\tilde{G} \xrightarrow{\tilde{p}} G \xrightarrow{\bar{p}} \bar{G}$ , where  $\tilde{G}$  is a simply connected  $k$ -group,  $\bar{G}$  is an adjoint  $k$ -group, and  $\tilde{p}$  and  $\bar{p}$  are central  $k$ -isogenies. The groups  $\tilde{G}$  and  $\bar{G}$  and the isogenies  $\tilde{p}$  and  $\bar{p}$  are determined uniquely up to  $k$ -isomorphism.*

1.2.6. Let  $k'$  be a finite separable field extension of  $k$ . We denote by  $R_{k'/k}$  the restriction of scalar functor from  $k'$  to  $k$ . (For definition, see [5, 6.17–6.21].)

PROPOSITION. (1) *If  $H$  is a  $k'$ -group, then the functor  $R_{k'/k}$  defines a bijection of the set of parabolic (resp. horospherical)  $k'$ -subgroups of  $H$  onto the set of parabolic (resp. horospherical)  $k$ -subgroups of the  $k$ -group  $R_{k'/k}(H)$  and if  $H$  is reductive, then  $\text{rank}_{k'}(H) = \text{rank}_k(R_{k'/k}(H))$ .*

(2) *If  $G$  is a simply connected (resp. adjoint) almost  $k$ -simple  $k$ -group, then there exists a finite separable field extension  $k'$  of  $k$  and a connected simply connected (resp. adjoint) absolutely almost simple  $k'$ -group  $G'$  such that  $G = R_{k'/k}G'$ .*

1.2.7. The next proposition is useful when we want to determine whether some algebraic groups and algebraic maps are defined over a subfield.

PROPOSITION (see [27, 3.1.8, 3.1.10]). (1) *Suppose that  $G \subset GL_n(\mathbb{C})$  is an algebraic group and that  $G \cap GL_n(k)$  is Zariski dense in  $G$  for some subfield  $k$  of  $\mathbb{C}$ . Then  $G$  is defined over  $k$ .*

(2) *Suppose that  $V, W$  are  $k$ -varieties and that  $f: V \rightarrow W$  is a regular map. Suppose also that there is a set  $A \subset V_k$  which is Zariski dense in  $V$  such that  $f(A) \subset W_k$ . Then  $f$  is defined over  $k$ .*

### 1.3. Adjoint Representation and Maximal Subgroups

1.3.1. Let  $G$  be a connected almost  $k$ -simple  $k$ -group,  $S$  a maximal  $k$ -split torus, and  $\Phi(S, G)$  a corresponding root system. We fix a basis  ${}_k\Delta$  of  $\Phi(S, G)$ . Since  $G$  is  $k$ -simple,  $\Phi(S, G)$  is an irreducible (but not necessarily reduced) root system. We note that a horospherical  $k$ -subgroup  $U$  is  $k$ -Heisenberg if and only if  $U$  is conjugate to  ${}_kV_\Theta$  such that  $[{}_kV_\Theta, {}_kV_\Theta] = Z({}_kV_\Theta)$  and  $Z({}_kV_\Theta) = U_{\alpha_h}$  where  $\alpha_h$  is the highest root in  $\Phi(S, G)$ . Note that the center of a  $k$ -Heisenberg subgroup is the root group of a highest  $k$ -root.

We note that if  $U_{\alpha_h}$  has dimension one, a  $k$ -Heisenberg subgroup is in fact a Heisenberg subgroup. If  ${}_kV_{\Theta}$  is  $k$ -Heisenberg (resp. commutative), we call  ${}_k\Delta - \Theta$  the set of  $k$ -Heisenberg (resp. commutative) roots. In an irreducible root system  $\Phi(S, G)$ , there exist a unique highest  $k$ -root  $\alpha_h$  and a unique set of simple roots  ${}_k\Delta_H \subset {}_k\Delta$  such that  ${}_kV_{{}_k\Delta - {}_k\Delta_H}$  is the unique  $k$ -Heisenberg standard horospherical  $k$ -subgroup of  $G$ .

It is not difficult to prove the following lemma which characterizes the commutative and  $k$ -Heisenberg roots in each irreducible root system.

LEMMA. (1) *A simple root  $\alpha$  is a commutative root if and only if its coefficient in  $\alpha_h$  is 1.*

(2) *The set  ${}_k\Delta_H$  is determined uniquely by the condition that if  $\alpha \in {}_k\Delta_H$ , then  $\alpha_h - \alpha$  is a root and the sum of the coefficients of  ${}_k\Delta_H$  in  $\alpha_h$  is 2.*

k-type	$\alpha_h$	${}_k\Delta_H$	Commutative roots
$A_n$	$\alpha_1 + \alpha_2 + \dots + \alpha_n$	$\{\alpha_1, \alpha_2\}$	$\{\alpha_1, \dots, \alpha_n\}$
$B_n$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_n$	$\alpha_2$	$\alpha_1$
$C_n$	$2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$	$\alpha_1$	$\alpha_n$
$D_n$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$\alpha_2$	$\{\alpha_1, \alpha_{n-1}, \alpha_n\}$
$E_6$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$\alpha_2$	$\{\alpha_1, \alpha_6\}$
$E_7$	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$\alpha_1$	$\alpha_7$
$E_8$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	$\alpha_8$	$\emptyset$
$F_4$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$\alpha_1$	$\emptyset$
$G_2$	$3\alpha_1 + 2\alpha_2$	$\alpha_2$	$\emptyset$
$BC_n$	$2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + 2\alpha_n$	$\alpha_1$	$\emptyset$

1.3.2 (cf. [7, 5.5.1]). Among maximal parabolic subgroups, the adjoint representation of a Levi component on the Lie algebra of the unipotent radical is absolutely irreducible if and only if the unipotent radical is commutative. These cases are quite restricted, and the list is well known (Table I).

1.3.3 (cf. [7, 5.5.2]). The commutator quotient of the Lie algebra of a Heisenberg horospherical subgroup carries a symplectic form defined in terms of the Lie bracket. This form is preserved by the action of the Levi component. Except for the groups of type  $A_n$ , the representation of the Levi component on the commutator quotient is absolutely irreducible (Table II).

1.3.4. The following two theorems of Dynkin classify the maximal connected subgroups of  $SL_n(\mathbb{C})$  into the three categories—reducible (as linear groups), irreducible non-simple, and irreducible simple. It is known that any irreducible connected Lie subgroup of  $SL_n(\mathbb{C})$  is semisimple [15, Theorem 1.1.1].

TABLE I  
List of Parabolic Subgroups with Commutative Unipotent Radical

Group	Levi component	Unipotent radical
$GL_n$	$GL_k \times GL_{n-k}$	$\mathbb{C}^k \otimes \mathbb{C}^{n-k}$
$SO_{2n+1}$	$O_{2n-1} \times GL_1$	$\mathbb{C}^{2n-1}$
$Sp_{2n}$	$GL_n$	$S^2(\mathbb{C}^n)$
$SO_{2n}$	$O_{2n-1} \times GL_1$	$\mathbb{C}^{2(n-1)}$
$SO_{2n}$	$GL_n$	$\Lambda(\mathbb{C}^n)$
$E_6$	$Spin_{10} \times GL_1$	$spin_{10}^+$
$E_7$	$E_6 \times GL_1$	$V_{27}$

THEOREM (cf. [15, 3.3.1–3.3.2]). *Let  $M$  be a maximal connected (complex) Lie subgroup of  $SL_n(\mathbb{C})$ .*

(1) *If  $M$  is reducible, then it is a maximal parabolic subgroup of  $SL_n(\mathbb{C})$ .*

(2) *If  $M$  is non-simple irreducible, then it is conjugate to the subgroup  $SL_s(\mathbb{C}) \otimes SL_t(\mathbb{C})$  where  $n = st$ ,  $s, t \geq 2$ .*

1.3.5. THEOREM (cf. [15, 3.3.3]). *Let  $R: M \rightarrow GL(V)$  be a non-trivial irreducible linear representation of a simply connected simple complex linear group  $M$ . If there are no nondegenerate bilinear forms in  $V$  invariant under  $R$ , then  $R(M)$  is a maximal connected subgroup of  $SL(V)$ , and if  $R$  is orthogonal or symplectic, then  $R$  is a maximal connected subgroup of  $SO(V)$  or  $Sp(V)$ , respectively. The only exceptions are the representations listed in Table 7 in [15].*

TABLE II  
List of Parabolic Subgroups with Heisenberg Unipotent Radical

Group	Levi component	Commutator quotient
$GL_n$	$GL_1 \times GL_{n-2} \times GL_1$	$\mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$
$O_n$	$O_{n-4} \times GL_2$	$\mathbb{C}^{2(n-4)}$
$Sp_{2n}$	$Sp_{2(n-1)} \times GL_1$	$\mathbb{C}^{2(n-1)}$
$E_6$	$GL_6$	$\Lambda^3(\mathbb{C}^6)$
$E_7$	$Spin_{12} \times GL_1$	$spin_{12}^+$
$E_8$	$E_7 \times GL_1$	$V_{56}$
$F_4$	$Sp_6 \times GL_1$	$\Lambda_{prim}^3(\mathbb{C}^6)$
$G_2$	$GL_2$	$S^3(\mathbb{C}^2)$

1.3.6. As a corollary of the above theorem and Subsections 1.3.2–1.3.3, we obtain the following two propositions. We denote by  $P$  a parabolic  $\mathbb{R}$ -subgroup of an adjoint absolutely simple algebraic group  $\mathbf{G}$ , by  $\mathbf{L}$  a Levi  $\mathbb{R}$ -subgroup of  $P$  and  $\mathbf{H} = [\mathbf{L}, \mathbf{L}]$ , and  $H = \mathbf{H}(\mathbb{R})^0$ . The notation  $\text{Ad}$  is as in Subsection 1.1.14.

PROPOSITION. *If  $R_u(P)$  is commutative, the representation  $\text{Ad}_i : \mathbf{H} \rightarrow SL(\mathcal{Z}_i)$  is absolutely irreducible and  $\text{Ad}_i(H)$  is a maximal connected closed subgroup of  $SL(\mathcal{Z}_i(\mathbb{R}))$ .*

*Proof.* The first statement is due to the well known fact that all the representations listed in Subsection 1.3.2 are absolutely irreducible. By Proposition 1.2.5, there exist a simply connected simple  $\mathbb{R}$ -group  $\tilde{\mathbf{H}}$  and an  $\mathbb{R}$ -isogeny  $p : \tilde{\mathbf{H}} \rightarrow \mathbf{H}$ . Consider the representation  $\text{Ad}_i \circ p$  of  $\tilde{\mathbf{H}}$ . By the previous theorem,  $\text{Ad}_i(p(\tilde{\mathbf{H}}))$ , which is the same as  $\text{Ad}_i(\mathbf{H})$ , is a maximal subgroup of  $SL(\mathcal{Z}_i)$ . Since  $\text{Ad}_i(\mathbf{H}(\mathbb{R})) \subset SL(\mathcal{Z}_i(\mathbb{R}))$ , it is clear that  $\text{Ad}_i(H)$  is a maximal connected closed subgroup of  $SL(\mathcal{Z}_i(\mathbb{R}))$ . ■

1.3.7. Consider the case when  $R_u(P)$  is Heisenberg. Except for groups of type  $A_n$ ,  $\mathcal{Z}_i$  has two  $\text{Ad}_i$ -invariant subspaces one of which is  $Z(\mathcal{Z}_i)$ . Denote by  $\mathcal{V}_i$  the other invariant subspace. For groups of type  $A_n$ ,  $\text{Ad}_i$  decomposes into the direct sum  $\rho \oplus \rho^* \oplus \text{id}$  where  $\rho$  is the  $(n - 1)$ -dimensional standard representation of  $A_{n-2}$ ,  $\rho^*$  is the dual of  $\rho$ , and  $\text{id}$  is the one-dimensional trivial representation. Let  $\mathcal{W}_i$  and  $\mathcal{W}_i^*$  be the invariant subspaces of  $\rho$  and  $\rho^*$ , respectively, and in this case, let  $\mathcal{V}_i$  denote  $\mathcal{W}_i \oplus \mathcal{W}_i^*$ . We shall denote by  $\text{Ad}'_i$  the restriction of  $\text{Ad}_i$  on  $\mathcal{V}_i$ . Set  $Sp(\mathcal{V}_i) = \{g \in SL(\mathcal{V}_i) \mid [gv, gw] = [v, w] \text{ for all } v, w \in \mathcal{V}_i\}$ .

PROPOSITION. (1) *If  $\mathbf{G}$  is not of type  $A_n$ ,  $\text{Ad}'_i$  is absolutely irreducible and  $\text{Ad}'_i(H)$  is a maximal connected closed subgroup of  $Sp(\mathcal{V}_i(\mathbb{R}))$  or  $Sp(\mathcal{V}_i(\mathbb{R}))$  itself.*

(2) *For  $A_n$ ,  $\text{Ad}'_i : \mathbf{H} \rightarrow Sp(\mathcal{V}_i)$  is equivalent to  $\rho \oplus \rho^*$  and  $\text{Ad}'_i(H)$  is the maximal connected semisimple closed subgroup of  $Sp(\mathcal{V}_i(\mathbb{R}))$ .*

*Proof.* Part (1) can be shown by the same argument as the proof of the previous proposition. To see (2), it is enough to note that  $\text{Ad}'_i(H)$  is the semisimple part of a Levi subgroup of the maximal parabolic subgroup  $\{g \in Sp(\mathcal{V}_i) \mid g\mathcal{W}_i = \mathcal{W}_i\}$  which stabilize  $\mathcal{W}_i$ . ■

1.3.8. LEMMA. *Let  $H$  be a connected simple Lie group and for each  $i = 1, 2$ ,  $\phi_i : H \rightarrow G_i$  an isomorphism for some Lie group  $G_i$ . Then  $\delta(H) = \{(\phi_1(h), \phi_2(h)) \mid h \in H\}$  is a maximal closed connected subgroup of  $G_1 \times G_2$ .*

*Proof.* If  $L$  is a closed connected subgroup which contains  $\delta(H)$  properly, then there exists an element  $(g, e) \in L$ . It follows that  $L$  contains the subgroup  $\{(\phi_1(h)g\phi_1(h^{-1}), e) \mid h \in H\}$ . Since  $H$  is simple

hence so is  $G_1$ ,  $\{(\phi_1(h)g\phi_1(h^{-1}), e) \mid h \in H\} = G_1 \times \{e\}$ . In the same way, we can show that  $L$  contains  $\{e\} \times G_2$ ; hence  $L = G_1 \times G_2$ . ■

**1.3.9. PROPOSITION.** *Set  $H = \{(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}) \mid A \in SL_n(\mathbb{C})\}$  and  $n \geq 4$ . Then a connected semisimple complex proper Lie subgroup of  $SL_{2n}(\mathbb{C})$  containing  $H$  properly is either  $\{(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}) \mid A, B \in SL_n(\mathbb{C})\}$  or  $SL_2(\mathbb{C}) \otimes SL_n(\mathbb{C})$ .*

*Proof.* Let  $M$  be a maximal closed semisimple subgroup of  $SL_{2n}(\mathbb{C})$  containing  $H$ . If  $M$  is reducible, then  $M$  is a semisimple part of a Levi subgroup, say  $L$ , of some maximal parabolic subgroup. Then  $L$  is the centralizer of a one-dimensional torus  $S$ . Since  $S$  should centralize  $H$ , we have that  $S = \{(\begin{smallmatrix} t & 0 \\ 0 & 1/t \end{smallmatrix}) \mid t \in \mathbb{C}^*\}$ . Therefore  $L = \{(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}) \in SL_{2n}(\mathbb{C}) \mid A, B \in GL_n(\mathbb{C})\}$  and hence  $M = [L, L]$ . Since  $H$  is a maximal subgroup of  $[L, L]$ , in this case  $[L, L]$  is the only semisimple subgroup containing  $H$ .

Suppose that  $M$  is irreducible and non-simple. Then  $M = gSL_m(\mathbb{C}) \otimes SL_k(\mathbb{C})g^{-1}$  for  $m > 1$  and  $k > 1$  such that  $mk = 2n$ . Since the Lie algebra of  $M$  is isomorphic to the direct sum of the Lie algebras of  $SL_m(\mathbb{C})$  and  $SL_k(\mathbb{C})$ , we have that  $n \leq m$  or  $n \leq k$ . It follows that  $k = n$  and hence  $m = 2$ . Now we will show that  $g \in SL_n(\mathbb{C}) \otimes SL_2(\mathbb{C})$ . Denote by  $T_0$  the diagonal subgroup of  $SL_{2n}(\mathbb{C})$  and set  $S = T_0 \cap SL_n(\mathbb{C}) \otimes SL_2(\mathbb{C})$ . By a direct calculation, we can see that  $C(S) = T_0$ . Let  $T_M$  be a maximal torus of  $M$  which contains  $S$ . Then  $T_M \subset C(S) = T_0$ . Since any two maximal tori are conjugate, there exists  $h \in SL_n(\mathbb{C}) \otimes SL_2(\mathbb{C})$  such that  $T_M = gh(T_0 \cap SL_n(\mathbb{C}) \otimes SL_2(\mathbb{C}))(gh)^{-1}$ . That  $gh(T_0 \cap SL_n(\mathbb{C}) \otimes SL_2(\mathbb{C}))(gh)^{-1} \subset T_0$  implies that  $gh \in N(T_0)$ . Since  $(gh)(SL_n(\mathbb{C}) \otimes SL_2(\mathbb{C}))(gh)^{-1}$  contains  $H$ , it follows that  $gh$  and hence  $g$  belongs to  $SL_n(\mathbb{C}) \otimes SL_2(\mathbb{C})$ .

We can now exclude the case when  $M$  is simple irreducible. We recall the well-known fact that the smallest possible dimensions of linear representations of the groups of types  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$ , and  $E_8$  are 7, 26, 27, 133, and 248 respectively. If  $M$  is of type  $G_2$ , it implies that  $2n$  is at least 8 and hence the rank of  $H$  is at least 4, yielding the contradiction. A similar argument can be applied for other types of exceptional groups to show that  $M$  is not of exceptional type. Now assume that  $M$  is of classical type. If it is of type  $B_k$ ,  $C_k$ , or  $D_k$ , then  $k$  is at least  $n - 1$  since the rank of  $H$  is  $n - 1$ . This excludes the type of  $B_k$  since the smallest possible dimension of linear representations of the groups of type  $B_{n-1}$  is  $2n + 1$ . If  $M$  is of type  $C_k$  or  $D_k$ , then we have  $k = n$  since the groups of type  $C_{n-1}$  or  $D_{n-1}$  cannot contain a subgroup of type  $A_{n-1}$  and  $k$  cannot be bigger than  $n$  since the dimension of the representation should be at most  $2n$ . It follows that  $M$  is either a symplectic or an orthogonal group. But by a simple calculation we can see that there is no bilinear form which is invariant under  $H$ . Therefore  $M$  can be neither  $C_n$  nor  $D_n$ . Now let  $M$  be of type  $A_k$ . Since  $H$  is reducible, there exists a maximal parabolic subgroup  $P_M$  of

$M$  which contains  $H$ . It is not difficult to see that a maximal parabolic subgroup of  $SL_{2n}(\mathbb{C})$  containing  $H$  is either

$$P_1 = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in SL_{2n}(\mathbb{C}) \mid A, B \in GL_n(\mathbb{C}) \right\} \quad \text{or}$$

$$P_2 = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \in SL_{2n}(\mathbb{C}) \mid A, B \in GL_n(\mathbb{C}) \right\}.$$

Denote by  $L$  the common Levi subgroup of  $P_1$  and  $P_2$ . We can also easily see that  $L$  is the only Levi subgroup of  $P_i$  which contains  $H$ . Therefore a Levi subgroup, say  $L_M$ , of  $P_M$  which contains  $H$  is contained in  $L$ . It is known that there is no semisimple subgroup of maximal rank in the groups of type  $A_k$ . It follows that  $L_M$  is either  $L_0 = \{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in SL_{2n}(\mathbb{C}) \mid A \in GL_n(\mathbb{C}) \}$  or  $L$  since  $H$  is a maximal semisimple subgroup of  $[L, L]$ . Hence the type of  $M$  is either  $A_n$  or  $A_{2n-1}$ , respectively. But we can check from the character formula that there is no irreducible representation of dimension  $2n$  for the groups of type  $A_n$  unless  $n = 3$ . ■

1.3.10. PROPOSITION. *Set  $H = \{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mid A \in SO_n(\mathbb{C}) \}$  and  $n \geq 8$ . Then a connected semisimple complex Lie subgroup of  $Sp_{2n}(\mathbb{C})$  containing  $H$  is either  $\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \in Sp_{2n}(\mathbb{C}) \mid A \in SL_n(\mathbb{C}) \}$  or  $SL_2(\mathbb{C}) \otimes SO_n(\mathbb{C})$ .*

The proof of the above proposition goes exactly the same as the proof of Proposition 1.3.9.

#### 1.4. $\mathbb{Q}$ -Forms of Algebraic Groups and $\mathbb{Q}$ -Rational Representations

We prove some basic propositions on the  $\mathbb{Q}$ -forms of semisimple groups and  $\mathbb{Q}$ -rational representations. The main references for this section are [21, 22].

1.4.1 (see [21, Chap. 2]). Let  $G$  be a semisimple algebraic  $k$ -group,  $\Gamma = \text{Gal}(K/k)$  where  $K$  is the separable closure of  $k$ , and  $S, T, \Delta$ , and  ${}_k\Delta$  be as in Subsection 1.2.1. Denote by  $\Delta_0$  the subset of  $\Delta$  which vanishes on  $S$ . We define the so-called  $*$ -action of  $\Gamma$  on  $\Delta$ . For  $\gamma \in \Gamma$ , there exists a unique element  $w$  in the Weyl group such that  $w({}^\gamma\Delta) = \Delta$  and we set  $\gamma(\alpha) = w({}^\gamma\alpha)$ . If the  $*$ -action is trivial, the  $k$ -form is called *inner* and otherwise, *outer*. The orbits of  $\Gamma$ , whose elements do not belong to  $\Delta_0$ , are called *distinguished orbits*.

The Tits index of a group  $G$  is the data consisting of  $\Delta$ , together with Dynkin diagram,  $\Delta_0$ , and the  $*$ -action of  $\Gamma$ . The group  $G$  is determined, up to  $k$ -isomorphism, by its  $K$ -isomorphism class, the Tits index, the commutator subgroup of  $C(S)$ , called the *semisimple anisotropic kernel* of  $G$ , given up to  $k$ -isogeny.

PROPOSITION [21, 2.5.4]. *Let  $P_\Theta$  denote the parabolic subgroup generated by  $T$  and  $U_\alpha$  ( $\alpha \in \Delta$ ) and  $U_{-\alpha}$  ( $\alpha \in \Theta$ ). Then  $P_\Theta$  is defined over  $k$  if and only if  $\Theta$  contains  $\Delta_0$  and is invariant under the  $*$ -action of  $\Gamma$ .*

1.4.2. For an adjoint semisimple  $k$ -group  $G$ , there exist an adjoint semisimple  $k$ -split group  $G^d$  and an adjoint semisimple  $k$ -quasi-split group  $G^q$  whose  $*$ -action of  $\Gamma$  on the Dynkin diagram is the one given by the index of  $G$ , both of which are  $K$ -isomorphic to  $G$ . The  $k$ -form of  $G$  is obtained by twisting  $G^d$  (resp.  $G^q$ ) by a cocycle  $c$  of  $\Gamma$  with values in  $\text{Aut}_K(G^d)$  (resp.  $\text{Int}_K(G^q)$ ) (see [21, 3.4.2]).

The proof of the following proposition is due to G. Prasad.

PROPOSITION. *Let  $G$  be an adjoint semisimple algebraic  $\mathbb{R}$ -group. Then for a given minimal parabolic  $\mathbb{R}$ -subgroup  $P$ , there exists a  $\mathbb{Q}$ -form on  $G$  with respect to which every parabolic  $\mathbb{R}$ -subgroup of  $G$  which contains  $P$  is defined over  $\mathbb{Q}$ .*

*Proof.* The group  $G$  decomposes into a direct product of adjoint  $\mathbb{R}$ -simple normal  $\mathbb{R}$ -groups by Proposition 1.2.4 and a parabolic  $\mathbb{R}$ -subgroup of  $G$  is a product of parabolic  $\mathbb{R}$ -subgroups of each  $\mathbb{R}$ -simple factor of  $G$ . Therefore it is enough to prove the proposition for an  $\mathbb{R}$ -simple group.

Case (1).  $G$  is absolutely simple.

Let  $G^q$  be the adjoint  $\mathbb{Q}$ -split  $\mathbb{Q}$ -group if the  $\mathbb{R}$ -form of  $G$  is of inner type, and the quasi-split  $\mathbb{Q}$ -form of  $G$ , splitting over  $k = \mathbb{Q}(i)$ , if the  $\mathbb{R}$ -form of  $G$  is of outer type. Let  $P = MU$  be a minimal parabolic  $\mathbb{R}$ -subgroup of  $G$  and  $P^q = M^q U^q$  be the corresponding parabolic  $\mathbb{Q}$ -subgroup of  $G^q$ , where  $M^q$  is a Levi  $\mathbb{Q}$ -subgroup of  $P^q$  and  $U^q$  is the unipotent radical. Set  $M' = [M, M]$  and  $M^{q'} = [M^q, M^q]$ . Since the two  $\mathbb{R}$ -forms of  $G$  and  $G^q$  differ only by their semisimple anisotropic kernels  $M'$  and  $M^{q'}$  and the indices of  $M'$  and  $M^{q'}$  coincide, the  $\mathbb{R}$ -form of  $G$  is the twist of the  $\mathbb{R}$ -form of  $G^q$  by a cocycle  $c$  of  $\text{Gal}(\mathbb{C}/\mathbb{R})$  with values in  $\text{Int}_{\mathbb{C}} M^q = M^{q'}$  (see [21, 3.4.2]). But it is known (see [3, Theorem 1.7]) that the natural homomorphism of  $H^1(\mathbb{Q}, M^{q'})$  to  $H^1(\mathbb{R}, M^{q'})$  is surjective. Thus there is an  $M^{q'}$ -valued cocycle  $d$  on  $\text{Gal}(\mathbb{C}/\mathbb{Q})$  cohomologous to  $c$  over  $\mathbb{Q}$ . The twist of  $G^q$  by the cocycle  $d$  is a  $\mathbb{Q}$ -form of  $G$  with the following properties: (i) over  $\mathbb{R}$ , it coincides with the  $\mathbb{R}$ -form of  $G$ ; (ii) its distinguished orbits of  $\text{Gal}(\mathbb{C}/\mathbb{Q})$  contain all the distinguished orbits of  $\text{Gal}(\mathbb{C}/\mathbb{R})$  of the  $\mathbb{R}$ -form of  $G$ . Therefore by Proposition 1.4.1, this  $\mathbb{Q}$ -form has the required property.

Case (2).  $G$  is  $\mathbb{R}$ -simple but not absolutely simple.

There exists an adjoint absolutely simple group  $G'$  such that  $G = R_{\mathbb{C}/\mathbb{R}} G'$  by Proposition 1.2.6. Let  $G'^d$  be the adjoint  $\mathbb{Q}(i)$ -split form of  $G'$

and  $G^d = R_{\mathbb{Q}(i)/\mathbb{Q}}G'^d$ . It is clear that  $G^d$  is a  $\mathbb{Q}$ -form of  $G$  with the desired property. ■

1.4.3. Let  $D$  be a central simple division algebra over  $k$  of degree  $d$ . For any  $r$ , there exists a  $k$ -form  $GL_{r,D}$  of  $GL_{rd}$  such that  $GL_{r,D}(k) = GL_r(D)$ . Let  $V$  be a finite dimensional (left) vector space over  $D$ . A  $D$ -representation or  $(D, k)$ -representation of  $G$  in  $V$  is a  $k$ -homomorphism  $G \rightarrow GL_{V,D}$ . A representation of  $G$  is called  $D$ -rational if it is equivalent to a  $D$ -representation.

1.4.4. PROPOSITION [22, Theorem 3.3]. *For any dominant weight  $\lambda$  fixed by  $\Gamma$ , there exists a division algebra  $D$ , unique up to  $k$ -isomorphism, such that an absolutely irreducible representation of  $G$  with the highest weight  $\lambda$  is  $D$ -rational.*

1.4.5. PROPOSITION. *Let  $G$  be an absolutely simple algebraic  $k$ -group of type  $C_n$ . If the standard  $2n$ -dimensional representation of  $G$  is  $k$ -rational, then  $G$  is split over  $k$ .*

*Proof.* By the classification of  $k$ -forms given in [21],  $G(k)$  is  $SU_{2n/d}(D, h)$  where  $D$  is a division algebra of degree  $d$  and  $h$  is a nondegenerate antihermitian sesquilinear form of index  $r$  relative to a first kind involution  $\sigma$  such that  $D^\sigma$  has dimension  $1/2d(d+1)$ . The canonical representation  $SU_{2n/d}(D, h) \rightarrow SL_{2n,D}$  is an absolutely irreducible representation with the same highest weight as the standard  $2n$ -dimensional representation. Since the standard representation is rational over  $k$ , by the uniqueness of such a  $D$ ,  $D = k$ . It implies that  $G$  is split over  $k$ . ■

1.4.6. Let  $\alpha$  be a dominant weight and  $k_\alpha$  the invariant field by the stabilizer of  $\alpha$  in  $\Gamma$ . Then Proposition 1.4.4 gives a central division algebra  $D_\alpha$  over  $k_\alpha$  such that  $\rho_\alpha : G \rightarrow GL_{n,D_\alpha}$  is an absolutely irreducible  $(D_\alpha, k_\alpha)$ -representation of dominant weight  $\alpha$ . Define  $rest_{D_\alpha/k_\alpha} : GL_{n,D_\alpha} \rightarrow GL_{nd^2}(K) = GL_n(D_\alpha \otimes_{k_\alpha} K)$  and similarly  $rest_{k_\alpha/k}$ . We put  $\rho_\alpha^k = rest_{k_\alpha/k}(rest_{D_\alpha/k} \circ \rho_\alpha)$ .

THEOREM [22, Theorem 7.2, Lemma 7.4]. (1) *For each dominant weight of  $G$ ,  $\rho_\alpha^k$  is irreducible over  $k$ .*

(2) *Each  $k$ -irreducible representation of  $G$  is  $k$ -equivalent to the representation of the form  $\rho_\alpha^k$ .*

(3) *Let  $\alpha$  and  $\alpha'$  be dominant weights. Then  $\rho_\alpha^k$  and  $\rho_{\alpha'}^k$  are  $k$ -equivalent iff there exists  $\gamma \in \Gamma$  such that  $\gamma(\alpha) = \alpha'$ .*

(4) *If  $\Gamma(\alpha) = \{\alpha_1, \dots, \alpha_n\}$ , then  $\rho_\alpha^k$  is equivalent over  $K$  to the direct sum of  $d$  irreducible representations of dominant weight  $\alpha_1, \dots, d$  irreducible representations of dominant weight  $\alpha_n$  where  $d^2 = [D_\alpha : k_\alpha]$ .*

1.4.7. Let  $n \geq 3$ , and  $G$  be either  $SL_n(\mathbb{C})$  or  $SO_{2n}(\mathbb{C})$ , and  $\psi$  be the representation of  $SL_2(\mathbb{C}) \times G$  defined by  $\psi(A, B) = A \otimes B$  for  $A \in SL_2(\mathbb{C})$  and  $B \in G$ .

PROPOSITION. Let  $H = SL_2(\mathbb{C}) \times G$ . Assume that the representation  $\psi$  is rational over  $\mathbb{Q}$ .

(1) If  $G = SL_n(\mathbb{C})$ , the group  $H$  is equipped with either a split  $\mathbb{Q}$ -form or an inner  $\mathbb{Q}$ -form such that  $H(\mathbb{Q})$  is conjugate to  $SL_1(D_{\mathbb{Q}}) \times SL_s({}^tD_{\mathbb{Q}})$  where  $D$  is a division algebra defined over  $\mathbb{Q}$  such that  $D_{\mathbb{Q}}$  is a central simple quaternion division algebra over  $\mathbb{Q}$ ,  $2s = n$ , and  ${}^tD_{\mathbb{Q}} := \{X \in M_2(\mathbb{C}) \mid X \in D_{\mathbb{Q}}\}$ .

(2) If  $G = SO_n(\mathbb{C})$ , the group  $H$  is equipped with either a split  $\mathbb{Q}$ -form or a  $\mathbb{Q}$ -form such that  $H(\mathbb{Q})$  is conjugate to  $SL_1(D_{\mathbb{Q}}) \times SU_s({}^tD_{\mathbb{Q}}, h)$  where  $D$  is a division algebra defined over  $\mathbb{Q}$  such that  $D_{\mathbb{Q}}$  is a central simple quaternion division algebra over  $\mathbb{Q}$ ,  $2s = n$ , and  $h$  is a non-degenerate hermitian form relative to an involution  $\sigma$  of the first kind such that  $D^{\sigma}$  has dimension 3.

In particular, with respect to those  $\mathbb{Q}$ -forms,  $\{(I, A) \mid A \in G\}$  is a  $\mathbb{Q}$ -subgroup of  $H$ .

*Proof.* (1) Let  $T$  be a maximal torus consisting of the diagonal elements of  $SL_2(\mathbb{C}) \times SL_n(\mathbb{C})$ . Then the highest weight  $\lambda_1$  of  $\psi$  sends  $t = \text{diag}(t_1, \dots, t_m) \times \text{diag}(s_1, \dots, s_k)$  to  $t_1 + s_1$ . It is known that if an absolutely irreducible representation  $\psi$  is rational over  $\mathbb{Q}$ , then its highest weight is invariant under the  $*$ -action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and the element  $\beta_{\psi}$  in the Brauer group  $\text{Br}(\mathbb{Q})$  associated to  $\psi$  is trivial (see [B-T, 12] for the definition of  $\beta_{\psi}$ ). Since  $n \neq 2$ ,  $H$  cannot be not almost simple over  $\mathbb{Q}$  and hence there exist two subgroups  $H_1$  and  $H_2$  which are  $\mathbb{Q}$ -forms of  $SL_2(\mathbb{C})$  and  $SL_n(\mathbb{C})$  respectively such that  $H = H_1 \times H_2$ . Since the highest weight  $\lambda_1$  is invariant under the  $*$ -action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , the  $*$ -action must be trivial on the Dynkin diagrams of both  $H_1$  and  $H_2$ , from which it follows that the  $\mathbb{Q}$ -forms of  $H_1$  and  $H_2$  are inner. Therefore  $H(\mathbb{Q})$  is conjugate to  $SL_r(D_{1\mathbb{Q}}) \times SL_s(D_{2\mathbb{Q}})$  where  $D_1$  and  $D_2$  are as described in (1). If  $\psi$  is rational having this  $\mathbb{Q}$ -structure of  $H$ , then the triviality of  $\beta_{\psi}$  implies that the two algebras  $(D_1)_{\mathbb{Q}}$  and  ${}^t(D_2)_{\mathbb{Q}}$  (resp.  $(D_2)_{\mathbb{Q}}$ ) represent the same element in the Brauer group  $\text{Br}(\mathbb{Q})$ . Since they are (central) division algebras, this implies that they are isomorphic. In the case when  $(D_1)_{\mathbb{Q}}$  and  ${}^t(D_2)_{\mathbb{Q}}$  are isomorphic, an isomorphism of  $(D_2)_{\mathbb{Q}}$  onto  ${}^t(D_1)_{\mathbb{Q}}$  extends to an isomorphism of the algebra  $M_s(D_2)_{\mathbb{Q}}$  onto the algebra  $M_s({}^t(D_1)_{\mathbb{Q}})$  and hence to an automorphism of the algebra  $M_k(\mathbb{C})$  through taking the tensor product with  $\mathbb{C}$  over  $\mathbb{Q}$ . It follows from the Skolem-Noether theorem that this automorphism is inner by an element  $g \in GL_k(\mathbb{C})$ . Thus we have  $gSL_s(D_2)_{\mathbb{Q}}g^{-1} = SL_s({}^t(D_1)_{\mathbb{Q}})$ .

The proof of (2) goes exactly as above, using the classification of  $\mathbb{Q}$ -forms of groups of  $D_n$ -type. ■

1.4.8. PROPOSITION. *Let  $G = SL_n(\mathbb{C})$  and let  $\psi: SL_n(\mathbb{C}) \rightarrow SL_{2n}(\mathbb{C})$  be the direct sum  $\rho \oplus \rho^*$  where  $\rho$  is the standard representation of  $SL_n(\mathbb{C})$  and  $\rho^*$  the dual of  $\rho$ , i.e.,  $\rho^*(A) = {}^t A^{-1}$  for  $A \in G$ . Then the  $\mathbb{Q}$ -forms of  $G$  with respect to which  $\psi$  is  $\mathbb{Q}$ -rational and irreducible over  $\mathbb{Q}$  are such that  $G(\mathbb{Q}) = SU_n(k, h)$  where  $k$  is a quadratic extension of  $\mathbb{Q}$  and  $h \in GL_n(k)$  such that  ${}^t h^\sigma = h$  where  $\sigma$  is the non-trivial element in  $\text{Gal}(k/\mathbb{Q})$ .*

*Proof.* Since the highest weights of  $\rho$  and  $\rho^*$  are obviously different, the  $*$ -action of  $\Gamma$  of the  $\mathbb{Q}$ -form is non-trivial, proving the form of the outer. That the division algebra attached to this outer  $\mathbb{Q}$ -form is in fact a field is a direct consequence of (4) in Theorem 1.4.6. ■

1.4.9. The following propositions are easy consequences of Theorem 1.4.6; their derivation from the theorem is similar to the proof of Proposition 1.4.8.

PROPOSITION. *Let  $H = SL_n(\mathbb{C}) \times SL_n(\mathbb{C})$  and let  $\psi$  be the representation such that  $\psi(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ . If  $H$  is equipped with a  $\mathbb{Q}$ -form with respect to which  $\psi$  is rational over  $\mathbb{Q}$ , then  $H(\mathbb{Q})$  is up to conjugation either  $SL_n(\mathbb{Q}) \times SL_n(\mathbb{Q})$  or  $R_{k/\mathbb{Q}} SL_n(k)$ .*

1.4.10. PROPOSITION. *Let  $G$  be  $SL_n(\mathbb{C})$  (resp.  $SO_{2n}(\mathbb{C})$ ) and  $\psi$  be the representation of  $G$  defined by  $\psi(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ . If  $G$  is equipped with a  $\mathbb{Q}$ -form with respect to which  $\psi$  is rational over  $\mathbb{Q}$ , then  $G(\mathbb{Q})$  is, up to conjugation, either  $SL_n(\mathbb{Q})$  (resp.  $SO_{2n}(\mathbb{Q})$ ) or  $SL_s(D_{\mathbb{Q}})$  (resp.  $SU_s(D_{\mathbb{Q}}, h)$ ) where  $D$  is a division algebra described in (1) (resp. where  $D$  and  $h$  are described in (2)) in the above proposition.*

### 1.5. Extension of $\mathbb{Q}$ -Forms

We show that for an adjoint absolutely simple group  $G$  and its parabolic subgroup  $P$  such that  $R_u(P)$  is commutative or Heisenberg, a  $\mathbb{Q}$ -form of  $[P, P]$  extends to a  $\mathbb{Q}$ -form of  $G$ . The following theorem of Raghunathan plays a key role.

1.5.1. THEOREM (Raghunathan [18, 3.31]). *Let  $\mathcal{G}$  be a complex semisimple Lie algebra and  $\mathcal{P}$  the Lie algebra corresponding to a parabolic subgroup  $P$  of  $G$  ( $=$  a semisimple Lie group with  $\mathfrak{g}$  its Lie algebra). Let  $\mathcal{P}_{\mathbb{Q}}$  be a Lie algebra over  $\mathbb{Q}$  and  $i: \mathcal{P}_{\mathbb{Q}} \rightarrow \mathcal{P}$  an injective homomorphism such that  $i \otimes \text{id}: \mathcal{P}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathcal{P}$  is an isomorphism. Then there exists a  $\mathbb{Q}$ -Lie algebra  $\mathfrak{g}_{\mathbb{Q}}$  and injective homomorphisms  $j: \mathfrak{g}_{\mathbb{Q}} \rightarrow \mathcal{G}$  and  $\alpha: \mathcal{P}_{\mathbb{Q}} \rightarrow \mathfrak{g}_{\mathbb{Q}}$  such that*

- (1)  $j \otimes \text{id}: \mathfrak{g}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \mathcal{G}$  is an isomorphism and
- (2)  $j \circ \alpha = \beta \circ i$  where  $\beta: \mathcal{P} \rightarrow \mathcal{G}$  is the natural inclusion.

1.5.2. Considering the adjoint representation of  $G$ , the following is a direct consequence of the above theorem.

**COROLLARY.** *Let  $G$  be an adjoint semisimple algebraic  $\mathbb{R}$ -group and  $P$  its parabolic  $\mathbb{R}$ -subgroup. If  $P$  is equipped with a  $\mathbb{Q}$ -form, then there exists a  $\mathbb{Q}$ -form of  $G$  which extends the  $\mathbb{Q}$ -form of  $P$ .*

1.5.3. It is well known that for every unipotent subgroup  $U$ , the logarithmic mapping  $\ln U \rightarrow \text{Lie}(U)$  is equivariant in the following sense: if  $f$  is a biregular automorphism of  $U$ , then  $\ln \circ f = df \circ \ln$  (see [11, 1.1.3]).

In particular, we have the following:

**LEMMA.** *Let  $G$  be a semisimple algebraic group and  $P$  a parabolic subgroup with a Levi subgroup  $L$ . Then  $\text{Ad}(h) \circ \ln = \ln \circ \text{Int}(h)$  where  $\text{Ad}: L \rightarrow \text{GL}(\mathcal{Z})$  is the representation as in Subsection 1.1.14.*

1.5.4. **LEMMA.** *Let  $G$  be an adjoint semisimple algebraic group and  $P, L, \mathcal{Z}$  be as in the above lemma. Then the representation  $\text{Ad}: L \rightarrow \text{GL}(\mathcal{Z})$  is faithful.*

*Proof.* Applying the previous lemma to the conjugation map of  $U$  by each element of  $L$ , we can see that the kernel of  $\text{Ad}: L \rightarrow \text{GL}(\mathcal{Z})$  is equal to  $C(U) \cap L$ . But it is known (e.g., [10, Lemma 11.16] or [16, Proposition 11.19]) that  $C(U) \subset U$ . Therefore  $C(U) \cap L$  is trivial, proving the lemma. ■

1.5.5. **PROPOSITION.** *Let  $G$  be an adjoint absolutely simple algebraic  $\mathbb{R}$ -group,  $P$  a parabolic  $\mathbb{R}$ -subgroup with an  $\mathbb{R}$ -Levi decomposition  $LR_u(P)$ , and  $H = [L, L]$ . Assume that  $H$  and  $R_u(P)$  have  $\mathbb{Q}$ -forms such that  $H(\mathbb{Q})$  normalizes  $R_u(P)(\mathbb{Q})$ . If  $R_u(P)$  is commutative or Heisenberg and  $H$  is non-trivial, there exists a  $\mathbb{Q}$ -form of  $G$  which extends the  $\mathbb{Q}$ -forms of  $H$  and  $R_u(P)$ .*

*Proof.* By Corollary 1.5.2, it is enough to extend a  $\mathbb{Q}$ -form of  $H$  and  $R_u(P)$  to a  $\mathbb{Q}$ -form of  $P$ . Consider the representation  $\text{Ad}: L \rightarrow \text{GL}(\mathcal{Z})$ ,  $\mathcal{Z} = \text{Lie}(R_u(P))$  defined in Subsection 1.1.14. The  $\mathbb{Q}$ -form on  $R_u(P)$  defines a  $\mathbb{Q}$ -structure on the vector space  $\mathcal{Z}$ , through the logarithm map, and hence a  $\mathbb{Q}$ -form on  $\text{GL}(\mathcal{Z})$ . Since  $H(\mathbb{Q})$  normalizes  $R_u(P)(\mathbb{Q})$  and the logarithm map is  $\mathbb{Q}$ -rational,  $\text{Ad}(H(\mathbb{Q}))$  preserves  $\mathcal{Z}(\mathbb{Q})$ . Since  $H$  is defined over  $\mathbb{Q}$ , and hence  $H(\mathbb{Q})$  is Zariski dense in  $H$ , it implies that  $\text{Ad}(H)$  is a  $\mathbb{Q}$ -subgroup of  $\text{GL}(\mathcal{Z})$ .

Case (1).  $\text{Ad}$  is absolutely irreducible, or equivalently,  $R_u(P)$  is commutative.

Since  $\text{Ad}$  is absolutely irreducible, it follows from Schur's lemma that the centralizer  $C(\text{Ad}(H))$  of  $\text{Ad}(H)$  coincides with  $Z(\text{GL}(\mathcal{Z}))$ . Since  $\text{Ad}(Z(L)) \subset C(\text{Ad}(H))$  and  $\text{Ad}(Z(L))$  has dimension 1,  $\text{Ad}(Z(L)) = Z(\text{GL}(\mathcal{Z}))$ . It follows that  $\text{Ad}(Z(L))$  is a  $\mathbb{Q}$ -subgroup of  $\text{GL}(\mathcal{Z})$ . Since

$\text{Ad}(H) \cap \text{GL}(\mathcal{Z}(\mathbb{Q}))$  and  $Z(L) \cap \text{GL}(\mathcal{Z}(\mathbb{Q}))$  are Zariski dense in  $\text{Ad}(H)$  and  $Z(L)$ , respectively,  $\text{Ad}(H)\text{Ad}(Z(L)) \cap \text{GL}(\mathcal{Z}(\mathbb{Q}))$  is Zariski dense in  $\text{Ad}(H)\text{Ad}(Z(L))$ . Since  $L = HZ(L)$ , we obtain that  $\text{Ad}(L)$  is a  $\mathbb{Q}$ -subgroup of  $\text{GL}(\mathcal{Z})$ . Since  $\text{Ad}$  is faithful,  $(\text{Ad}, \text{Ad}(L))$  now provides a  $\mathbb{Q}$ -form of  $L$ , which extends the given  $\mathbb{Q}$ -form of  $H$ . Since  $L(\mathbb{Q})$  normalizes  $R_u(\mathbb{Q})$  with respect to this  $\mathbb{Q}$ -form, we have a  $\mathbb{Q}$ -form of  $P$ .

*Case (2).*  $R_u(P)$  is Heisenberg and  $G$  is not of type  $A_n$ .

Then  $\mathcal{Z}$  decomposes into  $\mathcal{V} \oplus Z(\mathcal{Z})$  where  $\mathcal{V}$  is the  $\text{Ad}(H)$ -invariant subspace other than  $Z(\mathcal{Z})$ . Since  $Z(\mathcal{Z})$  has dimension one and  $H$  has no rational characters,  $\text{Ad}(H)$  acts trivially on  $Z(\mathcal{Z})$ . Denote by  $\text{Ad}'$  the restriction of  $\text{Ad}$  on  $\mathcal{V}$  so that  $\text{Ad} = \text{Ad}' \oplus \text{Id}$ . We note that  $\mathcal{V}$  is a  $\mathbb{Q}$ -subspace of  $\mathcal{Z}$ ; in fact, let  $(v, z) \in \mathcal{Z}(\mathbb{Q})$  where  $v \in \mathcal{V}$  and  $z \in Z(\mathcal{Z})$ , and  $h \in H(\mathbb{Q})$ . Then  $\text{Ad}(h)(v, z) = (\text{Ad}'(h)v, z) \in \mathcal{Z}(\mathbb{Q})$  and hence  $(\text{Ad}'(h)v - v, 0) \in \mathcal{Z}(\mathbb{Q})$ , giving a non-trivial element in  $\mathcal{V} \cap \mathcal{Z}(\mathbb{Q})$ . It follows from the fact that  $\text{Ad}'$  acts absolutely irreducibly on  $\mathcal{V}$  that  $\text{Ad}(H)(\mathbb{Q})(\mathcal{V} \cap \mathcal{Z}(\mathbb{Q}))$  generates a Zariski dense subspace of  $\mathcal{V}$ , proving the claim.

Since the restriction of  $\text{Ad}(H)$  on  $\mathcal{V}$  is absolutely irreducible, by the same argument as the previous case, we can extend the  $\mathbb{Q}$ -form of  $H$  to  $L$  with respect to which  $\text{Ad}(L(\mathbb{Q}))$  preserves  $\mathcal{V}(\mathbb{Q})$ . Since  $Z(\mathcal{Z}(\mathbb{Q})) = [\mathcal{V}(\mathbb{Q}), \mathcal{V}(\mathbb{Q})]$ ,  $\text{Ad}(L(\mathbb{Q}))$  also preserves  $Z(\mathcal{Z}(\mathbb{Q}))$ . It follows that  $L(\mathbb{Q})$  normalizes  $R_u(P)(\mathbb{Q})$ , yielding the desired  $\mathbb{Q}$ -form of  $P$ .

*Case (3).*  $R_u(P)$  is Heisenberg in a group of type  $A_n$ ,  $n \geq 3$ .

In this case,  $H$  is of type  $A_{n-2}$  and  $\text{Ad}$  is equivalent to the direct sum  $\rho \oplus \rho^* \oplus \text{id}$  where  $\rho$  (resp.  $\rho^*$ ) denotes the (resp. dual of) standard representation of  $A_{n-2}$ . By Proposition 1.4.7, the  $\mathbb{Q}$ -forms of  $H$  with respect to which  $\text{Ad}$  is  $\mathbb{Q}$ -rational are, up to conjugation and isogeny, such that  $H(\mathbb{Q}) = \text{SL}_{n-1}(\mathbb{Q})$ ,  $\text{SU}(h)(\mathbb{Q}) = \{g \in \text{SL}_{n-1}(k) \mid {}^t g^\sigma h g = h\}$  or  $(\text{SU}(h)(\mathbb{Q}))^{-1} = \{g \in \text{SL}_{n-1}(k) \mid {}^t g^{-1} \in \text{SU}(h)(\mathbb{Q})\}$  where  $k$  is a real quadratic extension field of  $\mathbb{Q}$ ,  $\sigma$  is a non-trivial element in  $\text{Gal}(k/\mathbb{Q})$ , and  $h \in \text{GL}_{n-1}(k)$  such that  ${}^t h^\sigma = h$ . We observe that each of those  $\mathbb{Q}$ -forms of  $H$  extends to a  $\mathbb{Q}$ -form of  $G$  with respect to which  $G(\mathbb{Q})$  is isogenous to  $\text{SL}_{n+1}(\mathbb{Q})$ ,  $\text{SU}(h')(\mathbb{Q}) = \{g \in \text{SL}_{n+1}(k) \mid {}^t g^{-1} h' g = h'\}$  or  $(\text{SU}(h')(\mathbb{Q}))^{-1}$  for

$$h' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

respectively. ■

1.5.6. *Remark.* If the rank of  $G$  is one,  $H$  is trivial. For the groups with rank at least 2,  $H$  is trivial only if  $G$  is of type  $A_2$  and  $P$  is a Borel subgroup, or equivalently,  $R_u(P)$  is Heisenberg.

## 2. THE SUBGROUPS OF THE FORM $\Gamma_{F_1, F_2}$ AND $\mathbb{Q}$ -FORMS

### 2.1. Discrete Subgroups in Algebraic Groups

2.1.1. The following two theorems are the well known Borel density theorem and a theorem of Borel and Harish-Chandra, respectively.

**THEOREM (Borel Density Theorem [2]).** *Any lattice in a connected semisimple Lie group without compact factors is Zariski dense.*

2.1.2. **THEOREM [4].** *Let  $G$  be a semisimple linear algebraic group defined over  $\mathbb{Q}$ . Then  $G(\mathbb{Z})$  is a lattice in  $G(\mathbb{R})$ .*

2.1.3. **LEMMA.** *Let  $\mathbf{G}$  be a connected semisimple algebraic  $\mathbb{R}$ -group,  $G = \mathbf{G}(\mathbb{R})^0$ , and  $\Gamma$  a discrete and Zariski dense subgroup in  $G$ . Then the normalizer  $N(\Gamma)$  of  $\Gamma$  in  $G$  is discrete.*

*Proof.* Let  $N^0$  be the connected component of the identity of  $N(\Gamma)$ . Since  $N^0$  normalizes  $\Gamma$  and  $\Gamma$  is discrete,  $N^0$  centralizes  $\Gamma$ . That is,  $N^0 \subset C(\Gamma)$  and therefore  $\Gamma \subset C(N^0)$ . Since  $C(N^0)$  is algebraic and  $\Gamma$  is Zariski dense,  $G \subset C(N^0)$  and hence  $N^0$  is contained in the center of  $G$ , which is finite. Thus  $N^0 = \{e\}$ , implying that  $N(\Gamma)$  is discrete. ■

2.1.4. **LEMMA.** (1) *If  $G$  is a locally compact group and  $\Gamma$  is a discrete subgroup of  $G$  containing a lattice in  $G$ , then  $\Gamma$  is a lattice in  $G$ .*

(2) *If  $\mathbf{G}$  is a semisimple algebraic  $\mathbb{Q}$ -group,  $G = \mathbf{G}(\mathbb{R})^0$ , and  $\Gamma$  is a discrete subgroup containing an arithmetic subgroup  $G(\mathbb{Z})$ , then  $\Gamma$  is also an arithmetic subgroup of  $G$ .*

(3) *Let  $G$  be as in (2) and in addition, suppose that  $G$  has no compact factors. Then the normalizer of an arithmetic subgroup of  $G$  is again an arithmetic subgroup of  $G$ .*

*Proof.* Part (1) is an immediate corollary of Lemma 1.6 of [16]. By Theorem 2.1.2,  $G(\mathbb{Z})$  is a lattice in  $G$ . Again by Lemma 1.6 of [16], the subgroup  $G(\mathbb{Z})$  is a subgroup of a finite index in  $\Gamma$ , proving (2). Part (3) is a direct consequence of Borel density theorem and Lemma 2.1.3. ■

2.1.5. We state a well known arithmeticity theorem of Margulis for the real field case, followed by his finiteness theorem.

**THEOREM (Margulis' Arithmeticity Theorem, see [13] or [27, Theorem 6.1.2]).** *Let  $G$  be a connected semisimple Lie group with trivial center and no compact factors. Suppose that the rank of  $G$  is at least 2 and that  $\Gamma$  is an irreducible lattice in  $G$ . Then there exists a semisimple algebraic  $\mathbb{Q}$ -group  $H$  and an epimorphism  $p: H(\mathbb{R})^0 \rightarrow G$  with compact kernel such that  $p(H(\mathbb{Z}) \cap H(\mathbb{R})^0)$  is commensurable to  $\Gamma$ . Furthermore if  $\Gamma$  is a non-uniform lattice,  $p$  can be taken as an isomorphism.*

2.1.6. THEOREM (Margulis' Finiteness Theorem, see [11, Chap. 8] or [27, Theorem 8.1.2]). *Let  $G$  and  $\Gamma$  be as in Theorem 2.1.5. Then any non-central normal subgroup of  $\Gamma$  has a finite index in  $\Gamma$ .*

2.1.7. Let  $G$  be a locally compact group and  $S_n$  a sequence of subsets of  $G$ . We say that  $S_n$  converges  $S$  if for every compact subset  $K \subset G$  and a neighborhood  $U$  of  $e$  in  $G$ , there exists an integer  $r = r(K, U)$  such that for all  $n \geq r$  and  $x \in S_n \cap K$ ,  $xU \cap S \neq \emptyset$  and for all  $y \in S \cap K$ ,  $yU \cap S_n \neq \emptyset$ .

THEOREM (Chabauty [16, Theorem 1.20]). *Let  $G$  be a Lie group and  $\Gamma_n$  a sequence of lattices in  $G$  such that for some open set  $W$  of  $G$  with  $e \in W$ ,  $W \cap \Gamma_n = \{e\}$  for all  $n$ . Then a subsequence  $\Gamma_{i_n}$  of  $\Gamma_n$  converges to  $\Gamma$  and  $\Gamma$  is a discrete subgroup. Furthermore, if  $\mu$  is a right Haar measure on  $G$ ,  $\mu(G/\Gamma) \leq \liminf \mu(G/\Gamma_{i_n})$ .*

2.1.8. A subgroup  $H$  of a Lie group  $G$  is said to have property (P) if every  $\text{Ad}(H)$ -stable subspace of  $\mathfrak{g}_{\mathbb{C}}$  is  $\text{Ad}(G)$ -stable where  $\text{Ad}$  denotes the adjoint representation of  $G$  in the complexification  $\mathfrak{g}_{\mathbb{C}}$  of the Lie algebra  $\mathfrak{g}$ .

THEOREM (Wang [16, Lemma 9.5]). *Let  $G$  be a semisimple Lie group and  $K$  a compact set of  $G$ . Then there is a neighborhood  $U$  of  $e$  in  $G$  such that the following holds: if  $\Gamma$  is any discrete subgroup of  $G$  such that  $\Gamma \cap U$  generates a subgroup with property (P), then  $\Gamma \cap U = \{e\}$ .*

For example, a Zariski dense subgroup in an algebraic group has property (P) since the adjoint representation is algebraic.

2.1.9. In the following, we list well known lemmas on lattices in unipotent algebraic groups (see [12, Chap. 3]).

LEMMA. *Let  $U$  be a unipotent algebraic  $\mathbb{R}$ -group and  $F$  a discrete subgroup of  $U(\mathbb{R})$ . Then  $F$  is Zariski dense in  $U$  if and only if the factor space  $U(\mathbb{R})/F$  is compact.*

2.1.10. LEMMA. *Assume further that  $F$  is a lattice in  $U(\mathbb{R})$ . Then*

- (1)  $U(\mathbb{R})/F$  is compact.
- (2)  $F$  is an arithmetic subgroup of  $U(\mathbb{R})$ .
- (3)  $F \cap Z(U(\mathbb{R}))$  is a lattice in  $Z(U(\mathbb{R}))$ .

2.1.11. LEMMA. *Let  $U$  be a unipotent  $\mathbb{Q}$ -group.*

- (1) For any  $n \in \mathbb{N}$ ,  $U(n\mathbb{Z})$  is a lattice in  $U(\mathbb{R})$ ;
- (2) if  $F \subset U(\mathbb{Q})$  and  $F$  is a lattice in  $U(\mathbb{R})$ , then  $F$  is commensurable to  $U(\mathbb{Z})$ ;
- (3) every subgroup of finite index in  $U(\mathbb{Z})$  contains  $U(n\mathbb{Z})$  for some  $n \in \mathbb{N}$ .

## 2.2. Generators of Arithmetic Subgroups

2.2.1. Let  $k$  be a number field and  $G$  a simply connected and absolutely almost simple algebraic group defined over  $k$  with respect to which  $U_1$  and  $U_2$  are opposite horospherical  $k$ -subgroups. Let  $S$  be a finite set of valuations of  $k$  containing all archimedean valuations and  $\Lambda$  be the ring of  $S$ -integers in  $k$  and the  $S$ -rank of  $G$  is at least 2. If  $U_1$  and  $U_2$  are maximal horospherical  $k$ -subgroups, then for any ideal  $A$  of  $\Lambda$ , the subgroup generated by  $U_1(A)$  and  $U_2(A)$  is of finite index in  $G(\Lambda)$  (for  $k$ -rank  $G \geq 2$  by Raghunathan [17] and for  $k$ -rank  $G = 1$  by Venkataramana [25]). This result was known for Chevalley groups for  $k$ -rank  $G \geq 2$  by Tits [23] and for classical groups of  $k$ -rank  $G \geq 2$  by Vaserstein [24].

In fact, this result holds for an arbitrary pair of opposite horospherical  $k$ -subgroups. The argument for this was explained to the author by T. N. Venkataramana.

2.2.2. COROLLARY. *Let  $G$  be as above and  $U_1, U_2$  a pair of opposite horospherical  $k$ -subgroups. Then for any ideal  $A$  of  $\Lambda$ , the subgroup generated by  $U_1(A)$  and  $U_2(A)$  is of finite index in  $G(\Lambda)$ .*

*Proof.* Let  $P'_1 \subset N(U_1)$  and  $P'_2 \subset N(U_2)$  be a pair of opposite minimal parabolic  $k$ -subgroups. This can be done as follows: if the pair  $N(U_1), N(U_2)$  is conjugate to  ${}_k P_\Theta, {}_k P_\Theta^-$  by  $g \in G(k)$  as in Lemma 1.2.2, set  $P'_i = gP_\Theta g^{-1}$  for each  $i = 1, 2$ . Set  $U'_i = R_u(P'_i)$  for each  $i = 1, 2$  and  $L = N(U_1) \cap N(U_2)$ . Then  $U'_i$  is the semi-direct product of  $(L \cap U'_i)$  and  $U_i$ . Denote by  $\Gamma_A$  the subgroup generated by  $U_1(A)$  and  $U_2(A)$ , and by  $\Gamma'_B$  the subgroup generated by  $U'_1(B)$  and  $U'_2(B)$ . By the result mentioned above for maximal horospherical  $k$ -subgroups,  $\Gamma'_B$  is an arithmetic subgroup. We note that  $(L \cap U'_i)(A)U'_i(A)$  has a finite index in  $U'_i(A)$ . Therefore there exists an ideal  $B \subset A$  such that  $U'_i(B) \subset (L \cap U'_i)(A)U'_i(A)$ . We claim that  $\Gamma'_B$  normalizes  $\Gamma_A$ . Let  $x \in U'_1(B)$  and write it as  $ly$  where  $l \in (L \cap U'_1)(A)$ ,  $y \in U_1(A)$ . For  $u_1 \in U_1(A)$ ,  $yu_1y^{-1} \in U_1(A)$  and hence  $lyu_1l^{-1}y^{-1} \in U_1(A)$  since  $L(A)$  normalizes  $U_1(A)$ . So,  $xU_1(A)x^{-1} = U_1(A)$ . For  $u_2 \in U_2(A)$ , since  $xu_2x^{-1} = (lyl^{-1})(lu_2l^{-1})(lyl^{-1})^{-1}$ ,  $lyl^{-1} \in U_1(A)$ , and  $lu_2l^{-1} \in U_2(A)$ , we have  $xu_2x^{-1} \in \Gamma_A$ . It shows that  $U'_1(B)$  normalizes  $\Gamma_A$  and similarly we can show that  $U'_2(B)$  normalizes  $\Gamma_A$ . Therefore  $\Gamma_A \subset N(\Gamma'_B)$ .

By the finiteness theorem of Margulis (Theorem 2.1.6),  $\Gamma_A$  has a finite index in  $N(\Gamma'_B)$  and hence an arithmetic subgroup. It is now clear that  $\Gamma_A$  has a finite index in  $G(A)$ . ■

2.2.3. COROLLARY. *Let  $G$  be a connected semisimple  $\mathbb{Q}$ -subgroup such that each  $\mathbb{Q}$ -simple factor has  $\mathbb{Q}$ -rank at least 1 and  $\mathbb{R}$ -rank at least 2. Let  $U_1, U_2$  be a pair of opposite horospherical  $\mathbb{Q}$ -subgroups and  $F_1, F_2$  lattices in*

$U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$  which are commensurable to  $U_1(\mathbb{Z})$  and  $U_2(\mathbb{Z})$ , respectively. If  $\Gamma_{F_1, F_2}$  is discrete, then it is an arithmetic subgroup of  $G(\mathbb{R})$ .

*Proof.* Suppose that  $G$  is simply connected. Using Proposition 1.2.4, we may assume that  $G$  is almost  $\mathbb{Q}$ -simple. By Proposition 1.2.6, there exist a finite separable extension  $k$  of  $\mathbb{Q}$ , a connected simply connected absolutely almost simple  $k$ -group  $G'$ , and a pair of opposite horospherical  $k$ -subgroups  $U'_1, U'_2$  of  $G'$  such that  $G = R_{k/\mathbb{Q}}G'$  and  $U_i = R_{k/\mathbb{Q}}U'_i$  for each  $i = 1, 2$ . We also note that  $\mathbb{R}$ -rank of  $G$  is equal to the  $S$ -rank of  $G'$  where  $S$  is the set of archimedean valuations of  $k$ . In fact, if  $k$  has  $r$  real embeddings and  $2s$  imaginary embeddings, then  $S$  has  $r + s$  elements. Both the  $\mathbb{R}$ -rank of  $G$  and the  $S$ -rank of  $G'$  are equal to  $(r + s)(k$ -rank of  $G')$ ). Since there exists  $n \in \mathbb{N}$  such that  $U_i(n\mathbb{Z}) \subset F_i$  by Lemma 2.1.11, we may assume that  $F_i = U_i(n\mathbb{Z})$  for each  $i = 1, 2$ . On the other hand, we can find an ideal  $A$  of  $\Lambda$ , the ring of the integers of  $k$ , such that  $R_{k/\mathbb{Q}}(U'_i(A)) \subset U_i(n\mathbb{Z})$ . By the previous corollary, the subgroup of  $G'$  generated by  $U'_1(A)$  and  $U'_2(A)$  has a finite index in  $G'(\Lambda)$ . Since  $\Gamma_{F_1, F_2}$  contains a subgroup generated by  $R_{k/\mathbb{Q}}(U'_1(A))$  and  $R_{k/\mathbb{Q}}(U'_2(A))$  and  $G(\mathbb{Z}) = R_{k/\mathbb{Q}}(G'(\Lambda))$ ,  $\Gamma_{F_1, F_2}$  contains a subgroup of finite index in  $G(\mathbb{Z})$ .

In general, there exists a simply connected semisimple  $\mathbb{Q}$ -group  $\tilde{G}$  and a central  $\mathbb{Q}$ -isogeny  $p : \tilde{G} \rightarrow G$  by Proposition 1.2.5. Then each  $\tilde{U}_i = p^{-1}(U_i)$  is a horospherical  $\mathbb{Q}$ -subgroup of  $\tilde{G}$  and there exists  $m \in \mathbb{N}$  such that  $p(\tilde{U}_i(m\mathbb{Z})) \subset U_i(n\mathbb{Z})$  for each  $i = 1, 2$ . Since  $p(\Gamma_{\tilde{U}_1(m\mathbb{Z}), \tilde{U}_2(m\mathbb{Z})}) \subset \Gamma_{U_1(n\mathbb{Z}), U_2(n\mathbb{Z})}$  and the image of an arithmetic subgroup under an isogeny map is an arithmetic subgroup, it only remains to apply the previous case to  $\Gamma_{\tilde{U}_1(m\mathbb{Z}), \tilde{U}_2(m\mathbb{Z})}$  to obtain that  $\Gamma_{U_1(n\mathbb{Z}), U_2(n\mathbb{Z})}$  is an arithmetic subgroup. ■

**2.2.4. EXAMPLE.** We present an example which shows that the assumption on the discreteness of  $\Gamma_{F_1, F_2}$  in the above corollary is essential.

Let  $U_1$  and  $U_2$  be the subgroups of the form  $(\begin{smallmatrix} I_m & M_{m \times k} \\ & I_k \end{smallmatrix}(\mathbb{R}))$  and  $(\begin{smallmatrix} I_m & & 0 \\ M_{k \times m}(\mathbb{R}) & & I_k \end{smallmatrix}(\mathbb{R}))$  for some  $m, k \in \mathbb{N}$ , respectively. By  $M_{m \times k}(\mathbb{R})$ , we mean the set of all  $m \times k$  matrices whose coefficients lie in  $\mathbb{R}$ . In fact any pair of minimal opposite horospherical subgroups in  $SL_n(\mathbb{R})$ ,  $n = m + k$ , is conjugate to such a pair  $U_1, U_2$ . Let  $F_1 = M_{m \times k}\mathbb{Z}$  and  $F_2 = (1/p)M_{k \times m}\mathbb{Z}$  for some  $p \in \mathbb{N}$ . Obviously  $F_2$  is commensurable to  $M_{k \times m}\mathbb{Z}$ .

We claim that unless  $p = 1$ , then  $\Gamma_{F_1, F_2}$  is not discrete. The subgroup  $\Gamma_{F_1, F_2}$  contains a subgroup, which is isomorphic to the subgroup  $\Gamma_p$  of  $SL_2(\mathbb{R})$  generated by  $(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix})$  and  $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ . Applying Jorgenson's inequality given below to  $\Gamma_p$ , we obtain that if  $\Gamma_p$  is discrete, then  $p \leq 1$ , yielding  $p = 1$  since  $p \in \mathbb{N}$ .

*Jorgenson's Inequality* (cf. [7]). Let  $A$  and  $B$  be matrices in  $SL_2(\mathbb{C})$  which generate an infinite subgroup. If the subgroup generated by  $A$  and  $B$  is discrete, then  $|\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2| \geq 1$ .

2.2.5. EXAMPLE. We close this section by giving some examples which demonstrate how the discreteness assumption on  $\Gamma_{F_1, F_2}$  restricts the choice of lattices  $F_1$  and  $F_2$  in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively.

(1) Let  $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ ,  $h = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ , and  $\Gamma_{x,y}$  be the subgroup of  $SL_2(\mathbb{R})$  generated by  $g$  and  $h$ . Jorgenson's inequality says that if  $\Gamma_{x,y}$  is discrete, then  $x^2y^2 \geq 1$ .

(2) Let  $U_1$  and  $U_2$  be as in Remark 2.2.4. Let  $F_1 = M_{m \times k}\mathbb{Z}$  and  $F_2 = \alpha M_{k \times m}\mathbb{Z}$  for some  $\alpha \in \mathbb{R}$ .

We claim that if  $\alpha$  is an irrational number, then  $\Gamma_{F_1, F_2}$  is not discrete. Let  $E_{ij}$  denote the elementary matrix whose entries are all 0 but 1 in  $(i, j)$ -entry. Note that the commutator  $[I + \alpha E_{m+1,1}, I + E_{1,n}]$  of two elements  $I + \alpha E_{m+1,1}$  and  $I + E_{1,n}$  is the element  $I + \alpha E_{m+1,n}$ . Let's denote this element by  $g$ . Then

$$g \begin{pmatrix} I_m & M_{m \times k}\mathbb{Z} \\ \mathbf{0} & I_k \end{pmatrix} g^{-1} = \begin{pmatrix} I_m & (M_{m \times k}\mathbb{Z})(I + \alpha E_{1,k}) \\ \mathbf{0} & I_k \end{pmatrix}.$$

Therefore  $\Gamma \cap U_1$  is not discrete if  $\alpha$  is an irrational number.

(3) Let  $\mathbf{G} = SL_{2n}(\mathbb{C})$  with an inner  $\mathbb{R}$ -form given by  $\mathbf{G}(\mathbb{R}) = SL_n(D_{\mathbb{R}})$  where  $D_{\mathbb{R}}$  is a Hamiltonian quaternion division algebra over  $\mathbb{R}$  and  $n \geq 3$ . Let  $U_1 = \begin{pmatrix} I_m & M_{m \times k}(D_{\mathbb{R}}) \\ \mathbf{0} & I_k \end{pmatrix}$  and  $U_2 = \begin{pmatrix} I_m & \mathbf{0} \\ M_{k \times m}(D_{\mathbb{R}}) & I_k \end{pmatrix}$  where  $m + k = n$ . Consider two quaternion division algebras  $D_1$  and  $D_2$  defined over  $\mathbb{Q}$  such that  $D_i(\mathbb{Q})$  is a central simple division algebra over  $\mathbb{Q}$  and  $D_i(\mathbb{R}) = D_{\mathbb{R}}$  for each  $i = 1, 2$ . The notation  $D_i(\mathbb{Z})$  denotes a  $\mathbb{Z}$ -order of the algebra  $D_i(\mathbb{Q})$ . Let  $F_1 = \begin{pmatrix} I_m & M_{m \times k}(D_1(\mathbb{Z})) \\ \mathbf{0} & I_k \end{pmatrix}$  and  $F_2 = \begin{pmatrix} I_m & \mathbf{0} \\ \alpha M_{k \times m}(D_2(\mathbb{Z})) & I_k \end{pmatrix}$ . We claim that if  $\Gamma_{F_1, F_2}$  is discrete, then  $D_1(\mathbb{Z}) = \alpha D_2(\mathbb{Z})$  up to commensurability. Since  $n$  is at least 3, we have either that  $m \geq 2$  or that  $k \geq 2$ . Set  $j = m$  if  $m \geq 2$ ;  $j = k$  otherwise. We denote by  $E_{ts}$  the elementary matrix in  $M_n(D_{\mathbb{R}})$  whose  $(t, s)$ -entry is the identity matrix and 0 elsewhere. Then for each  $x, y \in D_{\mathbb{R}}$ , the commutator  $C_{xy} := [I + xE_{1,n}, I + yE_{n,j}]$  is the element  $I + xyE_{1,j}$  since  $j \neq 1$ . For any  $x \in D_1(\mathbb{Z})$  and  $y \in \alpha D_2(\mathbb{Z})$ ,  $C_{xy} \in \Gamma_{F_1, F_2}$ . Since  $C_{xy}F_1C_{xy}^{-1} \subset F_1$ , we obtain that  $xy \in D_1(\mathbb{Z})$ . Therefore we obtain that  $\alpha D_1(\mathbb{Z})D_2(\mathbb{Z}) \subset D_1(\mathbb{Z})$ . It follows that  $\alpha D_2(\mathbb{Z}) = D_1(\mathbb{Z})$  up to commensurability.

### 2.3. Reflexive Horospherical Subgroups

2.3.1. Let  $\mathbf{G}$  be a connected semisimple algebraic  $\mathbb{R}$ -group without any  $\mathbb{R}$ -anisotropic factors and  $G = \mathbf{G}(\mathbb{R})^0$ . Let  $U_1, U_2$  be a pair of opposite horospherical  $\mathbb{R}$ -subgroups and  $F_1$  and  $F_2$  lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively.

LEMMA. (1)  $G$  is a normal subgroup of finite index in  $G(\mathbb{R})$ .

(2)  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$  generate  $G$ . In particular  $G$  is generated by one-parameter unipotent subgroups.

(3) The subgroup  $\Gamma_{F_1, F_2}$  is Zariski dense in  $G(\mathbb{R})$ .

*Proof.* Part (1) is well known. Since  $\mathbf{G}$  has no  $\mathbb{R}$ -anisotropic factors, (2) follows from [5, 6.14-15] (or see [11, Chap. 1, Theorem 2.3.1]). Since the Zariski closure of  $\Gamma_{F_1, F_2}$  contains  $U_1$  and  $U_2$ , (3) is a direct corollary of (2). ■

2.3.2. LEMMA. Suppose that  $P_1$  is a reflexive parabolic  $k$ -subgroup of a connected semisimple  $k$ -group  $G$  and that  $P_2$  is a parabolic  $k$ -subgroup opposite to  $P_1$ . Then the set  $M(P_1, P_2) = \{h \in R_u(P_2) \mid hP_1h^{-1} \text{ and } P_1 \text{ are opposite}\}$  is Zariski dense and open in  $R_u(P_2)$ . In particular,  $M(P_1, P_2) \cap R_u(P_2)(k)$  is non-empty.

*Proof.* It follows from Proposition 4.10 and Lemma 4.12 in [5] that the set  $M = \{h \in G \mid hP_1h^{-1} \text{ and } P_1 \text{ are opposite}\}$  is Zariski dense and open in  $G$ . Since  $R_u(P_2)P_1$  is open and  $M$  is right  $P_1$ -invariant, we have that  $R_u(P_2) \cap M$  is open in  $R_u(P_2)$ . ■

2.3.3. LEMMA. Suppose that  $U_1$  is reflexive, i.e.,  $wU_1w^{-1} = U_2$  for some  $w \in G(\mathbb{R})$ . If  $U_1$  is commutative (resp. Heisenberg), there exist  $u \in U_1(\mathbb{R})$ ,  $x \in (N(U_1) \cap N(U_2))(\mathbb{R})$  (independent of  $F_1$  and  $F_2$ ) such that  $u\Gamma_{F_1, F_2}u^{-1}$  contains  $\Gamma_{F_1, x(wF_1w^{-1})x^{-1}}$  (resp.  $\Gamma_{F_1', xF_2'x^{-1}}$  where  $F_1'$  and  $F_2'$  are lattices whose quotients by commutators are the same as those of  $F_1$  and  $wF_1w^{-1}$ , respectively).

*Proof.* Since  $\Gamma_{F_1, F_2}$  is Zariski dense, it follows from Lemma 2.3.2 that there exists  $\gamma \in \Gamma_{F_1, F_2}$  such that  $\gamma N(U_1)\gamma^{-1}$  is opposite to  $N(U_1)$ . Then there exists a unique element  $u \in U_1(\mathbb{R})$  such that  $u\gamma N(U_1)\gamma^{-1}u^{-1} = N(U_2)$ . Since  $wN(U_1)w^{-1} = N(U_2)$ , it follows that  $u\gamma w^{-1}$  is in  $N(U_2)$ . By Levi decomposition we can write  $u\gamma w^{-1} = xy$  for  $x \in N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$  and  $y \in U_2(\mathbb{R})$ . Now  $(u\gamma)F_1(u\gamma)^{-1} = (xy)wF_1w^{-1}(xy)^{-1}$ . Set  $F_1' = uF_1u^{-1}$  and  $F_2' = ywF_1w^{-1}y^{-1}$ . Then  $u\Gamma u^{-1} = u\gamma\Gamma\gamma^{-1}u^{-1}$  contains the subgroup generated by  $uF_1u^{-1} = F_1'$  and  $(u\gamma)F_1(u\gamma)^{-1} = xF_2'x^{-1}$ . If  $U_1$  is commutative, then  $F_1' = F_1$  and  $xF_2'x^{-1} = xF_2x^{-1}$ . If  $U_1$  is Heisenberg, it is clear that  $F_1'$  and  $F_2'$  satisfy the desired properties. ■

2.3.4. *Remark.* For an adjoint absolutely simple group  $G$  and its root system  $\Phi$  with a choice of basis  $\Delta$ , there exists a unique involutory permutation  $i$ , so-called *opposition involution* of  $\Delta$  such that  $\alpha \rightarrow -i(\alpha)$  extends to an operation of a Weyl element, say  $n$ . Then  $nP_\Theta n^{-1} = P_{i(\Theta)}^-$  for each subset  $\Theta \subset \Delta$ . Therefore if  $i(\Theta) = \Theta$ , then  $P_\Theta$  (resp.  $V_\Theta$ ) is conjugate to  $P_\Theta^-$  (resp.  $V_\Theta^-$ ).

It is known that  $i$  induces a non-trivial automorphism on  $\Delta$  if and only if  $G$  is type of  $A_n$ ,  $D_{2n+1}$ , or  $E_6$  (cf. [21, 1.5.1]). In such cases, we have non-reflexive commutative horospherical subgroups. On the other hand, it follows from the classification given in Subsection 1.3.1 that  $i(\Delta_H) = \Delta_H$ . Therefore every Heisenberg horospherical subgroup is reflexive.

#### 2.4. Margulis' Theorem on Representation Theory and Extension of $\mathbb{Q}$ -Structures

2.4.1. Let  $K$  be an algebraically closed field,  $H$  a connected algebraic  $K$ -group,  $\Lambda \subset H$  a Zariski dense subgroup,  $T$  a faithful linear representation of  $H$  into a finite dimensional linear space  $L$  over  $K$ ,  $M$  and  $N$  linear subspaces of  $L$ , and  $W$  a non-empty Zariski-open subset of  $H$ . We put  $d = \min_{h \in H} \dim(N \cap T(h)M)$  and assume that  $\dim(M \cap T(w)N) = \dim(N \cap T(w^{-1})M) = d$  for all  $w \in W$ . In addition, we assume that  $M$ ,  $N$ , and  $L$  are generated as linear subspaces by the unions  $\bigcup_{w \in W} (M \cap T(w)N)$ ,  $\bigcup_{w^{-1} \in W} (N \cap T(w)M)$  and  $\bigcup_{h \in H} T(h)M$ , respectively.

For a subfield  $k$  of  $K$ , any  $k$ -structure on  $M$  (resp.  $N$ ) induces in a natural way a  $k$ -structure on  $T(\lambda)M$  (resp.  $T(\lambda)N$ ) for any  $\lambda \in \Lambda$ . We say that two  $k$ -structures on  $M$  and  $N$  are compatible if  $M \cap T(\lambda)N$  for any  $\lambda \in \Lambda \cap W$  is a  $k$ -subspace of both  $M$  and  $T(\lambda)N$ .

**THEOREM (Margulis [12, Lemma 8.6.2]).** *If there exist compatible  $k$ -structures on  $M$  and  $N$ , then  $H$  can be given a  $k$ -structure such that  $\Lambda \subset H_k$ .*

2.4.2. **PROPOSITION.** *Let  $G$  be an adjoint semisimple algebraic  $\mathbb{R}$ -group,  $U_1, U_2$  a pair of opposite horospherical  $\mathbb{R}$ -subgroups and  $F_1, F_2$  lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively. Suppose that  $\Gamma_{F_1, F_2}$  is discrete and that there exists a non-trivial connected semisimple normal  $\mathbb{R}$ -algebraic subgroup  $S$  of  $N(U_1) \cap N(U_2)$  with a  $\mathbb{Q}$ -form with respect to which  $S(\mathbb{R}) \cap N(\Gamma_{F_1, F_2})$  is a Zariski dense arithmetic subgroup of  $S(\mathbb{R})$  and the projection of  $S$  onto each simple factor of  $G$  is infinite. Then there exists a  $\mathbb{Q}$ -form of  $G$  such that  $\Gamma_{F_1, F_2} \subset G(\mathbb{Q})$ .*

*Proof.* Set  $\Gamma = N(\Gamma_{F_1, F_2})$ ,  $B = \{g \in G \mid gN(U_2)g^{-1} \text{ and } N(U_1) \text{ are opposite}\}$ ,  $A_i = S \times U_i$ , and  $\mathfrak{A}_i = \text{Lie}(A_i)$  for each  $i = 1, 2$ . Denote by  $M$  and  $N$  the subspaces of the Lie algebra  $\mathfrak{g}$  of  $G$  generated, respectively, by  $\bigcup_{\gamma \in B \cap \Gamma} (\mathfrak{A}_1 \cap \text{Ad}(\gamma)\mathfrak{A}_2)$  and  $\bigcup_{\gamma^{-1} \in B \cap \Gamma} (\mathfrak{A}_2 \cap \text{Ad}(\gamma)\mathfrak{A}_1)$  and by  $V$  the

set of  $g_0 \in G$  for which  $\dim(M \cap \text{Ad}(g_0)N) = \min_{g \in G} \dim(M \cap \text{Ad}(g)N)$ . We set  $W = B \cap V$ .

We claim that the assumptions of Theorem 2.4.1 hold for  $G = H$ ,  $\Gamma = \Lambda$ ,  $\text{Ad} = T$ ,  $L = \mathfrak{g}$ ,  $M$ ,  $N$ , and  $W$ . First of all, since  $G$  is adjoint, equivalently  $Z(G)$  is trivial, the adjoint representation  $\text{Ad}$  of  $G$  is faithful. Since  $\mathfrak{A}_1 \cap \mathfrak{A}_2$  is the Lie algebra of  $S$ ,  $\mathfrak{A}_1 \cap \mathfrak{A}_2 \subset M$  is non-trivial. Since  $\mathfrak{g}$  is simple and the projection of  $S$  onto each simple factor of  $G$  is finite,  $\mathfrak{g}$  is generated by  $\text{Ad}(g)M$ ,  $g \in G$ . We observe that  $M \cap \text{Ad}(\gamma)N = \mathfrak{A}_1 \cap \text{Ad}(\gamma)\mathfrak{A}_2$  and  $N \cap \text{Ad}(\gamma)M = \mathfrak{A}_2 \cap \text{Ad}(\gamma)\mathfrak{A}_1$  for all  $\gamma \in B \cap \Gamma$ . Since the set  $B$  is of the form  $N(U_1)N(U_2)$  [5, Lemma 4.1.2] and  $A_i$  is a normal subgroup of  $P_i$ , we have that  $A_1 = (A_1 \cap \gamma A_2 \gamma^{-1}) \rtimes U_1$  for all  $\gamma \in B$ . This implies that  $\dim(M \cap \text{Ad}(\gamma)N) = \dim(\mathfrak{A}_1 \cap \text{Ad}(\gamma)\mathfrak{A}_2) = \dim(A_1 \cap \gamma A_2 \gamma^{-1}) = \dim(A_1) - \dim(U_1)$  for all  $\gamma \in B \cap \Gamma$ . But  $W \subset B$ ,  $W$  is open, and  $\Gamma$  is Zariski dense. Therefore  $d = \dim(A_1) - \dim(U_1)$  and hence  $B \cap \Gamma = W \cap \Gamma$ . It follows that  $M$  and  $N$  are generated by  $\cup_{\gamma \in W} (\mathfrak{A}_1 \cap \text{Ad}(\gamma)\mathfrak{A}_2)$  and  $\cup_{\gamma^{-1} \in W} (\mathfrak{A}_2 \cap \text{Ad}(\gamma)\mathfrak{A}_1)$ , respectively, proving the claim.

Since  $F_i$  is a Zariski dense arithmetic subgroup of  $U_i(\mathbb{R})$  for  $i = 1, 2$  and  $S \cap \Gamma$  is a Zariski dense arithmetic subgroup of  $S(\mathbb{R})$  normalizing  $F_1$  and  $F_2$ , we can give  $\mathbb{Q}$ -forms on  $A_1$  and  $A_2$  with respect to which  $A_1 \cap \Gamma$  and  $A_2 \cap \Gamma$  are arithmetic subgroups of  $A_1(\mathbb{R})$  and  $A_2(\mathbb{R})$  (e.g., Lemma 3.4.9). In a natural way, these  $\mathbb{Q}$ -forms induce  $\mathbb{Q}$ -structures on  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  and also on  $M$  and  $N$ . Since  $A_i \cap \Gamma$  is a lattice in  $A_i(\mathbb{R})$  for each  $i = 1, 2$ , and  $\Gamma$  is discrete, we have that for any  $\gamma \in \Gamma$ ,  $(A_1 \cap \gamma A_2 \gamma^{-1} \cap \Gamma) \rtimes F_1$  has a finite index in  $A_1 \cap \Gamma$ . Since  $A_1 \cap \Gamma$  is Zariski dense in  $A_1$ , the Zariski closure of  $A_1 \cap \gamma A_2 \gamma^{-1} \cap \Gamma$  is  $A_1 \cap \gamma A_2 \gamma^{-1}$  and hence  $A_1 \cap \gamma A_2 \gamma^{-1}$  is a  $\mathbb{Q}$ -subgroup of  $A_1$ . Since  $M \cap \text{Ad}(\gamma)N = \mathfrak{A}_1 \cap \text{Ad}(\gamma)\mathfrak{A}_2$  for all  $\gamma \in B \cap \Gamma$ , it follows that the  $\mathbb{Q}$ -structures on  $M$  and  $N$  are compatible. Now by Theorem 2.4.1, there exists a  $\mathbb{Q}$ -form of  $G$  such that  $\Gamma_{F_1, F_2} \subset G(\mathbb{Q})$ . ■

2.4.3. LEMMA. *If  $\Gamma_{F_1, F_2} \subset G(\mathbb{Q})$ , then  $U_i$  is defined over  $\mathbb{Q}$  and there exists  $n \in \mathbb{N}$  such that  $U_i(n\mathbb{Z}) \subset F_i$  for each  $i = 1, 2$ .*

*Proof.* Since  $F_i \subset U_i \cap G(\mathbb{Q})$  and  $F_i$  is Zariski dense in  $U_i$  by Lemma 2.1.9,  $U_i$  is defined over  $\mathbb{Q}$  by Proposition 1.2.7. The second claim follows from Lemma 2.1.11. ■

2.4.4. LEMMA. (1) *If  $\Gamma_{F_1, F_2} \subset G(\mathbb{Q})$  and for every infinite proper normal  $\mathbb{R}$ -subgroup  $G'$  of  $G$ ,  $F_i \cap G'(\mathbb{R})$  is finite for each  $i = 1, 2$  then  $G$  is  $\mathbb{Q}$ -simple.*

(2) *If  $G$  is  $\mathbb{Q}$ -simple and  $U_i$  is defined over  $\mathbb{Q}$ , then for every infinite proper normal  $\mathbb{R}$ -subgroup  $G'$  of  $G$ ,  $U_i \cap G'(\mathbb{R})$  is finite.*

*Proof.* (1) If  $G$  is not  $\mathbb{Q}$ -simple, there exist non-trivial semisimple normal  $\mathbb{Q}$ -subgroups  $G'$  and  $G''$  such that  $G$  is the direct product of  $G'$  and  $G''$  by Proposition 1.2.4. By Lemma 2.4.3,  $F_i$  contains  $U_i(n\mathbb{Z})$  for some  $n \in \mathbb{N}$ . Since  $U_i \cap G'$  is a  $\mathbb{Q}$ -subgroup,  $U_i \cap G'(n\mathbb{Z})$  is infinite. But  $U_i(n\mathbb{Z}) = (U_i \cap G'(n\mathbb{Z}))(U_i \cap G''(n\mathbb{Z}))$ , yielding that  $F_i \cap G'(\mathbb{R})$  is infinite. This contradicts the assumption.

(2) There exist a finite extension field  $k$  of  $\mathbb{Q}$ , an adjoint absolutely simple  $k$ -group  $G_0$ , and a horospherical  $k$ -group  $U'_i$  such that  $G = R_{k/\mathbb{Q}}(G_0)$  and  $U_i = R_{k/\mathbb{Q}}(U'_i)$  by Proposition 1.2.6. Any infinite normal algebraic  $\mathbb{R}$ -subgroup  $G'$  is then isomorphic to the product of suitable  $n$ -copies of  $G_0$ . Since  $U_i(\mathbb{Z}) = R_{k/\mathbb{Q}}(U'_i)(J)$ ,  $J$  the ring of the integers of  $k$ , the claim follows. ■

**2.4.5. COROLLARY.** *Let  $G$ ,  $U_1$ ,  $U_2$ ,  $F_1$ ,  $F_2$ , and  $S$  be as in Proposition 2.4.2. Assume that the real rank of each  $\mathbb{R}$ -simple factor of  $G$  is at least 2 and  $F_i \cap G'(\mathbb{R})$  is finite for every infinite proper normal algebraic  $\mathbb{R}$ -subgroup  $G'$  of  $G$  and each  $i = 1, 2$ . If  $\Gamma_{F_1, F_2}$  is discrete, then it is an arithmetic subgroup of  $G(\mathbb{R})$ .*

*Proof.* By Proposition 2.4.2 and Lemma 2.4.3, there exists a  $\mathbb{Q}$ -form of  $G$  such that  $\Gamma_{F_1, F_2} \subset G(\mathbb{Q})$  and  $U_i(n\mathbb{Z}) \subset F_i$  for some  $n \in \mathbb{N}$ . By Lemma 2.4.4,  $G$  is almost  $\mathbb{Q}$ -simple. It follows from Corollary 2.2.3 that the subgroup  $\Gamma$  generated by  $U_1(n\mathbb{Z})$  and  $U_2(n\mathbb{Z})$  is an arithmetic subgroup of  $G(\mathbb{R})$ . By Lemma 2.1.4,  $\Gamma_{F_1, F_2}$  is an arithmetic subgroup of  $G(\mathbb{R})$ . ■

**2.4.6. PROPOSITION.** *Let  $G$  be a  $\mathbb{Q}$ -simple algebraic  $\mathbb{Q}$ -group with real rank at least 2 and  $U_1, U_2$  be defined over  $\mathbb{Q}$ . Suppose that the semisimple part of  $N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$  does not have compact factors. If  $F_1$  and  $F_2$  are commensurable to  $U_1(\mathbb{Z})$  and  $wU_2(\mathbb{Z})w^{-1}$  for some  $w \in Z(L)(\mathbb{R})$ , respectively, and  $\Gamma_{F_1, F_2}$  is discrete,  $\Gamma_{F_1, F_2}$  is an arithmetic subgroup of  $G(\mathbb{R})$ .*

*Proof.* We may assume that  $F_1 = U_1(\mathbb{Z})$  and  $F_2 = wU_2(\mathbb{Z})w^{-1}$ . Let  $N = N(\Gamma_{F_1, F_2})$  and  $H$  the semisimple part of  $N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$ . Then  $N$  is discrete by Lemma 2.1.3 and  $H$  is a semisimple subgroup defined over  $\mathbb{Q}$ . Since  $w \in Z(L)$ ,  $H(\mathbb{Z})$  normalizes  $F_1$  and  $F_2$  and hence  $H(\mathbb{Z}) \subset N$ . Since  $H(\mathbb{Z})$  is a Zariski dense arithmetic subgroup of  $H(\mathbb{R})$ , it follows from Proposition 2.4.2 that there exists a  $\mathbb{Q}$ -form of  $G$  such that  $N \subset G(\mathbb{Q})$ . Since  $\Gamma_{F_1, F_2} \subset G(\mathbb{Q})$ , we can show the rest of claim by the same argument as the proof of Corollary 2.4.5. ■

## 2.5. The Subgroup Generated by $Z(U_1)$ and $Z(U_2)$

We prove some algebraic lemmas which we will need later in applying Proposition 2.4.2.

**2.5.1.** Let  $G$  be an adjoint  $k$ -simple algebraic  $k$ -group. This section has content only when the  $k$ -rank of  $G$  is at least 2. We continue the notation

from Subsection 1.3.1 for  $S, T, \Phi = \Phi(T, G), {}_k\Phi = \Phi(S, G), j: \Phi \rightarrow {}_k\Phi \cup \{0\}, \Delta, {}_k\Delta, {}_kV_\Theta, {}_kV_\Theta^-$  for  $\Theta \subset {}_k\Delta$  and so on. We recall that  $\alpha_h$  denotes the highest  $k$ -root in  ${}_k\Phi$  and  ${}_k\Delta_H$  the unique set of  $k$ -Heisenberg roots in  ${}_k\Delta$ .

The following lemma can be checked case by case in each irreducible root system.

LEMMA. (1) *The only positive  $k$ -root the sum of whose coefficients with respect to which the roots in  ${}_k\Delta_H$  is 2 is  $\alpha_h$ .*

(2) *If  $\beta \in {}_k\Phi^+$  is such that the sum of the coefficients with respect to  ${}_k\Delta_H$  is 1 and there exists a simple root in  ${}_k\Delta$  with respect to which the coefficient of  $\beta$  is 0, then  $\alpha_h - \beta \in {}_k\Phi^+$ .*

2.5.2. The commutator law over an algebraically closed field is well known. The following is a result of Vinberg which says that the commutator law holds for an arbitrary field. Hereafter, we shall also refer to this lemma as the commutator law.

LEMMA (e.g., [12, 4.5.1]). *If  $a, b, a + b \in {}_k\Phi$ , then  $[{}_k\mathcal{Z}_a, {}_k\mathcal{Z}_b] = {}_k\mathcal{Z}_{a+b}$  and moreover, for any nonzero  $x \in ({}_k\mathcal{Z}_a)$ , we have  $[x, {}_k\mathcal{Z}_b] \neq 0$ .*

2.5.3. We recall that the notation  $U_{\alpha_h}$  denotes the  $k$ -root group corresponding to the highest  $k$ -root  $\alpha_h$  and  $U_H$  denotes the  $k$ -Heisenberg horospherical  $k$ -subgroup  ${}_kV_{k\Delta - k\Delta_H}$ .

We observe that  $Z({}_kV_\Theta) = U_\psi$  where  $\psi = \{\beta \in {}_k\Phi^+ - [\Theta] \mid \beta + \alpha \notin {}_k\Phi \text{ for any } \alpha \in {}_k\Phi^+ - [\Theta]\}$ . In particular  $\alpha_h \in \psi$ .

LEMMA.  $Z({}_kV_\Theta) = U_{\alpha_h}$  if and only if  ${}_k\Delta_H \subset {}_k\Delta - \Theta$ .

*Proof.* If  $\gamma \in {}_k\Delta_H \cap \Theta$ ,  $\alpha_h - \gamma$  is a  $k$ -root by Lemma 2.5.1; hence  $\alpha_h - \gamma \in {}_k\Phi^+ - [\Theta]$ . We claim that  $(\alpha_h - \gamma) + \beta \notin {}_k\Phi^+$  for any  $\beta \in {}_k\Phi^+ - [\Theta]$ . Suppose not. Then by the height calculation, we have  $(\alpha_h - \gamma) + \beta = \alpha_h$ . It implies that  $\gamma = \beta$ , contradicting the assumption  $\gamma \in \Theta$ . Therefore if  $Z(U) = U_{\alpha_h}$ , or equivalently  $\psi = \{\alpha_h\}$  as the above notation, then  ${}_k\Delta_H \cap \Theta$  is empty.

To see the converse, suppose that  ${}_k\Delta_H \subset \Theta^c$ . Then  $U_H$  is contained in  ${}_kV_\Theta$ . Since the centralizer of  $U_H$  in  ${}_kV_\Theta$  is contained in  $R_u(N(U_H)) = U_H$  and hence coincides with  $Z(U_H) = U_{\alpha_h}$ , we have  $Z({}_kV_\Theta) \subset U_{\alpha_h}$ . ■

2.5.4. For  $a \in {}_k\Phi$ , denote by  ${}_k\mathcal{Z}_a$  the root space corresponding to  $a$  and put  ${}_kG_a = \exp({}_k\mathcal{Z}_a)$ . Note that for any subset  $\psi \subset {}_k\Phi$ , the subgroup  $G_{[\psi]}^*$  (as in the notation of Subsection 1.2.1) coincides with the subgroup generated by all the  ${}_kG_a$  ( $a \in \psi$ ).

A closed subset  $\Phi_0$  of  $\Phi$  is called an ideal if the following holds: for all  $a \in \Phi_0$  and  $\beta \in \Phi$  such that  $\alpha + \beta \in \Phi$ ,  $\alpha + \beta \in \Phi_0$ . It follows from the commutator law that if  $\Phi_0$  is an ideal of  $\Phi$ , then the subgroup  $G_{\Phi_0}^*$  is a normal subgroup of  $G$ .

In proving the following proposition, we use the fact (e.g., [26, Proposition 1.1.10]) that if  $\phi$  is a closed subset of  $\Phi$  such that if  $\alpha \in \phi$  then  $-\alpha \in \phi$ , then the subalgebra generated by  $\mathcal{U}_b$ ,  $b \in \phi$  is semisimple.

**PROPOSITION.** *Let  $U_1 = {}_kV_\Theta$  and  $U_2 = {}_kV_{\Theta^-}$  for some  $\Theta \subset {}_k\Delta$ . Denote by  $G_0$  the subgroup of  $G$  generated by  $Z(U_1)$  and  $Z(U_2)$ ,  $\mathbf{H}$  the commutator subgroup of  $N(U_1) \cap N(U_2)$  and  $S = \mathbf{H} \cap G_0$ . Then*

(1)  $G_0$  is a connected almost  $k$ -simple algebraic  $k$ -subgroup,  $Z(U_1)$  and  $Z(U_2)$  are opposite commutative horospherical  $k$ -subgroups of  $G_0$ , and  $S$  is a connected semisimple normal algebraic  $k$ -subgroup of  $\mathbf{H}$ . Moreover, if  $G$  is absolutely almost simple, so is  $G_0$ .

(2) If  $Z(U_1)$  is strictly bigger than  $U_{\alpha_h}$ , then  $S$  is isotropic over  $k$  and the  $k$ -rank of  $G_0$  is at least 2.

(3) If  $\mathbf{H}$  does not have any  $k$ -anisotropic factors, neither does  $S$ .

*Proof.* Set  $\psi = \{\beta \in {}_k\Phi^+ - [\Theta] \mid \beta + \alpha \notin {}_k\Phi \text{ for any } \alpha \in {}_k\Phi^+ - [\Theta]\}$ . Note that  $\psi$  is a closed subset of  ${}_k\Phi^+$  and  $Z(U_1) = U_\psi$  and  $Z(U_2) = U_{-\psi}$ .

(1) Note that  $[\psi]$  is a closed set of  ${}_k\Phi$  and the subgroup  $G_{[\psi]}^*$  generated by  $U_b$ ,  $b \in j^{-1}([\psi])$  (or equivalently,  $b \in [j^{-1}(\psi)]$ ) coincides with  $G_0$ . This group is algebraic, connected (any algebraic group generated by unipotent subgroups is connected) and semisimple (by the fact mentioned preceding the statement of this proposition). On the other hand,  $\psi$  is invariant under  $\text{Gal}(K/k)$  since  $Z(U_1)$  is defined over  $k$  (see Subsection 1.2.1). Therefore  $[\psi]$  is also invariant under  $\text{Gal}(K/k)$  and hence  $G_0$  is defined over  $k$ .

To show that  $G_0$  is almost  $k$ -simple, assume that  $\Psi = [\psi]$  is the disjoint union  $\Psi_1 \cup \Psi_2 \cup \dots \cup \Psi_s$  where each  $\Psi_i$  is a non-empty irreducible ideal of  $\Psi$ . We will show that each  $\Psi_i$  contains the unique highest  $k$ -root  $\alpha_h$ , which is an obvious contradiction if  $s \geq 2$ . Note that  $\Psi_i \cap \psi$  is non-empty for each  $i$ . For any  $\alpha \in \Psi_i \cap j^{-1}(\psi)$ , there exists simple  $k$ -roots  $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$  (not necessarily different) such that  $\alpha + \alpha_{i_1} + \dots + \alpha_{i_j} \in {}_k\Phi^+$  for each  $j = 1, 2, \dots, k$  and  $\alpha = \alpha_{i_1} + \dots + \alpha_{i_k} = \alpha_h$ .

Since  $\alpha + \alpha_{i_1} \in {}_k\Phi^+$ ,  $\alpha + \alpha_{i_1} \in \psi$ . It follows that  $\alpha_{i_1} \in \Psi$ . Since  $\Psi_i$  is an ideal of  $\Psi$ , we have  $\alpha + \alpha_{i_1} \in \Psi_i$ . Therefore by induction  $\alpha + \alpha_{i_1} + \dots + \alpha_{i_j} \in \Psi_i$  for each  $j = 1, 2, \dots, k$ ; hence  $\alpha_h \in \Psi_i$ . This proves that  $G_0$  is almost  $k$ -simple.

If  $G$  is absolutely almost simple, then  $\Phi$  has the unique highest root. Then using the same argument as above we can show that  $[j^{-1}(\psi)]$  cannot be a disjoint union of ideals. It follows that  $G_0$  is absolutely almost simple.

We note that  $\text{Lie } G_0$  is decomposed into  $(\text{Lie } L \cap \text{Lie } G_0) \oplus \text{Lie } Z(U_1) \oplus \text{Lie } Z(U_2)$ . Clearly  $Z(U_1)$  and  $Z(U_2)$  are opposite horospherical subgroups of  $G_0$ .

By the same argument we made for  $G_0$ , we can show that  $S$  is a semisimple algebraic  $k$ -group.

We now show that  $S$  is a normal subgroup of  $H$ . Let  $s \in S$  and  $h \in H$ . Then  $S = a_1 a_2 \cdots a_k$  for some  $a_i \in Z(U_1) \cup Z(U_2)$ . Since  $H \subset N(U_1) \cap N(U_2)$ ,  $ha_i h^{-1} \in U_1 \cup U_2$  for each  $i$ . Suppose that  $a_i \in Z(U_1)$ ; hence  $ha_i h^{-1} \in U_1$ . To see  $ha_i h^{-1} \in Z(U_1)$ , let  $u \in U_1$  and consider  $u(ha_i h^{-1}) = h(h^{-1}uh)a_i h^{-1}$ , which is equal to  $ha_i(h^{-1}uh)h^{-1}$  since  $\alpha_i \in Z(U_i)$ . So  $u(ha_i h^{-1}) = (ha_i h^{-1})u$ . It proves that  $H$  normalizes  $Z(U_1)$  and similarly  $Z(U_2)$ . Therefore  $hSh^{-1} \in G_0$ ; hence  $hSh^{-1} \in S$ , proving  $H$  normalizes  $S$ .

(2) Suppose that  $U_{\alpha_h}$  is a proper subgroup of  $Z(U_1)$ . By Lemma 2.5.3, we have an element  $\alpha \in \Theta \cap_k \Delta_H$ . Then  $\alpha_h - \alpha \in_k \Phi^+$  by Lemma 2.5.1. We claim that this root belongs to  $\psi$ . Otherwise, for some element  $\beta \in_k \Phi^+ - [\Theta]$ ,  $(\alpha_h - \alpha) + \beta$  is a  $k$ -root. By comparing height, it follows that  $(\alpha_h - \alpha) + \beta = \alpha_h$  hence  $\alpha = \beta$ , contradicting  $\alpha \in \Theta$ . Therefore  $\alpha = (\alpha_h) - (\alpha_h - \alpha) \in [\psi]$ , showing that  $S$  is isotropic over  $k$ , since  $\text{Lie } S$  clearly contains  $\mathcal{Z}_\alpha$  for all  $\alpha \in [\psi] \cap \Theta$ .

(3) Since  $S$  is a normal algebraic  $k$ -subgroup of  $\mathbf{H}$ ,  $S$  has no  $k$ -anisotropic factors unless  $\mathbf{H}$  does. ■

2.5.5. EXAMPLE. The following example illustrates the power of Proposition 2.4.2, which will be also one of the most fundamental ideas of the proof of the main theorem.

Let  $U_1, U_2$  be the pair of opposite horospherical subgroups in  $SL_n(\mathbb{R})$ , consisting of the elements of the form

$$\begin{pmatrix} I_l & X_1 & Z_1 \\ \mathbf{0} & I_m & Y_1 \\ \mathbf{0} & \mathbf{0} & I_k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_l & \mathbf{0} & \mathbf{0} \\ X_2 & I_m & \mathbf{0} \\ Z_2 & Y_2 & I_k \end{pmatrix},$$

respectively, where  $l + m + k = n$ . Note that  $U_1$  and  $U_2$  are two step nilpotent groups, i.e.,  $[U_i, U_i] = Z(U_i)$  for each  $i = 1, 2$ .

If  $lk \geq 2$  and  $F_1$  and  $F_2$  are lattices in  $U_1$  and  $U_2$  such that  $F_1 \cap Z(U_1) = M_{l \times k}(\mathbb{Z})$  and  $F_2 \cap Z(U_2) = M_{k \times l}(\mathbb{Z})$ , there exists a  $\mathbb{Q}$ -form of  $SL_n(\mathbb{R})$  such that  $\Gamma_{F_1, F_2} \subset G(\mathbb{Q})$ . To see this, note that the subgroup  $G_0$  generated by  $Z(U_1)$  and  $Z(U_2)$  consists of the elements

$$\begin{pmatrix} X & \mathbf{0} & X' \\ \mathbf{0} & I_m & \mathbf{0} \\ Z' & \mathbf{0} & Z \end{pmatrix}$$

in  $SL_n(\mathbb{R})$  and  $S = H \cap G_0$  consists of the elements

$$\begin{pmatrix} X & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & Z \end{pmatrix},$$

where  $H$  is the commutator subgroup of  $N(U_1) \cap N(U_2)$ . Observe that  $S \cap \Gamma_{F_1, F_2}$  is

$$\begin{pmatrix} M_{l \times l}(\mathbb{Z}) & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & M_{k \times k}(\mathbb{Z}) \end{pmatrix}.$$

It only remains to apply Proposition 2.4.2.

Since  $S(\mathbb{Q})$  is split in this case and hence at least one of  $\alpha_1$  and  $\alpha_{n-1}$  belongs to  $\Delta - \Delta_0$  (see the notation in Subsection 1.4.1), it follows from the classification of  $\mathbb{Q}$ -forms [21] that  $G(\mathbb{Q})$  is conjugate to  $SL_n(\mathbb{Q})$ .

### 3. ADJOINT ACTION ON THE SPACE OF LATTICES

#### 3.1. The Space of Lattices in Algebraic Unipotent Groups

We collect some well known lemmas.

**3.1.1. PROPOSITION** (e.g., [10, Lemma 1.3.1]). *The group of real points of a unipotent algebraic  $\mathbb{R}$ -group is a topologically connected, simply connected, nilpotent Lie group, and every unipotent subgroup of an algebraic group is nilpotent.*

**3.1.2.** We state a well known criterion of compactness due to Mahler.

**LEMMA (Mahler [6]).** *A set  $A$  of lattices of some Euclidean space is relatively compact if and only if there are positive constants  $d$  and  $\epsilon$  such that the  $\epsilon$ -neighborhood in the Euclidean space does not contain any nonzero element from any lattice belonging to  $A$  and  $\det x < d$  if  $x \in A$ .*

**3.1.3. LEMMA (Minkowski [6]).** *For any  $n$ , there is a constant  $c(n)$  such that in any quasi-lattice  $\Delta$  lying in an Euclidean space of dimension not exceeding  $n$ , there exists a basis  $e_1, e_2, \dots, e_k$  of  $\Delta$  ( $k = \dim \Delta$ ) such that for any  $l$  ( $1 \leq l \leq k$ ),  $\prod_{i=1}^l \|e_i\| < c(n)d(\Delta)^{1/k}$ .*

**3.1.4. LEMMA.** *Let  $W$  be the space of lattices of some fixed Euclidean space. For any relatively compact subset  $M \subset W$ , there are positive constants  $d_1$  and  $d_2$  such that in any lattice  $M$ , there is a basis consisting of elements of norm less than  $d_1$  and greater than  $d_2$ .*

*Proof.* It is easily deduced from Mahler's compactness criterion and the previous lemma. ■

3.1.5. PROPOSITION [10, Lemma 5.2]. *Let  $Z$  be a simply connected nilpotent Lie group and  $\mathfrak{Z}$  its Lie algebra. Then there exists an integer  $b$  such that if  $\Gamma$  is a subgroup of  $Z$  and  $\overline{\ln \Gamma}$  is the subring in  $\mathfrak{Z}$  generated by  $\ln \Gamma$ , then  $b(\overline{\ln \Gamma}) \subset \ln \Gamma$ . Furthermore  $b$  depends only on the length of the lower central series.*

3.1.6. Let  $\mathbf{G}$  be an algebraic  $\mathbb{R}$ -group,  $G = \mathbf{G}(\mathbb{R})^0$ , and  $\mathfrak{g}$  the Lie algebra of  $G$ . We choose a Euclidean metric  $\rho_{\mathfrak{g}}$  on  $\mathfrak{g}$  and a left invariant metric  $\rho_G$  on  $G(\mathbb{R})$  corresponding  $\rho_{\mathfrak{g}}$ . Denote by  $X$  the set of all discrete unipotent subgroups of  $G$ . Since in an algebraic  $\mathbb{R}$ -group, any two maximal unipotent  $\mathbb{R}$ -groups are conjugate to each other, there exists an integer  $N \in \mathbb{N}$  such that for any subgroup  $F \in X$ , the length of the lower central series of  $F$  is at most  $N$ . Therefore there exists an integer  $b \in \mathbb{N}$  such that if  $F \in X$ , then  $b(\overline{\ln F}) \subset \ln F$ . We denote  $b(\overline{\ln F})$  by  $\Delta_F$ . We introduce in the following topology on the space of quasi-lattices  $\Omega$  in that a subset  $B \subset \Omega$  is open if, for any  $F \in B$ , there exists some  $\epsilon > 0$  such that any quasi-lattice of the form  $T(F)$  belongs to  $B$  for each linear transformation  $T$  with  $\|T - Id\| \leq \epsilon$ . Let  $f: X \rightarrow \Omega$  be the mapping defined by  $f(F) = \Delta_F$ .

LEMMA [10, Lemma 5.7]. *For any subset  $L \subset X$  for which  $f(L)$  is relatively compact in  $\Omega$ , there are constants  $\epsilon > 0$  and  $c$  such that if  $F \in L$ , then*

- (1)  $\rho_G(e, x) > \epsilon$  for all non-trivial  $x \in F$ ;
- (2) there exists  $x_1, \dots, x_k \in F$  such that  $\rho_G(e, x_i) < c$  for any  $i$  and any algebraic subgroup of  $G$  containing  $x_1, \dots, x_k$  also contains the Zariski closure of  $F$  in  $G$ .

### 3.2. Adjoint Action

3.2.1. Let  $\mathbf{G}$  be a connected adjoint semisimple algebraic  $\mathbb{R}$ -group with no  $\mathbb{R}$ -anisotropic factors and  $U_1, U_2$  a pair of opposite horospherical  $\mathbb{R}$ -subgroups. Set  $\mathbf{L} = N(U_1) \cap N(U_2)$ ,  $\mathbf{H} = [\mathbf{L}, \mathbf{L}]$ , and  $H = \mathbf{H}(\mathbb{R})^0$  as well as  $G = \mathbf{G}(\mathbb{R})^0$ . As before set  $\mathcal{Z}_i = \text{Lie}(U_i)$  and denote by  $\text{Ad}_i: \mathbf{L} \rightarrow GL(\mathcal{Z}_i)$  the restriction of the adjoint representation of  $N(U_i)$ . We recall that  $\text{Ad}_i$  is faithful.

We mention that this section has no content if  $H$  is trivial.

3.2.2. Denote by  $\Omega_i$  the space of lattices in the Euclidean space  $\mathcal{Z}_i(\mathbb{R})$ . The group  $H$  acts through  $\text{Ad}_i$  on  $\mathcal{Z}_i(\mathbb{R})$  and hence on  $\Omega_i$ . The notation  $H.J_i$  denotes the orbit  $\{\text{Ad}_i(h)J_i \in \Omega_i \mid h \in H\}$  of  $J_i$  under  $H$  for a lattice  $J_i$  in  $\mathcal{Z}_i(\mathbb{R})$ . We also diagonalize the action of  $H$  on  $\Omega_i$  described above to

the space  $\Omega_1 \times \Omega_2$  of pairs of lattices in  $\mathcal{Z}_1(\mathbb{R})$  and  $\mathcal{Z}_2(\mathbb{R})$  by  $h(J_1, J_2) = (\text{Ad}_1(h)J_1, \text{Ad}_2(h)J_2)$ . the image of  $H$  under this action is the subgroup  $\{(\text{Ad}_1(h), \text{Ad}_2(h)) \in SL(\mathcal{Z}_1(\mathbb{R})) \times SL(\mathcal{Z}_2(\mathbb{R})) \mid h \in H\}$  and will be denoted by  $\delta(H)$ . The notation  $\delta(H).(J_1, J_2)$  denotes the  $H$ -orbit of the pair  $(J_1, J_2)$  via this action.

3.2.3. Let  $F_i$  be a lattice in  $U_i(\mathbb{R})$  and  $\Delta_{F_i}$  be as in Subsection 3.1.6 for  $i = 1, 2$ . Denote by  $\Lambda_{F_i}$  the stabilizer  $\{g \in H \mid \text{Ad}_i(g)\Delta_{F_i} = \Delta_{F_i}\}$  of  $\Delta_{F_i}$  and by  $\Lambda_{F_1, F_2}$  the stabilizer  $\{g \in H \mid \text{Ad}_1(g)\Delta_{F_1} = \Delta_{F_1}, \text{Ad}_2(g)\Delta_{F_2} = \Delta_{F_2}\}$  of  $(\Delta_{F_1}, \Delta_{F_2})$ .

The following lemma then immediately follows from the definition of  $\Delta_{F_i}$  and Lemma 1.5.3.

LEMMA. (1)  $\Delta_{hF_ih^{-1}} = \text{Ad}_i(h)\Delta_{F_i}$  for any  $h \in H$  and any lattice  $F_i$  in  $U_i(\mathbb{R})$ .

(2)  $\Lambda_{F_i} = \{g \in H \mid gF_i g^{-1} = F_i\}$ .

(3)  $\Lambda_{F_1, F_2} = \{g \in H \mid gF_1 g^{-1} = F_1, gF_2 g^{-1} = F_2\}$ .

3.2.4. We choose a Euclidean metric  $\rho_{\mathfrak{g}}$  on the Lie algebra  $\mathfrak{g}$  of  $G$  and a left invariant metric  $\rho_G$  on  $G$  corresponding to this metric. We define a topology on  $\Omega_i$  and  $\Omega_1 \times \Omega_2$  in the following way: a sequence of lattices  $\{\Delta_k \in \Omega_i \mid k = 1, 2, \dots\}$  converges to a lattice  $\Delta$  if and only if there is  $\epsilon > 0$  such that  $\rho_{\mathfrak{g}}(0, x) > \epsilon$  for all nonzero  $x \in \Delta_k$  and for all  $k$  and there exist bases in the lattices  $\Delta_k$  which converge to some basis in  $\Delta$ . It is not difficult to see that this topology coincides with the topology defined in Subsection 3.1.6.

We note that not every lattice in the Euclidean space  $\mathcal{Z}_i(\mathbb{R})$  is of the form  $\Delta_{F_i}$  for a lattice  $F_i$  in  $U_i(\mathbb{R})$  unless  $U_i$  is commutative. In fact any  $\mathbb{Z}$ -linear span of  $n$ -linearly independent vectors forms a lattice in  $\mathcal{Z}_i(\mathbb{R})$  where  $n = \dim(\mathcal{Z}_i)$ ; while a lattice in  $U_i(\mathbb{R})$  should satisfy certain relations from the group structure of  $U_i$  since by definition a lattice in  $U_i(\mathbb{R})$  is a subgroup. On the other hand, the following lemma says that any lattice in  $\overline{H.\Delta_{F_i}}$  is of the form  $\Delta_{E_i}$  for some lattice  $E_i$  in  $U_i(\mathbb{R})$ .

For a subset  $M$  of the space of lattices, we denote by  $\overline{M}$  the closure of  $M$ .

LEMMA. For any lattice  $J_i \in \overline{H.\Delta_{F_i}}$ , there exists a lattice  $E_i$  in  $U_i(\mathbb{R})$  such that  $\Delta_{E_i} = J_i$  and the determinant of  $J_i$  is equal to the determinant of  $\Delta_{F_i}$ .

*Proof.* Let  $x_n \in H$  be a sequence such that  $\text{Ad}_i(x_n)\Delta_{F_i}$  converges to  $J_i$ . Since  $\{\text{Ad}_i(x_n)\Delta_{F_i} \mid n \in \mathbb{N}\}$  is relatively compact, by Lemma 3.1.6, there is a neighborhood  $W$  of  $e$  such that  $W \cap x_n F_i x_n^{-1} = \{e\}$  for each  $n \in \mathbb{N}$ . By Chabauty's theorem (see Theorem 2.1.7), there exists a discrete subgroup  $E_i$  in  $U_i(\mathbb{R})$  to which a subsequence of  $x_n F_i x_n^{-1}$  converges. Since  $H$  is a semisimple group, the volume of  $U_i(\mathbb{R})/x_n F_i x_n^{-1}$  coincides with the

volume of  $U_i(\mathbb{R})/F_i$  for all  $n$ . It follows from Chabauty's theorem (Theorem 2.1.7) that  $E_i$  is a lattice in  $U_i(\mathbb{R})$ . Since a subsequence of  $\text{Ad}_i(x_n)\Delta_{F_i}$  converges to  $\Delta_{E_i}$ ,  $\Delta_{E_i} = J_i$ , proving the lemma. ■

**3.2.5. PROPOSITION.** *Suppose that  $\Gamma_{F_1, F_2}$  is discrete. Then for any pair  $(\Delta_{E_1}, \Delta_{E_2})$  lying in  $\overline{\delta(H) \cdot (\Delta_{F_1}, \Delta_{F_2})}$ , the subgroup  $\Gamma_{E_1, E_2}$  is discrete.*

*Proof.* Consider a sequence  $x_i \in H$  such that  $\text{Ad}(x_i)(\Delta_{F_1}, \Delta_{F_2})$  converges to  $(\Delta_{E_1}, \Delta_{E_2})$ . Since the set  $\{\text{Ad}_1(x_i)\Delta_{F_1}, \text{Ad}_2(x_i)\Delta_{F_2}, \Delta_{E_1}, \Delta_{E_2} \mid i \geq 1\}$  is relatively compact in the space of lattices, there exist  $d_1 > 0$  and  $d_2 > 0$  such that for any lattice in this set, there exists a basis consisting of elements of norm less than  $d_1$  and greater than  $d_2$  by Lemma 3.1.4. Let  $d'_1$  and  $d'_2$  be such that  $d'_2 \leq \rho_G(e, x) \leq d'_1$  implies that  $d_2 \leq \rho_{\mathfrak{g}}(\mathbf{0}, \ln x) \leq d_1$  for all  $x \in U_1 \cup U_2$ . Set  $K = \{g \in G \mid d'_2 \leq \rho_G(e, g) \leq d'_1\}$ . For each  $x_i \in H$ , the set  $x_i\Gamma_{F_1, F_2}x_i^{-1} \cap K$  generates the subgroup  $x_i\Gamma_{F_1, F_2}x_i^{-1}$ . Since  $\Gamma_{F_1, F_2}$  and hence  $x_i\Gamma_{F_1, F_2}x_i^{-1}$  are Zariski dense and hence have property (P), and  $K$  is compact, there exists a neighborhood  $V$  of  $e$  such that  $V \cap x_i\Gamma_{F_1, F_2}x_i^{-1} = \{e\}$  for each  $x_i$  by Theorem 2.1.8. Theorem 2.1.7 implies that the limiting subgroup  $\Gamma_{E_1, E_2}$  of the sequence  $x_i\Gamma_{F_1, F_2}x_i^{-1}$  is discrete. ■

**3.2.6.** We will need the following lemma for the proof of Proposition 3.2.7.

**LEMMA** (e.g., [10, Lemma 1.6.1]). *There is an  $\epsilon$ -neighborhood  $V$  of the identity  $e$  in  $G$  such that for any  $x, y \in V$ , we have  $\rho_G(e, xyx^{-1}y^{-1}) < \frac{1}{2} \min(\rho_G(e, x), \rho_G(e, y))$ .*

**3.2.7. PROPOSITION.** *Suppose that  $\Gamma_{F_1, F_2}$  is discrete. Then for a sequence  $x_i \in H$ , the set  $M_1 = \{\text{Ad}_1(x_i)\Delta_{F_1} \in H \cdot \Delta_{F_1} \mid i \geq 1\}$  is relatively compact if and only if the set  $M_2 = \{\text{Ad}_2(x_i)\Delta_{F_2} \in H \cdot \Delta_{F_2} \mid i \geq 1\}$  is relatively compact.*

*Proof.* Suppose that  $M_1$  is relatively compact and  $M_2$  is not. Choose a positive  $\epsilon$  and  $V$  as in Lemma 3.2.6. Since the set  $M_1$  is relatively compact, by Lemma 3.1.3, there exists a  $d > 0$  such that for any lattice  $\text{Ad}_1(x_i)F_1$  in  $M_1$ , there exists a basis  $\mathfrak{B}_i$  such that  $\rho_G(e, \exp x) < d$  for all  $x \in \mathfrak{B}_i$ . Let  $\mathfrak{B}$  be the union of all  $\mathfrak{B}_i$ 's. It is not difficult to see that we can find  $g \in C(H)$  such that  $\rho_G(e, gug^{-1}) < \epsilon$  for all  $u \in U_1(\mathbb{R})$  such that  $\rho_G(e, u) < d$ . By Mahler's criterion of compactness, there exists  $\epsilon_0 > 0$  such that  $\epsilon_0 < \epsilon$  and  $\text{Ad}_1(g)M_1$  does not contain any non-trivial element  $x$  such that  $\rho_G(e, \exp x) < \epsilon_0$ . Since  $\text{Ad}_2(g)M_2$  is not relatively compact and the determinants of the lattices in  $\text{Ad}_2(g)M_2$  are equal, there exists some  $j \in \mathbb{N}$  such that  $\text{Ad}_2(g)\text{Ad}_2(x_j)\Delta_{F_2}$  contains a non-trivial element  $y$  with  $\rho_G(e, \exp y) \leq \epsilon_0$ . Set  $n = x_{j-}$ . Set  $\tilde{U} = \{x \in U_1 \mid \ln x \in \mathfrak{B} \cap \text{Ad}_2(g)\text{Ad}_2(n)\Delta_{F_1}\}$ . With  $U_{-1} = g\tilde{U}g^{-1}$ ,  $U_0 = \{\exp y\}$  and  $U_i = \{sts^{-1}t^{-1} \mid$

$s \in U_i, t \in U_{i-1}$ ,  $\sup_{u \in U_i} \rho_G(e, u) = d_i$  tends to 0 as  $i$  goes to  $\infty$ , as  $U_{-1}$  and  $U_0$  lie in  $V$ . On the other hand, both  $U_0$  and  $U_{-1}$  belong to the discrete subgroup  $gn\Gamma_{F_1, F_2}n^{-1}g^{-1}$  and hence  $U_i \subset gn\Gamma_{F_1, F_2}n^{-1}g^{-1}$  for all  $i \geq 0$ . Thus there exists  $k > 0$  such that  $U_k$  is not trivial and  $U_{k+1}$  is trivial. Let  $z$  be a nontrivial element of  $U_k$ . Then  $\rho_G(e, z) \leq \epsilon_0$  and  $z$  belongs to  $C(U_{-1})$ . Since the centralizer of any subset is algebraic, the elements of  $gnU_1n^{-1}g^{-1}$  commute with  $z$  since  $U_1$  is the Zariski closure of  $\exp(\mathfrak{B})$ . Therefore,  $n^{-1}g^{-1}zgn \in C(U_1) \cap \Gamma_{F_1, F_2}$ . Since  $C(U_1) \subset U_1$  (see the proof of Lemma 1.5.4),  $C(U_1) \cap \Gamma_{F_1, F_2} = F_1$ . But  $\rho_G(e, z) < \epsilon_0$ , contradicting that the  $\epsilon_0$ -neighborhood intersects  $\text{Ad}(g)M_1$  trivially. This proves the proposition. ■

In fact, we have proved that if the sequence  $\text{Ad}_1(x_i)\Delta_{F_1}$  converges, then we can find a convergent subsequence of  $\text{Ad}_2(x_i)\Delta_{F_2}$ . The following is an immediate corollary of Propositions 3.2.5 and 3.2.7.

**3.2.8. PROPOSITION.** *If  $\Gamma_{F_1, F_2}$  is discrete, then for any lattice  $\Delta_{E_1}$  in  $\overline{H \cdot \Delta_{F_1}}$ , there exists a lattice  $E_2$  in  $U_2(\mathbb{R})$  such that  $(\Delta_{E_1}, \Delta_{E_2})$  lies in  $\delta(H) \cdot (\Delta_{F_1}, \Delta_{F_2})$  and the subgroup  $\Gamma_{E_1, E_2}$  is discrete.*

### 3.3. Ratner's Theorem and Orbit Closures

**3.3.1.** Let  $G$  be a connected Lie group and  $\Gamma$  a lattice in  $G$ . Consider the natural action of  $G$  by left translation on the homogeneous space  $G/\Gamma$ . For  $x \in G/\Gamma$ ,  $G_x = \{g \in G \mid g.x = x\}$  is the stabilizer of  $x$ . M. Ratner proved the following theorem which was known as Raghunathan's conjecture.

**THEOREM (Ratner [19]).** *Let  $H$  be a connected closed subgroup of  $G$  generated by unipotent one-parameter subgroups in it. Then for  $x \in G/\Gamma$ , there exists a connected and closed subgroup  $L$  of  $G$  containing  $H$  such that  $\overline{H.x}$  coincides with  $L.x$  and  $L \cap G_x$  is a lattice in  $L$ .*

**3.3.2. THEOREM (see [16, Theorem 1.13]).** *Let  $G$  and  $\Gamma$  be as above and  $H$  be a closed subgroup of  $G$ . Then if  $H \cap G_x$  is a lattice for some  $x \in G/\Gamma$ , then  $H.x$  is closed.*

**3.3.3. PROPOSITION [20, Proposition 3.2].** *Let  $\mathbf{G} \subset SL_n(\mathbb{C})$  be an algebraic  $\mathbb{Q}$ -group,  $G = \mathbf{G}_{\mathbb{R}}^0$ ,  $\Gamma = G \cap SL_n(\mathbb{Z})$ , and  $H$  be a subgroup generated by algebraic unipotent one-parameter subgroups of  $G$  contained in  $H$ . Suppose that  $\overline{H \cdot \Gamma} = M \cdot \Gamma$  for a connected Lie subgroup  $M$  of  $G$ . Let  $\mathbf{M}$  be the smallest algebraic  $\mathbb{Q}$ -subgroup containing  $M$ . Then the radical of  $\mathbf{M}$  is a unipotent algebraic  $\mathbb{Q}$ -subgroup and  $M = \mathbf{M}_{\mathbb{R}}^0$ . Furthermore, if  $\mathbf{M} = \mathbf{S}\mathbf{U}$ ,  $\mathbf{U} = R_u(\mathbf{M})$  is an  $\mathbb{R}$ -Levi-decomposition of  $\mathbf{M}$ , then  $M = \mathbf{S}(\mathbb{R})^0 \mathbf{U}(\mathbb{R})^0$ .*

In particular, we note that the Levi component of  $\mathbf{M}$  is semisimple in the above proposition since the radical of  $\mathbf{M}$  is unipotent.

**3.3.4. PROPOSITION.** *With the same notation as the above proposition, there exists an element  $u \in \mathbf{U}(\mathbb{R})$  such that the orbit  $S(\mathbb{R})^0.(u\Gamma)$  is closed.*

*Proof.* Since  $\mathbf{M}$  is defined over  $\mathbb{Q}$ , there exists  $u \in \mathbf{U}(\mathbb{R})$  such that  $u\mathbf{S}u^{-1}$  is defined over  $\mathbb{Q}$ . It follows that  $u^{-1}(\mathbf{S})\mathbb{R}^0u.\Gamma$  is closed. Therefore its left-translation by  $u$ ,  $\mathbf{S}(\mathbb{R})^0.(u\Gamma)$ , is closed. ■

**3.4. Closedness of Some Orbits and  $\mathbb{Q}$ -Structures**

**3.4.1.** We keep the notation for  $\mathbf{G}$ ,  $\mathbf{L}$ ,  $G$ , and  $H$  from Section 3.2.1. In addition, we assume that the  $\mathbb{R}$ -rank of  $\mathbf{G}$  is at least 2 and  $H$  is non-compact. We fix a non-trivial semisimple normal  $\mathbb{R}$ -subgroup  $\mathbf{H}'$  of  $\mathbf{H}$  without any  $\mathbb{R}$ -anisotropic factors and set  $H' = \mathbf{H}'(\mathbb{R})^0$ . Then  $H'$  is generated by unipotent one-parameter subgroups by Lemma 2.3.1.

In this section, we fix lattices  $F_1$  and  $F_2$  in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively. Let  $\Omega_i$  denote the space of lattices in  $\mathcal{Z}_i(\mathbb{R})$  of the same determinant as  $\Delta_{F_i}$  for each  $i = 1, 2$ . Since  $SL(\mathcal{Z}_i(\mathbb{R}))$  acts transitively on  $\Omega_i$ ,  $\Omega_i$  can be identified with the homogeneous space  $SL_l(\mathbb{R})/SL_l(\mathbb{Z})$ , through the isomorphism of  $\mathcal{Z}_i$  with the Euclidean space  $\mathbb{R}^l$  where  $l = \dim(\mathcal{Z}_i(\mathbb{R}))$ .

**3.4.2. PROPOSITION.** *The orbit  $H'.\Delta_{F_i}$  (resp.  $\delta(H').(\Delta_{F_1}, \Delta_{F_2})$ ) is closed if and only if  $\Lambda_{F_i} \cap H'$  (resp.  $\Lambda_{F_1, F_2} \cap H'$ ) is a Zariski dense arithmetic subgroup of  $H$ .*

*Proof.* Suppose that  $H'.\Delta_{F_i}$  is closed. Choose a representative of  $\Delta_{F_i}$  in  $SL_l(\mathbb{R})/SL_l(\mathbb{Z})$ , say  $g_iSL_l(\mathbb{Z})$  so that  $\text{Ad}_i(\Lambda_{F_i} \cap H') \subset g_iSL_l(\mathbb{Z})g_i^{-1}$ . By Ratner's theorem,  $\Lambda_{F_i}$  is a lattice in  $H'$ . Since  $H'$  has no compact factors by the assumption, it follows from Borel density theorem that  $g_i^{-1}\text{Ad}_i(\Lambda_{F_i} \cap H')g_i \subset SL(\mathbb{Q})$  is Zariski dense in  $g_i^{-1}\text{Ad}_i(\mathbf{H}')g_i$ . Therefore by Proposition 1.2.7,  $g_i^{-1}\text{Ad}_i(\mathbf{H}')g_i$  is a  $\mathbb{Q}$ -subgroup of  $SL_n(\mathbb{C})$ , providing a  $\mathbb{Q}$ -form of  $H'$  such that  $H'(\mathbb{Z})$  is commensurable to  $\Lambda_{F_i} \cap H'$ . If  $\delta(H').(\Delta_{F_1}, \Delta_{F_2})$  is closed and  $\Delta_{F_i} = g_iSL_l(\mathbb{Z})$  for  $i = 1, 2$ , by the same argument as above,  $(g_1^{-1}, g_2^{-1})\{(\text{Ad}_1(h), \text{Ad}_2(h)) \mid h \in H'\}(g_1, g_2)$  is a  $\mathbb{Q}$ -subgroup of  $SL_l(\mathbb{C}) \times SL_l(\mathbb{C})$ , providing a  $\mathbb{Q}$ -form of  $\delta(H')$  and hence of  $H'$  such that  $H'(\mathbb{Z})$  is commensurable to  $(\Lambda_{F_1, F_2} \cap H')$ . The converse is a direct consequence of Theorem 2.1.2 and Theorem 3.3.2. ■

**3.4.3. COROLLARY.** *Let  $F_i$  and  $E_i$  be commensurable lattices in  $U_i(\mathbb{R})$ . Then  $H'.\Delta_{F_i}$  is closed if and only if  $H'.\Delta_{E_i}$  is closed.*

*Proof.* If  $F_i$  and  $E_i$  are commensurable, then  $\Lambda_{F_i}$  and  $\Lambda_{E_i}$  are commensurable. Therefore  $\Lambda_{F_i} \cap H'$  is a lattice in  $H'$  if and only if  $\Lambda_{E_i} \cap H'$  is a lattice in  $H'$ . The corollary now follows from Theorem 3.4.2. ■

**3.4.4. PROPOSITION.** *Assume that for any infinite proper normal  $\mathbb{R}$ -subgroup  $G'$  of  $G$ ,  $F_i \cap G'(\mathbb{R})$  is finite and that  $\Gamma_{F_1, F_2}$  is discrete. Then  $\delta(H').(\Delta_{F_1}, \Delta_{F_2})$  is closed for some non-trivial semisimple normal  $\mathbb{R}$ -subgroup*

$H'$  of  $H$  without any compact factors if and only if  $\Gamma_{F_1, F_2}$  is an arithmetic subgroup (also, irreducible lattice) of  $G$ .

*Proof.* Suppose that  $\delta(H').(F_1, F_2)$  is closed. Then the stabilizer  $\Lambda_{F_1, F_2} \cap H'$  is a Zariski dense arithmetic subgroup in  $H'$  by Proposition 3.4.2. Let  $N$  be the normalizer of  $\Gamma_{F_1, F_2} \cap H'$ . Then  $N$  contains  $\Lambda_{F_1, F_2} \cap H'$  by Lemma 3.2.3. By Corollary 2.4.5, there exists a  $\mathbb{Q}$ -form of  $G$  such that  $\Gamma_{F_1, F_2}$  is commensurable to  $G(\mathbb{Z})$ . Since  $G$  is  $\mathbb{Q}$ -simple by the assumption on  $F_i$  (Lemma 2.4.4), it follows that  $\Gamma_{F_1, F_2}$  is an irreducible lattice.

Suppose that  $\Gamma_{F_1, F_2}$  is an arithmetic subgroup of  $G$ , that is, there exists a  $\mathbb{Q}$ -form of  $G$  such that  $\Gamma_{F_1, F_2}$  is commensurable to  $G(\mathbb{Z})$ . It follows that  $U_1$  and  $U_2$  are defined over  $\mathbb{Q}$  and  $F_i$  is commensurable to  $U_i(\mathbb{Z})$  for each  $i = 1, 2$ . Therefore  $H'$  is also defined over  $\mathbb{Q}$  and  $\Lambda_{F_1, F_2} \cap H'$  is commensurable to  $H'(\mathbb{Z})$ . By Theorem 3.4.2,  $\delta(H').(F_1, F_2)$  is closed.  $\blacksquare$

**3.4.5. COROLLARY.** *Suppose that  $\mathbf{G}$  is a  $\mathbb{Q}$ -simple group with respect to which  $U_1$  and  $U_2$  are  $\mathbb{Q}$ -subgroups and that  $E_1 = U_1(\mathbb{Z})$  and  $E_2 = zU_2(\mathbb{Z})z^{-1}$  for some  $z \in C(H')$ . If  $\Gamma_{E_1, E_2}$  is discrete, the orbit  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  is closed.*

*Proof.* Since  $z \in C(H')$ ,  $\Lambda_{U_1(\mathbb{Z}), U_2(\mathbb{Z})} \cap H' = \Lambda_{U_1(\mathbb{Z}), zU_2(\mathbb{Z})z^{-1}} \cap H'$ . On the other hand,  $\Lambda_{U_1(\mathbb{Z}), U_2(\mathbb{Z})} \cap H'$  is commensurable to  $H'(\mathbb{Z})$ . By Proposition 3.4.2, the orbit  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  is closed.  $\blacksquare$

**3.4.6.** Since  $H'$  is a normal subgroup of finite index in  $\mathbf{H}'(\mathbb{R})$ , the following is another corollary of Theorem 3.3.2 and Proposition 3.4.2.

**LEMMA.** (1)  $\mathbf{H}'(\mathbb{R}).\Delta_{F_i}$  is closed if and only if  $H'.\Delta_{F_i}$  is closed.

(2)  $\delta(\mathbf{H}')(\mathbb{R}).(\Delta_{F_1}, \Delta_{F_2})$  is closed if and only if  $\delta(H').(\Delta_{F_1}, \Delta_{F_2})$  is closed.

**3.4.7. PROPOSITION.** *Suppose that  $\Gamma_{F_1, F_2}$  is discrete and that  $H'.\Delta_{F_1}$  and  $H'.\Delta_{F_2}$  are closed. If  $\overline{\delta(\mathbf{H}'(\mathbb{R})).(\Delta_{F_1}, \Delta_{F_2})}$  contains a closed orbit  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  and  $\Gamma_{E_1, E_2}$  is an irreducible lattice in  $G$ , then  $\delta(H').(\Delta_{F_1}, \Delta_{F_2})$  is closed.*

*Proof.* Let  $\mathbf{H}'(\mathbb{R}) = \bigcup_{i=1, \dots, n} g_i H'$  and  $\delta(H').(\Delta_{E_1}, \Delta_{E_2}) \subset \overline{\delta(\mathbf{H}'(\mathbb{R})).(\Delta_{F_1}, \Delta_{F_2})}$  be a closed orbit. Then  $\delta(H').(\Delta_{E_1}, \Delta_{E_2}) \subset \delta(H').(\text{Ad}_1(g_k)\Delta_{F_1}, \text{Ad}_2(g_k)\Delta_{F_2})$  for some  $1 \leq k \leq n$ . Let  $\Delta'_{E_i} = \text{Ad}_i(g_k^{-1})\Delta_{E_i}$  for each  $i = 1, 2$ . Since  $H'$  is a normal subgroup of  $\mathbf{H}'(\mathbb{R})$ , we have that  $\delta(H').(\Delta'_{E_1}, \Delta'_{E_2}) \subset \overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$ . Since  $g_k^{-1}(\Gamma_{E_1, E_2})g_k = \Gamma_{E'_1, E'_2}$ , the subgroup  $\Gamma_{E'_1, E'_2}$  is an irreducible lattice in  $G$ .

Therefore we may replace  $\mathbf{H}'(\mathbb{R})$  by  $H'$  in the statement of the proposition. By Ratner's theorem, there exists a connected closed subgroup  $M'$  of  $SL(\mathcal{Z}_1(\mathbb{R})) \times SL(\mathcal{Z}_2(\mathbb{R}))$  containing  $\delta(\text{Ad}(H'))$  such that  $\overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$  is  $M'.(\Delta_{F_1}, \Delta_{F_2})$ . Since  $H'.\Delta_{F_1}$  and  $H'.\Delta_{F_2}$  are closed by the assumption,

$M' \subset H' \times H'$ . Assume that  $\delta(H').(\Delta_{F_1}, \Delta_{F_2})$  is not closed. We claim that  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  is open in  $M'.(\Delta_{E_1}, \Delta_{E_2})$ . It is enough to show that for a sequence  $(\text{Ad}_1(x_i)\Delta_{E_1}, \text{Ad}_2(y_i)\Delta_{E_2})$ ,  $i \geq 1$  in  $M'.(\Delta_{E_1}, \Delta_{E_2})$  converging to  $(\Delta_{E_1}, \Delta_{E_2})$ , there exists  $n \in \mathbb{N}$  such that  $(\text{Ad}(x_i)\Delta_{E_1}, \text{Ad}(y_i)\Delta_{E_2}) \in \delta(H').(\Delta_{E_1}, \Delta_{E_2})$  for all  $i > n$ . Denote by  $\Gamma_i$  the subgroup generated by  $x_i E_1 x_i^{-1}$  and  $y_i E_2 y_i^{-1}$ . Since  $M'.(\Delta_{E_1}, \Delta_{E_2})$  lies inside  $M'.(\Delta_{F_1}, \Delta_{F_2})$ , each  $\Gamma_i$  is discrete by Proposition 3.2.5 and the sequence of  $\Gamma_i$ 's converges to  $\Gamma_{E_1, E_2}$ . Therefore there exists a neighborhood  $V$  of the identity  $e$  such that  $V \cap \Gamma_i = \{e\}$  (see the proof of Proposition 3.2.5). It is well known that an irreducible lattice in a semisimple Lie group with real rank at least 2 is finitely presentable and locally rigid (for example, see [11]). Let  $h_1, h_2, \dots, h_s$  be generators of  $\Gamma_{E_1, E_2}$  with relations  $w_j(h_1, h_2, \dots, h_s) = e$ ,  $j = 1, 2, \dots, t$ . We may assume the generators lie in the set  $E_1 \cup E_2$ .

Denote by  $h_{ik} \in x_i E_1 x_i^{-1} \cup y_i E_2 y_i^{-1}$  the element in  $\Gamma_i$  such that  $x_i h_{ik} x_i^{-1} \rightarrow h_k$  if  $h_k \in E_1$ , and  $y_i h_{ik} y_i^{-1} \rightarrow h_k$  if  $h_k \in E_2$  as  $i \rightarrow \infty$ , for each  $i \geq 1$  and  $k = 1, 2, \dots, s$ . Since the number of generators is finite, we can find a sufficiently large  $n$  such that  $w_j(h_{i1}, h_{i2}, \dots, h_{is}) \in V \cap \Gamma_i = \{e\}$  for each  $j = 1, 2, \dots, t$ , and  $i > n$ . Thus we can define a homomorphism  $r_i : \Gamma_{E_1, E_2} \rightarrow \Gamma_i$  which carries  $h_k$  to  $h_{ik}$  for each  $k = 1, 2, \dots, s$  and for all  $i > n$ . Note that the sequence  $\{r_i \mid i > n\}$  tends to the identity map  $\text{id}$  of  $\Gamma_{E_1, E_2}$  in the limit. It follows from the local rigidity of  $\Gamma_{E_1, E_2}$  that there is  $n_0 > n$  such that  $r_i$  is an inner automorphism for each  $i > n_0$ . Now if  $r_i = \text{int}(g_i)$  for  $g_i \in G$ ,  $g_i \in N(U_1) \cap N(U_2) = \mathbf{L}$ , since  $g_i E_1 g_i^{-1} \subset U_1(\mathbb{R})$  and  $g_i E_2 g_i^{-1} \subset U_2(\mathbb{R})$ .

Therefore  $\text{Ad}(g_i) = \text{Ad}_1(x_i)$  and  $\text{Ad}(g_i) = \text{Ad}_2(y_i)$ , and hence  $(x_i, y_i) \in (H' \times H') \cap \delta(\mathbf{L}(\mathbb{R})) = \delta(H')$  for all  $i > n_0$ . It follows that  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  is open. On the other hand  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  is also closed by the assumption. Since both  $\delta(H')$  and  $M'$  are connected,  $\delta(H') = M'$ . This contradiction establishes the closedness of the orbit  $\delta(H').(\Delta_{F_1}, \Delta_{F_2})$ . ■

**3.4.8. PROPOSITION.** *There exists a lattice  $E_i$  in  $U_i(\mathbb{R})$  such that  $\Delta_{E_i} \in \text{SL}(\mathcal{Z}_i(\mathbb{R})).\Delta_{F_i}$  and the orbit  $H'.\Delta_{E_i}$  is closed.*

*Proof.* By Proposition 1.4.2, there exists a  $\mathbb{Q}$ -form of  $\mathbf{G}$  such that every parabolic  $\mathbb{R}$ -subgroup is defined over  $\mathbb{Q}$ . Therefore  $N(U_1)$  and  $N(U_2)$  are defined over  $\mathbb{Q}$ . We may assume that the determinant of  $\Delta_{U_i(\mathbb{Z})}$  is the same as  $\Delta_{F_i}$  by modifying the  $\mathbb{Q}$ -form by the inner automorphism by an element of  $\mathbf{L}$ . It is now enough to set  $E_i = U_i(\mathbb{Z})$  to conclude the proof since  $H' \cap H(\mathbb{Z})$  is a lattice in  $H'$ . ■

**3.4.9. LEMMA.** *Let  $H$  be an algebraic group,  $H = L \rtimes N$ ,  $\Lambda$  a Zariski dense arithmetic subgroup of  $L$ , and  $F$  a Zariski dense arithmetic subgroup of  $N$  which normalizes  $\Lambda$ . Then  $\Lambda \rtimes F$  is a Zariski dense arithmetic subgroup of  $H$ .*

*Proof.* Consider the  $\mathbb{Q}$ -forms of  $L$  and  $N$  such that  $\Lambda$  and  $F$  are commensurable to  $L(\mathbb{Z})$  and  $N(\mathbb{Z})$ , respectively. Since  $\Lambda$  normalizes  $F$ ,  $L(\mathbb{Q})$  normalizes  $N(\mathbb{Q})$  and hence they define a  $\mathbb{Q}$ -form of  $H$ . The lemma now follows from the well-known fact that  $L(\mathbb{Z})N(\mathbb{Z})$  is of finite index in  $H(\mathbb{Z})$ . ■

**3.4.10. PROPOSITION.** *Suppose that  $G$  is absolutely simple, that  $U_1$  is either commutative or Heisenberg and that  $H$  has no compact factors, i.e.,  $H = H'$ . Then the orbit  $H.\Delta_{F_i}$  is closed if and only if there exists a  $\mathbb{Q}$ -form of  $G$  with respect to which  $U_i$  and  $N(U_1) \cap N(U_2)$  are defined over  $\mathbb{Q}$  and  $F_i$  is commensurable to  $U_i(\mathbb{Z})$ .*

*Proof.* Suppose that the orbit  $H.\Delta_{F_i}$  is closed. By Proposition 3.4.2, there exists a  $\mathbb{Q}$ -form of  $H$  such that  $\Lambda_{F_i}$  is commensurable to  $H(\mathbb{Z})$ . Since  $F_i$  is a Zariski dense arithmetic subgroup of  $U_i$  and  $\Lambda_{F_i}$  normalizes  $F_i$ , we obtain a  $\mathbb{Q}$ -form on  $\mathbf{H}U_i$ . Since  $\mathbf{H}U_i = [N(U_i), N(U_i)]$ , it follows from Proposition 1.5.3 that there exists a  $\mathbb{Q}$ -form of  $G$  which we are looking for, proving the claim since in the proof we see  $N(U_1) \cap N(U_2)$  is defined over  $\mathbb{Q}$ .

Let  $G$  have a  $\mathbb{Q}$ -form with respect to which  $U_i$  and  $H$  are defined over  $\mathbb{Q}$ . If  $F_i$  is commensurable to  $U_i(\mathbb{Z})$ , then  $\Lambda_{F_i}$  is commensurable to  $H(\mathbb{Z})$ . It follows from Proposition 3.4.2 that  $H.\Delta_{F_i}$  is closed. ■

**3.4.11. LEMMA.** *Let  $G$  be a semisimple  $\mathbb{Q}$ -group such that  $U_1$  and  $N(U_1) \cap N(U_2)$  are defined over  $\mathbb{Q}$ . Then  $U_2$  is also defined over  $\mathbb{Q}$ .*

*Proof.* The map  $P' \rightarrow P' \cap P$  is a bijection between the set of opposite parabolic subgroups to a parabolic subgroup  $P$  and the Levi subgroups of  $P$  [5, 4.8]. It follows that an opposite parabolic subgroup  $P'$  to  $P$  is defined over  $\mathbb{Q}$  if  $P$  and  $P \cap P'$  are defined over  $\mathbb{Q}$ . Therefore  $N(U_2)$  is defined over  $\mathbb{Q}$ ; hence so is  $U_2$ . ■

**3.4.12.** We close this section by presenting specific lattices  $F_i$  such that the orbit  $H'.\Delta_{F_i}$  is closed, which will be also used in the proof of the main theorem.

Let  $\mathbf{G} = SL_{2(k+1)}(\mathbb{C})$  with the  $\mathbb{R}$ -form defined by  $G = SL_{k+1}(D_{\mathbb{R}})$ ,  $k \geq 2$ ,  $D_{\mathbb{R}}$  is a Hamiltonian quaternion algebra over  $\mathbb{R}$ ,

$$U_1(\mathbb{R}) = \begin{pmatrix} 1 & M_{1 \times k} D_{\mathbb{R}} \\ 0 & I_k \end{pmatrix},$$

$$U_2(\mathbb{R}) = \begin{pmatrix} 1 & 0 \\ M_{k \times 1} D_{\mathbb{R}} & I_k \end{pmatrix}$$

and

$$H = \begin{pmatrix} SL_1(D_{\mathbb{R}}) & \mathbf{0} \\ \mathbf{0} & SL_k(D_{\mathbb{R}}) \end{pmatrix}.$$

Then

$$H' = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & SL_k(D_{\mathbb{R}}) \end{pmatrix}.$$

Let  $F_1$  be a lattice in  $U_1(\mathbb{R})$ . We will show that if the orbit  $H'.\Delta_{F_1}$  is closed, then there exists a central simple division algebra  $D_1$  over  $\mathbb{Q}$  and  $D_1$  is isomorphic to  $D_{\mathbb{R}}$  over  $\mathbb{R}$  such that  $F_1 = (I_0^{\alpha M_1 \times k D_1(\mathbb{Z})})$  for some  $\alpha \in \mathbb{R}$  and up to conjugation by an element in  $H'$ .

That the orbit  $H'.\Delta_{F_1}$  is closed implies that there exists a  $\mathbb{Q}$ -form of  $\mathbf{H}'$  such that the representation  $\text{Ad}_1$  restricted to  $\mathbf{H}'$  to  $SL_{4k}(\mathbb{C})$  given by  $A \rightarrow (A, A)$  is  $\mathbb{Q}$ -rational. By Proposition 1.4.10, the  $\mathbb{Q}$ -form of  $\mathbf{H}'$  is up to conjugation such that  $\mathbf{H}'(\mathbb{Q}) = SL_k(D_1)(\mathbb{Q})$  for some  $D_1$  described above. Without loss of generality, we may assume that  $\mathbf{H}'(\mathbb{Q}) = SL_k(D_1)(\mathbb{Q})$ . This  $\mathbb{Q}$ -form of  $\mathbf{H}'$  obviously extends to a  $\mathbb{Q}$ -form of  $\mathbf{H}$  such that  $\mathbf{H}(\mathbb{Q}) = SL_1(D_1)(\mathbb{Q}) \times SL_k(D_1)(\mathbb{Q})$ . Now set  $E_1 = (I_0^{M_1 \times k D_1(\mathbb{Z})})$ . Then  $H(\mathbb{Z})$  preserves  $E_1$  by conjugation. Now the claim that  $E_1 = F_1$  up to a constant multiple and commensurability follows from the fact that any two irreducible  $\mathbb{Q}$ -rational representations which are isomorphic over  $\mathbb{C}$  are isomorphic over  $\mathbb{Q}$ .

#### 4. THE PROOF OF THE MAIN THEOREM

For the entire Section 4, we assume that  $\mathbf{G}$  is an adjoint absolutely simple algebraic  $\mathbb{R}$ -group with real rank at least 2 and  $U_1, U_2$  is a pair of opposite horospherical  $\mathbb{R}$ -subgroups. We recall the notation  $G = \mathbf{G}(\mathbb{R})^0$ ,  $\mathbf{L} = N(U_1) \cap N(U_2)$ ,  $\mathbf{H} = [\mathbf{L}, \mathbf{L}]$ , and  $H = \mathbf{H}(\mathbb{R})^0$  from Subsection 3.2.1 and the terminology Property (A) from Subsection 1.1.16.

##### 4.1. Commutative Horospherical Subgroup Cases

We are now ready to give the proof of the main theorem for commutative horospherical subgroup cases. A main feature in this case is the fact that  $\text{Ad}_i(H)$  is a maximal connected closed subgroup of  $SL(\mathcal{Z}_i(\mathbb{R}))$ ; hence by Ratner's theorem  $H.\Delta_{F_i}$  is either closed or dense in  $\Omega_i$ . Moreover, unless  $G$  is of type  $A_n$ ,  $\delta(H)$  is a maximal connected closed subgroup of  $\text{Ad}_1(H) \times \text{Ad}_2(H)$ ; hence  $\delta(H).(\Delta_{F_1}, \Delta_{F_2})$  is either closed or dense in

$H.\Delta_{F_1} \times H.\Delta_{F_2}$ . This feature facilitates finding a closed  $\delta(H)$ -orbit lying in  $\overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$ , which is enough to prove the main theorem by Propositions 3.4.4 and 3.4.7.

4.1.1. *Remark.* The case when  $H$  does have compact factors occurs only when the  $\mathbb{R}$ -type of  $G$  is  ${}^1A_n$ ,  $n = 2k + 1$ , and  $U_1$  is conjugate to the minimal parabolic subgroup  $V_\Theta$  where  $\Theta$  is either  $\Delta - \alpha_1$  or  $\Delta - \alpha_n$ .

4.1.2. PROPOSITION. *Let  $U_1, U_2$  be a pair of commutative horospherical  $\mathbb{R}$ -subgroups of  $\mathbf{G}$  and  $F_1$  and  $F_2$  be lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$  respectively. Let  $\mathbf{H}'$  be the maximal semisimple normal  $\mathbb{R}$ -subgroup of  $\mathbf{H}$  without any  $\mathbb{R}$ -anisotropic factors and  $H' = \mathbf{H}'(\mathbb{R})^0$ . Then there exists a lattice  $E_i$  in  $U_i(\mathbb{R})$  such that  $H'.\Delta_{E_i} \in \overline{H'.\Delta_{F_i}}$  and  $H'.\Delta_{E_i}$  is closed for each  $i = 1, 2$ .*

*Proof.* Suppose that  $H' = H$ . Since  $\text{Ad}_i(H)$  is a maximal connected closed subgroup of  $SL(\mathcal{Z}_i(\mathbb{R}))$  (see Proposition 1.3.6) it follows from Ratner's theorem that  $\overline{H.\Delta_{F_1}}$  is either  $H.\Delta_{F_1}$  or  $SL(\mathcal{Z}_i(\mathbb{R})).\Delta_{F_1}$ . It then follows from Proposition 3.4.8 that  $\overline{H.\Delta_{F_1}}$  contains a closed orbit  $H.\Delta_{E_i}$  for some lattice  $E_i$  in  $U_i(\mathbb{R})$ .

According to the previous remark, it remains to consider when the  $\mathbb{R}$ -form of  $\mathbf{G}$  is of type  ${}^1A_{2k+1}$ , and  $H$  and  $H'$  are of types  ${}^1A_1 \times {}^1A_{2k-1}$  and  ${}^1A_{2k-1}$  respectively. Without loss of generality, we may realize  $\mathbf{G}$  as follows:  $\mathbf{G} = SL_{k+1}(D_{\mathbb{R}})$ ,  $D_{\mathbb{R}}$  is a Hamiltonian quaternion division algebra over  $\mathbb{R}$ ,  $\mathbf{H} = SL_1(D_{\mathbb{R}}) \times SL_k(D_{\mathbb{R}})$ , and  $\mathbf{H}' = SL_k(D_{\mathbb{R}})$ . By Theorem 3.3.1, there exists a connected closed subgroup  $M'$  of  $SL(\mathcal{Z}_1(\mathbb{R}))$  containing  $\text{Ad}_1(H')$  such that  $\overline{H'.\Delta_{F_1}} = M'.\Delta_{F_1}$ . Denote by  $S'$  a Levi subgroup of  $M'$  which contains  $\text{Ad}_1(H')$ . By Proposition 3.3.3,  $S'$  is a semisimple subgroup, and for some  $u \in R_u(M')$ ,  $S'.(u\Delta_{F_1})$  is closed. We claim that  $H'.(u\Delta_{F_1})$  is closed. Note that  $\text{Ad}_1(\mathbf{H}') = I_2 \otimes SL_{2k}(\mathbb{C})$  and since  $\text{Ad}_1$  is rational over  $\mathbb{R}$ ,  $\text{Ad}_1(H') \subset xSL_{4k}(\mathbb{R})x^{-1}$  where  $x$  is a basis transformation of  $\mathcal{Z}_1(\mathbb{R}) = D_{\mathbb{R}}^k$  to  $\mathbb{R}^{4k}$ . Let  $g \in xSL_{4k}(\mathbb{R})x^{-1}$  be a representative of the lattice  $u\Delta_{F_1}$  in the space  $xSL_{4k}(\mathbb{R})x^{-1}/xSL_{4k}(\mathbb{Z})x^{-1}$ . We consider the  $\mathbb{Q}$ -form of  $SL(\mathcal{Z}_1) = SL_{4k}(\mathbb{C})$  given by this choice of this lattice  $u\Delta_{F_1}$  in  $\mathcal{Z}_1$ . Denote by  $\mathbf{M}'$  the smallest  $\mathbb{Q}$ -subgroup of  $SL_{4k}(\mathbb{C})$  containing  $M'$  and by  $\mathbf{S}'$  the smallest  $\mathbb{Q}$ -subgroup containing  $S'$ . Then  $M' = \mathbf{M}'(\mathbb{R})^0$  and  $S' = \mathbf{S}'(\mathbb{R})^0$ . Denote by  $\rho_1$  the block diagonal embedding of  $S_1 = SL_{2k}(\mathbb{C}) \times SL_{2k}(\mathbb{C})$  into  $SL_{4k}(\mathbb{C})$ . Also denote by  $\rho_2$  the tensor product representation of  $S_2 = SL_2(\mathbb{C}) \times SL_{2k}(\mathbb{C})$ . By Proposition 1.3.9, we have that  $\mathbf{S}'$  is either  $\rho_1(S_1)$  or  $\rho_2(S_2)$ . That  $\mathbf{S}' = \rho_i(S_i)$  would imply that there exists a  $\mathbb{Q}$ -form of  $S_i$ , compatible with the  $\mathbb{R}$ -form of  $S_i$  (i.e.,  $S_i(\mathbb{R})^0 = S'$ ) such that  $\rho_i$  becomes  $\mathbb{Q}$ -rational in such a way that  $\rho_i(S_i(\mathbb{Q})) \subset gSL_{4k}(\mathbb{Q})g^{-1}$ . We first observe that this condition on compatibility with the  $\mathbb{R}$ -form cannot be satisfied by any  $\mathbb{Q}$ -form of  $S_1$ . By Proposition 1.4.9, the  $\mathbb{Q}$ -forms of  $S_1$  with respect to which  $\rho_1$  is rational over  $\mathbb{Q}$  are such that  $S_1(\mathbb{Q})$  is

conjugate to the subgroup  $SL_{2k}(\mathbb{Q}) \times SL_{2k}(\mathbb{Q})$  or to the subgroup  $R_{k/\mathbb{Q}}SL_n(k)$  where  $k$  is a quadratic extension of  $\mathbb{Q}$ . Since  $\text{Ad}_1(\mathbf{H}')$  must be an  $\mathbb{R}$ -subgroup of  $\mathbf{S}'$ , it follows that  $\text{Ad}_1(H') = \{(A, A) \mid A \in SL_k(D_{\mathbb{R}})\}$  must be conjugate to  $\{(A, A) \mid A \in SL_{2k}(\mathbb{R})\}$  or  $R_{\mathbb{C}/\mathbb{R}}SL_{2k}(\mathbb{R})$  by an element in  $S_1$ , which cannot be true.

Now, for the  $\mathbb{Q}$ -forms of  $S_2$ , which makes  $\rho_2$  be  $\mathbb{Q}$ -rational, it follows from Proposition 1.4.10 that  $\rho_2^{-1}(\text{Ad}_1(H'))$  is also defined over  $\mathbb{Q}$ . It follows also that  $H'.u\Delta_{F_1}$  is closed. ■

**4.1.3. LEMMA.** *Let  $U_1, U_2, F_1, F_2, \mathbf{H}'$ , and  $H'$  be the same as in Proposition 4.1.2. Assume that  $\Gamma_{F_1, F_2}$  is discrete. Then there exists  $(\Delta_{E_1}, \Delta_{E_2})$  inside  $\overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$  such that  $H'.\Delta_{E_1}$  and  $H'.\Delta_{E_2}$  are closed.*

*Proof.* By the previous proposition,  $\overline{H'.\Delta_{F_1}}$  contains a closed orbit  $H'.\Delta_{E_1}$  for some lattice  $E_1$  in  $U_1(\mathbb{R})$ . By Proposition 3.2.8, there exists a lattice  $E_2$  in  $U_2(\mathbb{R})$  such that  $(\Delta_{E_1}, \Delta_{E_2}) \in \overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$ . By the remark preceding Proposition 3.2.8, in fact  $(\Delta_{E_1}, \Delta_{E_2}) \in \overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$ . If  $H'.\Delta_{E_2}$  is not closed, then let  $E'_2$  be a lattice in  $U_2(\mathbb{R})$  such that  $\Delta_{E'_2} \in \overline{H'.\Delta_{E_2}}$  and  $H'.\Delta_{E'_2}$  is closed. Again we get a lattice  $E'_1$  in  $U_1(\mathbb{R})$  such that  $\Delta_{E'_1} \in \overline{H'.\Delta_{E_1}}$  such that  $(\Delta_{E'_1}, \Delta_{E'_2}) \in \overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$ . Since  $H'.\Delta_{E_1}$  is closed, so is  $H'.\Delta_{E'_1}$ , proving the lemma. ■

**4.1.4. THEOREM.** *Let  $U_1, U_2$  be a pair of commutative horospherical  $\mathbb{R}$ -subgroups of  $\mathbf{G}$ . Assume that if  $\text{rank}_{\mathbb{R}}(\mathbf{G}) = 2$ , then  $\mathbf{G}$  is not of type  $E_6$ . Then the triple  $(\mathbf{G}, U_1, U_2)$  has property (A).*

*Proof.* Let  $F_1, F_2$  be arbitrary lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$  respectively. To prove that  $\Gamma_{F_1, F_2}$  is arithmetic subgroup, it is enough to find a closed orbit  $\overline{\delta(H').(\Delta_{E_1}, \Delta_{E_2})}$  lying in  $\overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$  by Propositions 3.4.4 and 3.4.7. By the previous lemma, there exists  $(\Delta_{E_1}, \Delta_{E_2}) \in \overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$  such that  $H'.\Delta_{E_1}$  and  $H'.\Delta_{E_2}$  are closed.

*Case 1:  $U_1$  Is Reflexive.* In this case, we always have that  $H' = H$ . Since  $H.\Delta_{E_1}$  is closed, by Proposition 3.4.10 there exists a  $\mathbb{Q}$ -form of  $G$  with respect to which  $U_1$  and  $H$  are defined over  $\mathbb{Q}$  and  $E_1$  is commensurable to  $U_1(\mathbb{Z})$ . Since  $U_2$  is also defined over  $\mathbb{Q}$  by Lemma 3.4.11, there exists  $w \in G(\mathbb{Q})$  such that  $wU_1(\mathbb{Q})w^{-1} = U_2(\mathbb{Q})$  by Lemma 1.2.2. By Lemma 2.3.3, we may assume that  $E_2 = x(wE_1w^{-1})x^{-1}$  for some  $x \in (\mathbf{L})(\mathbb{R})$ . Write  $x = yz$  for  $y \in H(\mathbb{R})$  and  $z \in Z(L)(\mathbb{R})$ . Suppose that  $\overline{\delta(H).(\Delta_{E_1}, \Delta_{E_2})}$  is not closed, and denote by  $M$  a connected closed subgroup of  $\text{Ad}_1(H) \times \text{Ad}_2(H)$  such that  $\overline{\delta(H)} \subset M$  and  $\overline{\delta(H).(\Delta_{E_1}, \Delta_{E_2})} = M.(\Delta_{E_1}, \Delta_{E_2})$  (Theorem 3.3.1).

*Subcase (i):*  $M = \text{Ad}_1(H) \times \text{Ad}_2(H)$ . In that case, it is clear that  $\delta(\mathbf{H}(\mathbb{R})) \cdot (\Delta_{E_1}, \Delta_{E_2})$  is  $\mathbf{H}(\mathbb{R}) \cdot \Delta_{E_1} \times \mathbf{H}(\mathbb{R}) \cdot \Delta_{E_2}$ . Therefore we have found a closed orbit  $\delta(H) \cdot (\Delta_{U_1(\mathbb{Z})}, \Delta_{zU_2(\mathbb{Z})z^{-1}})$  lying inside  $\overline{\delta(\mathbf{H}(\mathbb{R})) \cdot (\Delta_{E_1}, \Delta_{E_2})}$  up to commensurability. It follows from Proposition 3.4.5 that  $\delta(H) \cdot (\Delta_{E_1}, \Delta_{E_2})$  is closed

*Subcase (ii):*  $M$  Is a Proper Subgroup of  $\text{Ad}_1(H) \times \text{Ad}_2(H)$ . If the absolute type of  $\mathbf{G}$  is not  $A_n$ , then  $H$  is a simple group and hence  $\delta(H)$  is a maximal closed connected subgroup of  $\text{Ad}_1(H) \times \text{Ad}_2(H)$ . Therefore the case when  $M$  is a proper subgroup of  $\text{Ad}_1(H) \times \text{Ad}_2(H)$  can happen only when  $\mathbf{G}$  has type  $A_n$ .

Let us now suppose that  $\mathbf{G}$  has type  $A_n$ . From Fig. 1.3.2.,  $\mathbf{H}$  is the product of two absolutely simple  $\mathbb{R}$ -groups, say,  $\mathbf{N}_1 \times \mathbf{N}_2$ . Also  $\text{Ad}_1(\mathbf{H}) = \mathbf{N}_1 \otimes \mathbf{N}_2$  and  $\text{Ad}_2(\mathbf{H}) = \mathbf{N}_2 \otimes \mathbf{N}_1$ . There are only two proper connected closed subgroups of  $\text{Ad}_1(H) \times \text{Ad}_2(H)$  which properly contain  $\delta(H)$ :  $M_1 = \{(\text{Ad}_1(A, B), \text{Ad}_2(C, B)) \mid A, C \in \mathbf{N}_1(\mathbb{R})^0, B \in \mathbf{N}_2(\mathbb{R})^0\}$  and  $M_2 = \{(\text{Ad}_1(A, B), \text{Ad}_2(A, D)) \mid A \in \mathbf{N}_1(\mathbb{R})^0, B, D \in \mathbf{N}_2(\mathbb{R})^0\}$ . Assume that  $M = M_1$ . If  $y \in H(\mathbb{R})$  is  $(a_1, a_2)$  where  $a_i \in \mathbf{N}_i(\mathbb{R})$  for each  $i = 1, 2$ , then  $(\Delta_{U_1(\mathbb{Z})}, \text{Ad}_2(a_2)\text{Ad}_2(z)\Delta_{U_2(\mathbb{Z})}) \in M_1 \cdot (\Delta_{E_1}, \Delta_{E_2})$ . Recall that  $E_1 = U_1(\mathbb{Z})$  and denote by  $E_2''$  the lattice  $(a_2 z)U_2(\mathbb{Z})(a_2 z)^{-1}$ . We claim that the  $M_2$ -orbit of  $(\Delta_{E_1}, \Delta_{E_2''})$  is closed. Observe that the subgroup  $\{(\text{Ad}_1(A, B), \text{Ad}_2(A, D)) \mid \text{Ad}_1(A, B)\Delta_{E_1} = \Delta_{E_1}, \text{Ad}_2(A, D)\Delta_{E_2''} = \Delta_{E_2''}\}$  is equal to  $\{(\text{Ad}_1(A, B), \text{Ad}_2(A, a_2 D a_2^{-1})) \mid (A, B) \in H(\mathbb{Z}), (A, D) \in H(\mathbb{Z})\}$ . Since this group, which is the intersection of the stabilizer of  $(\Delta_{E_1}, \Delta_{E_2''})$  and  $M_2$ , is a lattice in  $M_2$ , it follows that the  $M_2$ -orbit of  $(\Delta_{E_1}, \Delta_{E_2''})$  is closed. Since  $M_1 \cap M_2 = \delta(H)$  and  $M_1 \cdot (\Delta_{E_1}, \Delta_{E_2''})$  and  $M_2 \cdot (\Delta_{E_1}, \Delta_{E_2''})$  are closed,  $\delta(H) \cdot (\Delta_{E_1}, \Delta_{E_2''})$  is closed. It follows that  $\delta(H) \cdot (\Delta_{E_1}, \Delta_{E_2})$  is closed since  $\delta(H) \cdot (\Delta_{E_1}, \Delta_{E_2''}) \subset M \cdot (\Delta_{E_1}, \Delta_{E_2})$ . The case when  $M = M_2$  can be dealt with in the same way.

This establishes the proof of the theorem when  $U_1$  is not reflexive.

*Case 2:  $U_1$  Is Not Reflexive and  $H = H'$ .* First we observe that the only  $\mathbb{Q}$ -forms having a nonreflexive conjugacy class of a commutative horospherical  $\mathbb{Q}$ -subgroup are the following:  ${}^1A_n$  with  $\mathbb{Q}$ -rank  $\geq 2$ ,  $D_{2m+1}$  with  $\mathbb{Q}$ -rank  $2m + 1$ ,  ${}^1E_{6,2}^{28}$ , and  ${}^1E_{6,6}^0$  [21]. Since  $H \cdot \Delta_{E_1}$  and  $H \cdot \Delta_{E_2}$  are closed, there are  $\mathbb{Q}$ -forms  $\bar{G}$  and  $\tilde{G}$  of  $G$  such that  $E_1$  and  $E_2$  are commensurable to  $U_1 \cap \bar{G}(\mathbb{Z})$  and  $U_2 \cap \tilde{G}(\mathbb{Z})$  respectively. Suppose that the  $\mathbb{Q}$ -rank of  $\bar{G}$  is less than or equal to that of  $\tilde{G}$ .

*Subcase (i):  $\bar{G}$  and  $\tilde{G}$  Are Isomorphic over  $\mathbb{Q}$ .* Since  $U_1$  and  $U_2$  are defined over  $\mathbb{Q}$  with respect to both of the  $\mathbb{Q}$ -forms, we may assume that  $\tilde{G}(\mathbb{Q}) = x\bar{G}(\mathbb{Q})x^{-1}$  for some  $x \in N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$ . It follows that  $E_2$  is commensurable to  $x(U_2 \cap \bar{G}(\mathbb{Z}))x^{-1}$ . Let  $x = yz$  for  $z \in Z(N(U_1) \cap N(U_2))(\mathbb{R})$  and  $y \in \mathbf{H}(\mathbb{R})$ . Then  $(\Delta_{E_1}, \Delta_{yE_2y^{-1}}) \in H \cdot \Delta_{E_1} \times H \cdot \Delta_{E_2}$  while  $E_1$  and  $yE_2y^{-1}$  are commensurable to  $U_1 \cap \bar{G}(\mathbb{Z})$  and  $z(U_2 \cap \bar{G}(\mathbb{Z}))z^{-1}$  re-

spectively. By the same argument as in the previous case, we can show that  $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$  is closed.

*Subcase (ii):  $\bar{G}$  and  $\tilde{G}$  Are Not Isomorphic over  $\mathbb{Q}$ .* By the uniqueness of  $\mathbb{Q}$ -split forms of  $G$ ,  $\bar{G}$  is not split over  $\mathbb{Q}$ ; hence the  $\mathbb{Q}$ -form of  $\bar{G}$  has type either  ${}^1A_n$  or  ${}^1E_{6,2}^{28}$ .

There exist a maximal torus  $\bar{T}$  containing a maximal  $\mathbb{Q}$ -split torus  $\bar{S}$  and  $\bar{G}$  and a basis  $\Delta$  of  $\Phi(\bar{T}, G)$  such that  $U_1 = V_{\Delta-\alpha}$  and  $U_2 = V_{\Delta-\alpha}^-$  for some commutative root  $\alpha \in \Delta$ . Since  $U_1$  is a  $\mathbb{Q}$ -subgroup of  $\bar{G}$ , by Proposition 1.4.1,  $\alpha$  is fixed by the  $*$ -action (in the Tits index of  $\bar{G}$ ) of  $\text{Gal}(\mathbb{C}/\mathbb{Q})$  and  $\alpha \notin \Delta_0, \Delta_0$  the subset of  $\Delta$  whose elements vanish on  $\bar{S}$ . Set  $U'_1 = V_{\Delta-\{\alpha, i(\alpha)\}}$  and  $U'_2 = V_{\Delta-\{\alpha, i(\alpha)\}}^-$  where  $i$  is the opposition involution on  $\Delta$  (see 2.3.4). Since  $i$  commutes with the  $*$ -action of the Galois group,  $i(\alpha)$  is also fixed by the  $*$ -action and  $i(\alpha) \notin \Delta_0$ . It follows that  $U'_1$  and  $U'_2$  are  $\mathbb{Q}$ -subgroups of  $\tilde{G}$ . In particular,  $U'_1$  is a maximal horospherical  $\mathbb{Q}$ -subgroup of  $\bar{G}$  when  $\bar{G}$  is of type  ${}^1E_{6,2}^{28}$ .

Since  $U_1$  is a  $\mathbb{Q}$ -subgroup of  $G$ , it is enough to note the fact that  $i$  is invariant under an isomorphism between two Dynkin diagrams to see that  $U'_1$  and  $U'_2$  are also  $\mathbb{Q}$ -subgroups of  $\tilde{G}$ . Therefore  $E'_1 = \bar{G}(\mathbb{Z}) \cap U'_1(\mathbb{R})$  and  $E'_2 = \tilde{G}(\mathbb{Z}) \cap U'_2(\mathbb{R})$  are lattices in  $U'_1(\mathbb{R})$  and  $U'_2(\mathbb{R})$  respectively. Note that for each  $k = 1, 2$ ,  $E'_k \cap U_k(\mathbb{R}) = E_k$  up to commensurability.

Set  $i(U_1) = V_{\Delta-i(\alpha)}$  and  $i(U_2) = V_{\Delta-i(\alpha)}^-$ . Then  $U_k \cap i(U_k) = Z(U'_k)$  for each  $k = 1, 2$  (see the following Lemma 4.1.5). In particular, we have  $Z(U'_k) \subset U_k$ . Then  $Z(U'_k(\mathbb{R})) \cap E_k$  is a lattice in  $Z(U'_k(\mathbb{R}))$  for each  $k = 1, 2$  since  $Z(U'_k)$  is a  $\mathbb{Q}$ -subgroup with respect to both  $\mathbb{Q}$ -forms  $\bar{G}$  and  $\tilde{G}$ . Denote by  $\mathbf{G}_0$  the algebraic  $\mathbb{R}$ -subgroup of  $\mathbf{G}$  generated by  $Z(U'_1)$  and  $Z(U'_2)$ ,  $G_0 = \mathbf{G}_0(\mathbb{R})^0$ , and by  $\Gamma_0$  the subgroup generated by  $Z(U'_1(\mathbb{R})) \cap E_1$  and  $Z(U'_2(\mathbb{R})) \cap E_2$ . Note that  $\Gamma_0$  is discrete since  $\Gamma_0 \subset \Gamma_{E_1, E_2}$ .

By Proposition 2.5.4,  $\mathbf{G}_0$  is an absolutely simple algebraic  $\mathbb{R}$ -group and  $Z(U'_1)$  and  $Z(U'_2)$  are opposite commutative horospherical  $\mathbb{R}$ -subgroups of  $\mathbf{G}_0$ . Since  $i$  obviously preserves the set  $\{\alpha, i(\alpha)\}$ ,  $U'_1$  is reflexive and it follows that  $Z(U'_1)$  is also reflexive.

We now claim that the commutator subgroup, say  $S'$ , of  $N(Z(U'_1)) \cap N(Z(U'_2)) \cap \mathbf{G}_0$  has no  $\mathbb{R}$ -anisotropic factors. Recall from Proposition 2.5.4 that  $S'$  is a semisimple normal algebraic  $\mathbb{R}$ -subgroup of  $N(U'_1) \cap N(U'_2)$ . If the absolute type of  $\mathbf{G}$  is  $E_\beta$ , the  $\mathbb{R}$ -form of  $G$  is split by the assumption that the  $\mathbb{R}$ -form of  $\mathbf{G}$  is not  ${}^1E_{6,2}^{28}$ , and hence so are the  $\mathbb{R}$ -forms of  $\mathbf{G}_0$  and  $S'$ , proving the claim in that case. For reference the types of  $\mathbf{G}_0$  and  $S'$  are in this case  $D_5$  and  $D_4$  respectively. Now if the absolute type of  $\mathbf{G}$  is  $A_n$  and hence  $\mathbf{H}$  is  $H_1 \times H_2$ , where  $H_1$  and  $H_2$  are respectively of types  $A_k$  and  $A_{n-k-1}$ ,  $k > 1$  (from the assumption that  $\bar{G}$  is not  $\mathbb{Q}$ -split), then in that case  $G_0$  is of type  $A_m$  for some  $m < n$  and the type of  $S'$  is  $H_j \times H_j$

where  $j = 1$  if  $k < n - k - 1$ ; otherwise,  $j = 2$ . The claim follows from the assumption that  $\mathbf{H}$  has no  $\mathbb{R}$ -anisotropic factors.

It also follows that the  $\mathbb{R}$ -rank of  $\mathbf{G}_0$  is at least 2 since the  $\mathbb{R}$ -rank of  $S'$  is at least 2.

Therefore there exists a  $\mathbb{Q}$ -form of  $\mathbf{G}_0$  with respect to which  $\Gamma_0$  is commensurable to  $\mathbf{G}_0(\mathbb{Z})$  by the previous result (Case 1) on the case of reflexive horospherical subgroup cases. Note that  $\Gamma_0 \subset \Gamma_{E_1, E_2}$  and  $\Gamma_{E_1, E_2} \cap S'(\mathbb{R}) = S' \cap \mathbf{G}_0(\mathbb{Z})$  is a Zariski dense arithmetic subgroup of  $S'$  since  $S'(\mathbb{R})$  has no compact factors. Observe that  $\overline{G}(\mathbb{Q}) \cap S' = \tilde{G}(\mathbb{Q}) \cap S'$  since  $S' \cap G_0(\mathbb{Z}) \subset \Gamma_{E_1, E_2}$ , up to commensurability. In a group  $G$  of type  $A_n$ , an inner  $\mathbb{Q}$ -form of  $G$  is completely determined by a  $Q$ -form of  $S'$  up to  $\mathbb{Q}$ -isomorphism. In a group  $G$  of type  $E_6$ , it implies that both  $\overline{G}$  and  $\tilde{G}$  are of type  ${}^1E_{6,2}^{28}$  with the same semisimple anisotropic kernel determined by  $S' \cap G_0(\mathbb{Z})$ . In conclusion, there exists a  $\mathbb{Q}$ -isomorphism  $F: \overline{G} \rightarrow \tilde{G}$ . By the same argument as that in Subcase (i), we can prove that the orbit  $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$  is closed.

*Case 3:  $U_1$  Is Not Reflexive and  $H \neq H'$ .* Without loss of generality, we may realize  $\mathbf{G}$  as follows (see Section 3.4.12):  $\mathbf{G} = SL_{2(k+1)}(\mathbb{C})$ ,  $k \geq 2$ ,  $G = SL_{k+1}(D_{\mathbb{R}})$ ,  $D_{\mathbb{R}}$  is a Hamiltonian quaternion division algebra over  $\mathbb{R}$ ,  $\mathbf{H} = SL_1(D_{\mathbb{R}}) \times SL_k(D_{\mathbb{R}})$ , and  $\mathbf{H}' = SL_k(D_{\mathbb{R}})$ . The closedness of the orbit  $H'.\Delta_{E_i}$  implies that there exists a  $\mathbb{Q}$ -form of  $\mathbf{H}'$  such that the representation of  $\mathbf{H}'$  to  $SL_{4k}(\mathbb{C})$  defined by  $A \rightarrow (A, A)$  is  $\mathbb{Q}$ -rational. It follows that the  $\mathbb{Q}$ -form of  $\mathbf{H}'$  is up to conjugation such that  $\mathbf{H}'(\mathbb{Q}) = SL_k(D_i)(\mathbb{Q})$ , where  $D_i$  is a central simple division algebra over  $\mathbb{Q}$  and  $D_i$  is isomorphic to  $D_{\mathbb{R}}$  over  $\mathbb{R}$ . Assume that  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  is not closed. Since  $H'$  is simple, then it follows that the orbit is dense in  $H'.\Delta_{E_1} \times H'.\Delta_{E_2}$ . Therefore the pair  $(\Delta_{F_1}, \Delta_{F_2}) \in \overline{\delta(H').(\Delta_{E_1}, \Delta_{E_2})}$  where

$$E_1 = \begin{pmatrix} 1 & M_{1 \times k} D_1(\mathbb{Z}) \\ 0 & I_k \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 1 & 0 \\ \lambda M_{k \times 1} D_2(\mathbb{Z}) & I_k \end{pmatrix}.$$

Since the subgroup  $\Gamma_{E_1, E_2}$  is discrete, by Example 2.2.5, we have that up to commensurability  $D_1(\mathbb{Z}) = \lambda D_2(\mathbb{Z})$ . Denote  $D' = D_1$ . Therefore  $\Gamma_{E_1, E_2}$  is an arithmetic subgroup commensurable to  $SL_{k+1}(D)(\mathbb{Z})$ . By Proposition 3.4.4, this proves that  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  as well as that  $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$  is closed. ■

4.1.5. We now give a proof of the lemma used in the proof of the above theorem.

LEMMA.  $U_k \cap i(U_k) = Z(U'_k)$  for each  $k = 1, 2$ .

*Proof.* It suffices to prove that  $(\Phi^+ - [\Delta - \alpha]) \cap (\Phi^+ - [\Delta - i(\alpha)]) = \{\beta \in \Phi^+ - [\Delta - \{\alpha, i(\alpha)\}] \mid \beta + \gamma \notin \Phi^+ \text{ for any } \gamma \in \Phi^+ - [\Delta - \{\alpha, i(\alpha)\}]\}$ . That the coefficient of  $\beta$  is nonzero with respect to each  $\alpha$  and

$i(\alpha)$  implies that the coefficient is 1 in each case by Lemma 1.3.1 and by the assumption that  $\alpha$  is a commutative root hence so is  $i(\alpha)$ . Since the highest root has the coefficient 1 with respect to each  $\alpha$  and  $i(\alpha)$ ,  $\beta + \gamma$  cannot be a root for any  $\gamma \in_{\mathbb{R}} \Phi^+ - [\Delta - \{\alpha, i(\alpha)\}]$ , proving the inclusion  $\subset$ . Suppose that  $U_\beta \subset Z(U'_1)$  and the coefficient of  $\beta$  with respect to  $\alpha$  is zero. There exist simple roots  $\alpha_{i_1}, \dots, \alpha_{i_m}$  such that  $\beta + \alpha_{i_1} + \dots + \alpha_{i_j} \in_{\mathbb{R}} \Phi^+$  for each  $j = 1, 2, \dots, m$  and  $\beta + \alpha_{i_1} + \dots + \alpha_{i_m}$  is the highest root. Then for some  $j$ ,  $\alpha_{i_j} = \alpha$ . Since  $\alpha \in \Phi^+ - [\Delta - \{\alpha, i(\alpha)\}]$  and  $U_{\beta + \alpha_{i_1} + \dots + \alpha_{i_{j-1}}} \subset Z(U'_1)$ , we obtain a contradiction to the assumption  $U_\beta \subset Z(U'_1)$ . Similarly we can show that the coefficient of  $\beta$  with respect to  $i(\alpha)$  is nonzero. This proves the other inclusion relation. ■

4.1.6. *Remark.* We remark that in [14], we proved the closedness of the orbits  $H.\Delta_{F_1}$  and  $H.\Delta_{F_2}$  in a direct way, in the case of  $G(\mathbb{R}) = SL_n(\mathbb{R})$  and this method works for the  $\mathbb{R}$ -split groups of classical types

## 4.2. Heisenberg Horospherical Subgroup Cases

4.2.1. In this section, let  $U_1$  and  $U_2$  be a pair of opposite Heisenberg horospherical  $\mathbb{R}$ -subgroups. We recall the representations  $\text{Ad}_i : \mathbf{L} \rightarrow GL(\mathcal{Z}_i)$  from Subsection 1.1.14 and  $\text{Ad}'_i : \mathbf{H} \rightarrow SL(\mathcal{V}_i)$  from Subsection 1.3.7. Denote by  $\pi$  the projection of  $\mathcal{Z}_i$  onto  $\mathcal{V}_i$ .

The following lemma easily follows from Lemma 2.1.10.

LEMMA. *Let  $F_i$  be a lattice in  $U_i(\mathbb{R})$  and  $\Delta'_{F_i} = \pi(\Delta_{F_i})$ . Then  $\Delta_{F_i} \cap Z(\mathcal{Z}_i)$  and  $\Delta'_{F_i}$  are lattices in  $Z(\mathcal{Z}_i)$  and  $\mathcal{V}_i$ , respectively.*

The group  $H$  acts through  $\text{Ad}_i$  and  $\text{Ad}'_i$  on the space of lattices in  $\mathcal{Z}_i$  and  $\mathcal{V}_i$ , respectively. The notation  $H.J_i$  and  $H.J'_i$  denotes the orbits  $\text{Ad}_i(H)(J_i)$  and  $\text{Ad}'_i(H)(J'_i)$ , respectively, under this action.

4.2.2. To prove the main theorem in this case, we investigate the closures of the orbits  $H'.\Delta_{F_i}$  and  $\delta(H').(\Delta_{F_1}, \Delta_{F_2})$  as we did in Section 4.1. Unlike the case when  $U_1$  and  $U_2$  are commutative, we have various closed subgroups between  $\text{Ad}_i(H)$  and  $SL(\mathcal{Z}_i(\mathbb{R}))$ . The key point of narrowing the possibilities for the closures of those orbits will be that (1) we may assume that the closures are the homogeneous spaces under semisimple subgroups of  $SL(\mathcal{Z}_i(\mathbb{R}))$ , after replacing  $\Delta_{F_1}$  and  $\Delta_{F_2}$  by suitable lattices inside the closures and (2)  $\text{Ad}'_i(H)$  is a maximal connected closed (resp. semisimple) subgroup of  $Sp(\mathcal{V}_i(\mathbb{R}))$  if  $G$  is not of type  $A_n$  (resp. if  $G$  is of type  $A_n$ ).

4.2.3. Since  $[\mathcal{V}_i, \mathcal{V}_i] = Z(\mathcal{Z}_i)$ , we have a skew-symmetric bilinear product  $\langle \cdot, \cdot \rangle : \mathcal{V}_i(\mathbb{R}) \times \mathcal{V}_i(\mathbb{R}) \rightarrow Z(\mathcal{Z}_i(\mathbb{R}))$  defined by  $\langle v, w \rangle = [v, w]$ . We set  $Sp(\mathcal{V}_i) = \{g \in SL(\mathcal{V}_i) \mid \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in \mathcal{V}_i\}$ .

LEMMA. Let  $F_i$  be a lattice in  $U_i(\mathbb{R})$  and  $\Phi(F_i) = \{g \in SL(\mathcal{Z}_i) \mid \langle g(v), g(w) \rangle \in \Delta_{F_i} \cap Z(\mathcal{Z}_i) \text{ for all } v, w \in \Delta_{F_i}\}$ . Then the connected component of identity in  $\Phi(F_i)$  is  $Sp(\mathcal{Z}_i)$

*Proof.* In a Lie group, the connected components coincide with the path-wise connected components. Denote by  $\Phi^0$  the path-wise connected component of the identity in  $\Phi(F_i)$  and let  $g \in \Phi^0$ . Then there exists a continuous family  $g_t \in \Phi^0$ ,  $t \in [0, 1]$ , such that  $g_0 = e$  and  $g_1 = g$ . For any  $v, w \in \Delta_{F_i}$ ,  $\langle g_t(v), g_t(w) \rangle \in \Delta_{F_i} \cap Z(\mathcal{Z}_i)$ . Since  $\Delta_{F_i} \cap Z(\mathcal{Z}_i)$  is discrete, it follows that  $\langle g_t(v), g_t(w) \rangle = \langle v, w \rangle$  for all  $t \in [0, 1]$ . In particular, it implies that  $\langle g(v), g(w) \rangle = \langle v, w \rangle$  for all  $v, w \in Z(\mathcal{Z}_i)$  since  $\Delta_{F_i}$  is a lattice in  $\mathcal{Z}_i$ . Hence  $g \in Sp(\mathcal{Z}_i)$ . ■

4.2.4. Denote by  $P_i$  the parabolic subgroup of  $SL(\mathcal{Z}_i)$  which stabilizes the line  $Z(\mathcal{Z}_i)$ . Then  $[P_i, P_i]$  is isomorphic to  $SL(\mathcal{Z}_i)W_i$  where  $W_i = R_u(P_i)$ .

LEMMA. (1)  $\overline{H' \cdot \Delta_{F_i}} \subset Sp(\mathcal{Z}_i(\mathbb{R})) \cdot \Delta_{F_i}$ .

(2)  $\overline{H' \cdot \Delta_{F_i}} \subset Sp(\mathcal{Z}_i(\mathbb{R}))W_i(\mathbb{R}) \cdot \Delta_{F_i}$ .

*Proof.* By Ratner's theorem, there exists a connected closed subgroup  $L'_i \subset SL(\mathcal{Z}_i(\mathbb{R}))$  containing  $\text{Ad}'_i(H')$  such that  $\overline{H' \cdot \Delta_{F_i}} = L'_i \cdot \Delta_{F_i}$ . Since  $\text{Ad}'_i(H') \subset Sp(\mathcal{Z}_i(\mathbb{R}))$ , it is not difficult to see that  $L'_i \subset \Phi(F_i)$  and hence (1) follows from Lemma 4.2.3. Since  $\Delta_{F_i} \cap Z(\mathcal{Z}_i)$  is a lattice in  $Z(\mathcal{Z}_i)$ , it follows that  $P_i$  is defined over  $\mathbb{Q}$  with respect to the  $\mathbb{Q}$ -form of  $SL(\mathcal{Z}_i)$  given by  $\Delta_{F_i}$ . Then (2) follows from (1) and Proposition 3.3.3. ■

4.2.5. PROPOSITION. For any lattice  $E_i$  in  $U_i(\mathbb{R})$ , the orbit  $Sp(\mathcal{Z}_i(\mathbb{R})) \cdot \Delta_{E_i}$  is closed.

*Proof.* Similarly to proof of Lemma 4.2.4, we can show that if  $\overline{Sp(\mathcal{Z}_i(\mathbb{R})) \cdot \Delta_{E_i}} = M' \cdot \Delta_{E_i}$  for some connected closed subgroup  $M'$  of  $SL(\mathcal{Z}_i(\mathbb{R}))$ ,  $M' \subset Sp(\mathcal{Z}_i(\mathbb{R}))$ . Therefore  $M' = Sp(\mathcal{Z}_i(\mathbb{R}))$ , proving that the orbit is closed. ■

4.2.6. PROPOSITION. If  $Sp(\mathcal{Z}_i(\mathbb{R})) \cdot \Delta_{F_i}$  is closed, it contains a closed orbit  $H' \cdot \Delta_{E_i}$ .

*Proof.* By Proposition 1.4.2, there exists a  $\mathbb{Q}$ -form of  $G$  such that  $U_1$  and  $U_2$  are defined over  $\mathbb{Q}$ . Set  $E_i = U_i(\mathbb{Z})$ . Since  $\Lambda_{E_i} = H(\mathbb{Z})$  and  $\Lambda_{E_i} \cap H'$  are lattices in  $H$  and  $H'$ , respectively,  $H \cdot \Delta_{E_i}$  and  $H' \cdot \Delta_{E_i}$  are closed. We will show that  $\Delta_{E_i}$  is contained in  $Sp(\mathcal{Z}_i(\mathbb{R})) \cdot \Delta_{F_i}$ . It is enough to show that  $\Delta_{E_i}$  is contained in  $Sp(\mathcal{Z}_i(\mathbb{R})) \cdot \Delta_{F_i}$ . Fixing an isomorphism of  $\mathcal{Z}_i(\mathbb{R})$  to  $\mathbb{R}^n$ , let  $J_i$  be the standard lattice  $\mathbb{Z}^n$ . Since  $Sp(\mathcal{Z}_i(\mathbb{R})) \cdot \Delta_{F_i}$  is closed, it follows that  $\Delta_{F_i}$  is commensurable to  $gSp_n(\mathbb{Z})g^{-1}$  for some  $g \in Sp(\mathcal{Z}_i(\mathbb{R}))$ . It implies that  $\Delta_{F_i}$  is commensurable to  $g\mathbb{Z}^n$ . For the same reason,  $\Delta_{E_i}$  is commensurable to  $h\mathbb{Z}^n$  for some  $h \in Sp(\mathcal{Z}_i(\mathbb{R}))$ , proving the claim. ■

4.2.7. If we assume that the real rank of  $\mathbf{G}$  is at least 3, then the cases when  $H$  has compact factors occur only when  $\mathbf{G}$  is of type  $D_{n+2}$ ,  $n \geq 4$ . Furthermore, the nonsplit real forms of  $\mathbf{G}$  admitting Heisenberg horospherical  $\mathbb{R}$ -subgroups are such that  $G = SU_{n+2}(D_{\mathbb{R}}, h)$  where  $D$  is a Hamiltonian quaternion division algebra over  $\mathbb{R}$  and  $h \in GL_{n+2}(D_{\mathbb{R}})$  is a nondegenerate hermitian form relative to an involution  $\sigma$  of the first kind such that the space of  $\sigma$ -symmetric elements in  $D_{\mathbb{R}}$  has dimension 3. Since the real rank of  $G$  is at least 2, we may assume that

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where  $h_0 \in GL_n(D_{\mathbb{R}})$  is a non-degenerate hermitian form relative to  $\sigma$ . In this case,

$$H = \left\{ \left( \begin{array}{ccc} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \sigma A \end{array} \right) \mid A \in SL_1(D_{\mathbb{R}}), B \in SU_n(D_{\mathbb{R}}, h_0) \right\}$$

and

$$H' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & SU_n(D_{\mathbb{R}}, h_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.2.8. PROPOSITION. *Suppose that  $H = H'$  or that the real rank of  $\mathbf{G}$  is at least 3. Then  $\overline{H' \cdot \Delta_{F_i}}$  contains a closed orbit  $H' \cdot \Delta_{E_i}$ .*

*Proof.* By Theorem 3.3.1, there exists a connected closed subgroup  $M$  containing  $\text{Ad}_i(H')$  such that  $\overline{H' \cdot \Delta_{F_i}} = M' \cdot \Delta_{F_i}$ . By Proposition 3.3.4, we may assume that  $M'$  is a semisimple subgroup of  $Sp(\mathcal{Z}'_i(\mathbb{R}))$ . Suppose that  $H = H'$ . Then  $M$  is either  $\text{Ad}_i(H)$  or  $Sp(\mathcal{Z}'_i(\mathbb{R}))$ . Therefore the claim follows from Proposition 4.2.6. Now if  $H \neq H'$ , we consider the realization of the  $\mathbb{R}$ -form of  $\mathbf{G}$  given in the previous section. Note that  $\text{Ad}'_i(\mathbf{H}') = \{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mid A \in SO_{2n}(\mathbb{C}) \}$  and hence the Zariski closure of  $M'$  is either  $M_1 = \{ \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \mid A \in SL_n(\mathbb{C}) \}$  or  $M_2 = SL_2(\mathbb{C}) \times SO_{2n}(\mathbb{C})$ . By the same argument as in the proof of Proposition 4.1.2, we can show that  $M'$  cannot be  $M_1$ , and when  $M = M_2$ , the closedness of the orbit  $M \cdot \Delta_{F_i}$  implies that  $H' \cdot \Delta_{F_i}$  is closed.

4.2.9. LEMMA. *If  $H \cdot \Delta_{F_i}$  is closed, then  $\Delta_{F_i} \cap \mathcal{Z}'_i(\mathbb{R})$  is a lattice in  $\mathcal{Z}'_i(\mathbb{R})$ . Furthermore  $\Delta_{F_i}$  is, up to commensurability, determined by  $\Delta_{F_i} \cap \mathcal{Z}'_i(\mathbb{R})$ .*

*Proof.* It follows from the  $\mathbb{Q}$ -rationality of  $\text{Ad}_i$  that  $\mathcal{Z}'_i$  is a  $\mathbb{Q}$ -subspace with respect to the  $\mathbb{Q}$ -structure of  $\mathcal{Z}_i$  given by  $\Delta_{F_i}$  (see the proof of

Proposition 1.5.5). It implies that  $\Delta_{F_i} \cap \mathcal{Z}_i(\mathbb{R})$  is Zariski dense and hence is a lattice in  $\mathcal{Z}_i(\mathbb{R})$ . The second claim follows from the fact that  $[\Delta_{F_i} \cap \mathcal{Z}_i(\mathbb{R}), \Delta_{F_i} \cap \mathcal{Z}_i(\mathbb{R})]$  is commensurable to  $Z(\mathcal{Z}_i) \cap \Delta_{F_i}$ . ■

4.2.10. *Remark.* (1) The subgroup  $H$  is trivial only when  $G$  is  $\mathbb{R}$ -split of type  $A_2$ .

(2) The subgroup  $H$  has compact factors only when  $G$  is of the (absolute) type  $B_n$  with the real rank 2 and  $D_n$ .

The following theorem covers the cases not only when  $H$  has no compact factors but also when  $\mathbf{G}$  is of type  $D_n$  as long as the real rank of  $\mathbf{G}$  is greater than 2. For example, it covers all Heisenberg horospherical subgroup cases for the groups with real rank at least 3.

4.2.11. **THEOREM.** *Let  $U_1, U_2$  be a pair of opposite Heisenberg horospherical  $\mathbb{R}$ -subgroups. Assume that if  $\text{rank}_{\mathbb{R}} \mathbf{G} = 2$ , the (absolute) type of  $\mathbf{G}$  is none of  $A_2, B_n$  and  $D_n$  for some  $n$ . Then  $(\mathbf{G}, U_1, U_2)$  has property (A).*

*Proof.* By the same argument as in Theorem 4.1.4, it is enough to find a closed orbit  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  inside  $\overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$ . By Proposition 4.2.8,  $\overline{H'.\Delta_{F_1}}$  contains a closed orbit  $H'.\Delta_{E_1}$ . By the same argument as the proof of Theorem 4.1.3, we may assume that there exists a lattice  $E_2$  such that  $H'.\Delta_{E_2}$  is closed and  $(\Delta_{E_1}, \Delta_{E_2}) \in \overline{\delta(H').(\Delta_{F_1}, \Delta_{F_2})}$ . Therefore there exist a  $\mathbb{Q}$ -form of  $G$  with respect to which  $U_1, U_2$ , and  $H'$  are defined over  $\mathbb{Q}$  and  $E_1$  is commensurable to  $U_1(\mathbb{Z})$ .

If  $H = H'$ , this follows from Proposition 3.4.10. It only remains to consider the case described in 4.2.7. That the orbit  $H'.\Delta_{E_1}$  is closed implies that there exists a  $\mathbb{Q}$ -form of  $H'$  such that the representation  $A \rightarrow \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$  is rational over  $\mathbb{Q}$ . It follows that  $H'(\mathbb{Q}) = SU_n(D'(\mathbb{Q}), h'_0)$  where  $D'$  and  $h'_0$  are described as in Proposition 1.4.10. Therefore it is enough to observe that this  $\mathbb{Q}$ -form of  $H'$  naturally extends to a  $\mathbb{Q}$ -form of  $\mathbf{G}$  by declaring that  $\mathbf{G}(\mathbb{Q}) = SU_{n+2}(D'(\mathbb{Q}), h')$  where

$$h' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h'_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Assume that the orbit  $\delta(H').(\Delta_{E_1}, \Delta_{E_2})$  is not closed. Denote by  $M'$  a connected closed subgroup of  $\text{Ad}_1(H') \times \text{Ad}_2(H')$  such that  $\delta(H') \subset M'$  and  $\overline{\delta(H').(\Delta_{E_1}, \Delta_{E_2})} = M'.(\Delta_{E_1}, \Delta_{E_2})$ .

*Case (1).*  $M = \text{Ad}_1(H') \times \text{Ad}_2(H')$ . Since a Heisenberg horospherical subgroup is reflexive (see Remark 2.3.4), there exists  $w \in G(\mathbb{Q})$  such that  $wU_1(\mathbb{Q})w^{-1}$  is conjugate to  $U_2(\mathbb{Q})$ . By Lemma 2.3.3, we may assume that for some  $x \in N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$ ,  $\Delta_{E_2} \cap \mathcal{Z}_2$  is commensurable to  $\Delta_{xU_2(\mathbb{Z})x^{-1}} \cap \mathcal{Z}_2$ . By Lemma 2.3.3,  $\Delta'_{E_2}$  is commensurable to  $\Delta'_{xU_2(\mathbb{Z})x^{-1}}$ .

Write  $x$  as  $yw$  where  $y \in H$  and  $w \in Z(\mathbf{L}(\mathbb{R}))$ . Again  $y = y_1y_2$  where  $y_1 \in H'$  and  $y_2 \in C(H')$  since  $H'$  commutes with compact factors of  $H$ . Set  $z = y_2w$ . Then  $(\delta_{U_1(\mathbb{Z})}, \Delta_{z^{-1}E_2z} \in \delta(\mathbf{H}'(\mathbb{R})) \cdot (\Delta_{E_1}, \Delta_{E_2})$ . Since the  $H'$ -orbits of  $\Delta_{y^{-1}E_2y}$  and  $\Delta_{zU_2(\mathbb{Z})z^{-1}}$  are closed and their projections onto  $\mathcal{Z}'_i(\mathbb{R})$  are commensurable, by Lemma 4.2.9,  $(\Delta_{U_1(\mathbb{Z})}, \Delta_{zU_2(\mathbb{Z})z^{-1}}) \in \delta(\mathbf{H}'(\mathbb{R})) \cdot (\Delta_{E_1}, \Delta_{E_2})$  up to commensurability, contradicting that  $\delta(H') \cdot (\Delta_{E_1}, \Delta_{E_2})$  is not closed by Proposition 3.4.5.

*Case (2).*  $M'$  is a proper subgroup of  $\text{Ad}_1(H') \times \text{Ad}_2(H')$ .

This case happens only when the type of  $G$  is  $B_n$  or  $D_n$ ; otherwise  $H$  is a simple Lie group, and hence  $\delta(H')$  is a maximal connected closed subgroup of  $\text{Ad}_1(H') \times \text{Ad}_2(H')$ . Since in this case  $H$  is a product of two simple groups, by the same argument as Case (1)(ii) in Theorem 4.1.4, we can show that  $\delta(H') \cdot (\Delta_{E_1}, \Delta_{E_2})$  contains a closed  $\delta(H')$ -orbit. This concludes the proof. ■

### 4.3. Non- $\mathbb{R}$ -Heisenberg Horospherical Subgroup Cases

We deal with non- $\mathbb{R}$ -Heisenberg subgroup cases by reducing into commutative horospherical subgroup cases. If  $Z(U_1)$  is not the root group of a highest real root, this can be done by considering  $Z(U_1)$  and  $Z(U_2)$ . Otherwise we show that we can replace  $U_1, U_2$  by another pair  $U'_1, U'_2$  of opposite horospherical subgroups such that  $Z(U'_1)$  is not a root group of a highest real root.

4.3.1. In Subsections 4.3.1–4.3.2, we assume that  $G$  is almost  $\mathbb{R}$ -simple. We define the pair of horospherical  $\mathbb{R}$ -subgroups  $U'_1, U'_2$  mentioned in the beginning of this section and prove their needed properties.

For each  $i = 1, 2$ , set  $\mathcal{Z}_i = \text{Lie}(U_i)$ ,  $\tilde{\mathcal{Z}}_i = \{X \in \mathcal{Z}_i \mid [X, \mathcal{Z}_i] \subset Z(\mathcal{Z}_i)\}$  and  $\mathcal{Z}'_i = \{X \in \mathcal{Z}_i \mid [X, \mathcal{Z}_i] = 0\}$ . Denote by  $\tilde{U}_i$  and  $U'_i$  the connected subgroups of  $U_i$  such that  $\text{Lie}(\tilde{U}_i) = \tilde{\mathcal{Z}}_i$  and  $\text{Lie}(U'_i) = \mathcal{Z}'_i$ . Equivalently,  $\tilde{U}_i = \{u \in U_i \mid gug^{-1}u^{-1} \in Z(U_i) \text{ for all } g \in U_i\}$  and  $U'_i$  the centralizer of  $\tilde{U}_i$  in  $U_i$ .

PROPOSITION. (1) The subgroups  $\tilde{U}_i$  and  $U'_i$  are  $\mathbb{R}$ -subgroups of  $U_i$ .

(2) If  $U_i$  is defined over  $\mathbb{Q}$ , then  $\tilde{U}_i$  and  $U'_i$  are  $\mathbb{Q}$ -subgroups.

(3) If  $F_i$  is a lattice in  $U_i(\mathbb{R})$ ,  $F_i \cap U'_i(\mathbb{R})$  is a lattice in  $U'_i(\mathbb{R})$ .

*Proof.* It is well known that the normal subgroups in the ascending central series of a unipotent  $\mathbb{R}$ -group are defined over  $\mathbb{Q}$ . Hence  $\tilde{U}_i$  is defined over  $\mathbb{R}$ . Since  $U'_i$  is the centralizer of  $\tilde{U}_i$  in  $U_i$ , it is defined over  $\mathbb{R}$ . This argument works in the same way for the field  $\mathbb{Q}$ . Part (3) directly follows from (2) and Lemma 2.1.11. ■

4.3.2. The notation in this section is the same as in Subsection 2.5.1 with  $k = \mathbb{R}$ . Set  $\Psi_1 = \{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid \text{the coefficient of } \alpha \text{ with respect to } {}_{\mathbb{R}}\Delta - (\Theta \cup {}_{\mathbb{R}}\Delta_H) \text{ is } 0\}$  and  $\Psi_2 = ({}_{\mathbb{R}}\Phi^+ - [\Theta]) - \Psi_1$ . Then  $\Psi_1$  and  $\Psi_2$  are closed sets of  ${}_{\mathbb{R}}\Phi^+$ . It is easy to see that  $\Psi_2 = {}_{\mathbb{R}}\Phi^+ - [\Theta \cup {}_{\mathbb{R}}\Delta_H]$ .

To simplify the notation, let  ${}_{\mathbb{R}}\Delta = \Delta$  and  ${}_{\mathbb{R}}\Delta_H = \Delta_H$  in the following proposition.

PROPOSITION. For  $U_1 = {}_{\mathbb{R}}V_{\Theta}$  and  $U_2 = {}_{\mathbb{R}}V_{\Theta}^-$ , suppose that  $Z(U_1) = U_{\alpha_h}$  and  $[U_1, U_1] \neq Z(U_1)$ . Then

$$(1) \quad \tilde{U}_1 = U_{\psi} \text{ where } \psi = \{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid \alpha_h - \alpha \in \Psi_1\}.$$

(2)  $U'_1 = {}_{\mathbb{R}}V_{\Theta \cup \Delta_H}$ ,  $U'_2 = {}_{\mathbb{R}}V_{\Theta \cup \Delta_H}^-$ . In particular,  $U'_1$  and  $U'_2$  are opposite horospherical  $\mathbb{R}$ -subgroups of  $G$  and  $Z(U'_1)$  is strictly bigger than  $U_{\alpha_h}$ .

*Proof.* The last claim directly follows from Lemma 2.5.3. Since  $Z(U_1) = U_{\alpha_h}$ , we have that  $\Delta_H \subset \Theta^c$  and that  $\Delta - (\Theta \cup \Delta_H)$  is non-empty by the hypothesis that  $U_1 \neq [U_1, U_1]$ . We observe that  $\{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid U_{\alpha} \subset \tilde{U}_1\} = \{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid \text{if } \alpha + \beta \in {}_{\mathbb{R}}\Phi^+ \text{ for } \beta \in {}_{\mathbb{R}}\Phi^+ - [\Theta], \text{ then } \alpha + \beta = \alpha_h\}$ , which we will denote by  $\tilde{\Phi}$ . Also  $\{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid U_{\alpha} \subset U'_1\} = \{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid \alpha + \beta \notin {}_{\mathbb{R}}\Phi^+ \text{ for all } \beta \text{ with } U_{\beta} \subset \tilde{U}_1\}$ , which we will denote by  $\Phi'$ . It is enough to show that  $\tilde{\Phi} = \psi$  for (1) and  $\Phi' = \Psi_2$  for (2).

(1) To show that  $\psi \subset \tilde{\Phi}$ , let  $\beta = \alpha_h - \alpha$  with  $\alpha \in \Psi_1$ . If  $\alpha + \gamma \in {}_{\mathbb{R}}\Phi^+$  for some  $\gamma \in {}_{\mathbb{R}}\Phi^+ - [\Theta]$ , then the coefficient of  $\gamma$  with respect to each simple  $\mathbb{R}$ -root in  $\Delta - (\Theta \cup \Delta_H)$  must be 0. Since  $\gamma \in {}_{\mathbb{R}}\Phi^+ - [\Theta]$ , the sum of the coefficients of  $\gamma$  with respect to  $\Delta_H$  should be non-zero and hence 1. Therefore the sum of all coefficients of  $\beta + \gamma$  with respect to  $\Delta_H$  is 2, yielding  $\beta + \gamma = \alpha_h$  by Proposition 1.3.1.

To show the converse, assume that  $\alpha \in \tilde{\Phi}$ . If  $\alpha \notin \psi$ , then  $\text{ht}(\alpha) \leq \text{ht}(\alpha_h) - 2$ . We can find  $\mathbb{R}$ -simple roots  $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$  (not necessarily different) such that  $\alpha + \alpha_{i_1} + \dots + \alpha_{i_j} \in {}_{\mathbb{R}}\Phi^+$  for each  $j = 1, 2, \dots, k$  and  $\alpha + \alpha_{i_1} + \dots + \alpha_{i_k} = \alpha_h$ . Let  $j$  be the smallest number such that  $\alpha_{i_j} \in \Delta - \Theta$ . Then  $j < k$  since  $\alpha \notin \psi$ . Set  $\alpha' = \alpha + \alpha_{i_1} + \dots + \alpha_{i_{j-1}}$ . It follows from the property of ascending series of  $U$  that if  $U_{\alpha} \subset \tilde{U}$  and  $\alpha + \alpha_m \in {}_{\mathbb{R}}\Phi^+$  for  $\alpha_m \in \Delta$ , then  $U_{\alpha + \alpha_m} \subset \tilde{U}$ . Therefore by induction we have  $U_{\alpha'} \subset \tilde{U}$ . But  $\alpha' + \alpha_{i_j} \in {}_{\mathbb{R}}\Phi^+$ ,  $\alpha_{i_j} \in {}_{\mathbb{R}}\Phi^+ - [\Theta]$  while  $\alpha' + \alpha_{i_j} \neq \alpha_h$ . Therefore  $\alpha' \notin \tilde{\Phi}$ , contradicting  $U_{\alpha'} \subset \tilde{U}$ .

(2) To show that  $\Psi_2 \subset \Phi'$ , assume  $\alpha \in \Psi_2 - \Phi'$ . Then  $\alpha + \beta \in {}_{\mathbb{R}}\Phi^+$  for some  $\beta \in \tilde{\Phi}$ . Therefore  $\alpha + \beta = \alpha_h$ . Since  $\beta = \alpha_h - \alpha \in \psi$  by (1),  $\alpha \in \Psi_1 = {}_{\mathbb{R}}\Phi^+ - \Psi_2$ , contradicting the assumption.

To show the converse, suppose that  $\alpha \in \Phi' - \Psi_2$ . Since  $\Delta - (\Theta \cup \Delta_H)$  is non-empty and  $\alpha \in \Psi_1$ , the sum of the coefficients of  $\alpha$  with respect to  $\Delta_H$  is 1 and there exists a simple root in  $\Delta$  with respect to which the coefficient of  $\alpha$  is 0. It follows from Lemma 2.5.1 that  $\alpha_h - \alpha \in {}_{\mathbb{R}}\Phi^+$ .

Hence  $\alpha_h - \alpha \in \psi$ ;  $U_{\alpha_h - \alpha} \subset \tilde{U}$ . But  $\alpha = \alpha + (\alpha_h - \alpha)$ , contradicting the assumption that  $\alpha \subset \Phi$  by (1). ■

**4.3.3. PROPOSITION.** *Let  $G$  be a connected adjoint  $\mathbb{R}$ -simple algebraic  $\mathbb{R}$ -group with real rank at least 2 and  $U_1, U_2$  a pair of opposite horospherical  $\mathbb{R}$ -subgroups. Let the notation  $G_0, \mathbf{H}$ , and  $S$  be the same as Proposition 2.5.4 and  $Z_i = Z(U_i)$  for each  $i = 1, 2$ . Suppose that  $S$  is non-trivial and has no  $\mathbb{R}$ -anisotropic factors. If the triple  $(G_0, Z_1, Z_2)$  has property (A), then so does  $(G, U_1, U_2)$ .*

*Proof.* By Proposition 2.5.4,  $G_0$  is a connected almost  $\mathbb{R}$ -simple  $\mathbb{R}$ -group with real rank at least 2 and  $S$  is a semisimple normal algebraic  $\mathbb{R}$ -group of  $\mathbf{H}$ . Now let  $F_1$  and  $F_2$  be lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively, such that the subgroup  $\Gamma_{F_1, F_2}$  generated by them is discrete. Note that  $Z_i(\mathbb{R}) \cap F_i$  is a lattice in  $Z_i(\mathbb{R})$  for  $i = 1, 2$ , by Lemma 2.1.10. By the assumption that  $(G_0, Z_1, Z_2)$  has property (A), there exists a  $\mathbb{Q}$ -form of  $G_0$  with respect to which  $Z_1$  and  $Z_2$  are defined over  $\mathbb{Q}$  and the subgroup  $\Gamma_0$  generated by  $Z_1 \cap F_1$  and  $Z_2 \cap F_2$  is commensurable to the subgroup  $G_0(\mathbb{Z})$ . Since  $S(\mathbb{R})$  has no compact factors,  $S \cap \Gamma_0(\mathbb{Z}) \subset \Gamma_{F_1, F_2}$  is a Zariski dense arithmetic subgroup of  $S(\mathbb{R})$ . Therefore by Corollary 2.4.5,  $\Gamma_{F_1, F_2}$  is an arithmetic subgroup of  $G$ . ■

**4.3.4. THEOREM.** *Let  $\mathbf{G}$  be an adjoint absolutely simple algebraic  $\mathbb{R}$ -group with real rank at least 2 and  $U_1, U_2$  a pair of opposite horospherical  $\mathbb{R}$ -subgroups which satisfies either of the following condition:*

- (1)  $[U_1, U_1] = Z_1, Z_1$  is not the root group of a highest real root, and if  $\text{rank}_{\mathbb{R}}(G_0) = 2, G_0$  is not of type  $E_6$ , where  $G_0$  is the subgroup generated by  $Z_1$  and  $Z_2$ .
- (2)  $[U_1, U_1] \neq Z_1, Z_1$  is the root group of a highest real root, and if  $\text{rank}_{\mathbb{R}}(G'_0) = 2, G'_0$  is not of type  $E_6$ , where  $U'_i$  is the centralizer of  $\tilde{U}_i = \{g \in U_i \mid gug^{-1}u^{-1} \in Z_i \text{ for all } u \in U_i\}$  in  $U_i$ , and  $G'_0$  is the subgroup generated by the center of  $U'_1$  and of  $U'_2$ .

*Then the triple  $(\mathbf{G}, U_1, U_2)$  has Property A.*

*Proof.* (1) Let  $U_1, U_2$  satisfy the first condition. By Theorem 4.1.4 and Proposition 2.5.4, the triple  $(G_0, Z(U_1), Z(U_2))$  has property (A). This proves the theorem by the previous proposition.

(2) Since  $U_1, U_2$  is conjugate to the pair  ${}_{\mathbb{R}}V_{\Theta}$  and  ${}_{\mathbb{R}}V_{\Theta}^-$  for some  $\Theta \subset_{\mathbb{R}}\Delta$ , by Proposition 4.3.2,  $U'_1$  and  $U'_2$  are opposite horospherical  $\mathbb{R}$ -subgroups and  $Z(U'_1)$  is not the root group of a highest real root. By part (1), we have that the triple  $(G'_0, Z(U'_1), Z(U'_2))$  has property (A).

Now let  $F_1$  and  $F_2$  be lattices in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively, and  $\Gamma_{F_1, F_2}$  be discrete. Then  $F'_i = F_i \cap U'_i(\mathbb{R})$  is a lattice in  $U'_i(\mathbb{R})$  by Proposi-

tion 4.3.1 and  $\Gamma_{F'_1, F'_2}$  is discrete since  $\Gamma_{F'_1, F'_2} \subset \Gamma_{F_1, F_2}$ . Therefore the subgroup  $\Gamma_{F'_1, F'_2}$  is an arithmetic subgroup; hence so is  $\Gamma_{F_1, F_2}$ . ■

4.4. Arithmetic Subgroups of the Form  $\Gamma_{F_1, F_2}$

We discuss, for a given pair  $U_1$  and  $U_2$  of opposite horospherical subgroups, how specifically we can determine the discrete subgroups of the form  $\Gamma_{F_1, F_2}$  from the main theorem.

4.4.1. COROLLARY. *Let  $G$  be an absolutely simple algebraic  $\mathbb{Q}$ -group with the Tits index  $(\Delta, \Delta_0, *$ -action of  $\text{Gal}(\mathbb{C}/\mathbb{Q})$ ) and  $U_1 = V_\Theta, U_2 = V_\Theta^-$  for some  $\Theta \subset \Delta$ . If  $\Gamma_{F_1, F_2}$  is a subgroup commensurable to  $G(\mathbb{Z})$  for some lattices  $F_1$  and  $F_2$  in  $U_1(\mathbb{R})$  and  $U_2(\mathbb{R})$ , respectively, then  $\Theta$  contains  $\Delta_0$  and is invariant under the  $*$ -action of  $\text{Gal}(\mathbb{C}/\mathbb{Q})$ .*

*Proof.* It follows from the assumption that  $\Gamma_{F_1, F_2}$  is commensurable to  $G(\mathbb{Z})$  that  $F_1$  is commensurable to  $U_1(\mathbb{Z}) = U_1 \cap G(\mathbb{Z})$ . Since  $F_1$  is a lattice in  $U_1(\mathbb{R})$ , it follows that  $U_1(\mathbb{Z})$  is Zariski dense in  $U_1$ . By Proposition 1.2.7,  $U_1$  is defined over  $\mathbb{Q}$ . Therefore the normalizer  $N(U_1)$  is defined over  $\mathbb{Q}$ . Since  $N(U_1) = P_\Theta$ , the corollary follows from Proposition 1.4.1. ■

4.4.2. Let  $U_1, U_2$  be as in the Example 2.2.4 and  $F_1$  and  $F_2$  lattices in  $M_{m \times k}(\mathbb{R})$  and  $M_{k \times m}(\mathbb{R})$ , respectively.

COROLLARY. *Let  $n \geq 3$  and  $G = SL_n(\mathbb{R})$ . If the subgroup  $\Gamma_{F_1, F_2}$  is discrete, there exist elements  $g \in GL_m(\mathbb{R})$  and  $h \in GL_k(\mathbb{R})$  such that  $gF_1h^{-1}$  and  $hF_2g^{-1}$  are, up to commensurability, one of the following pairs:*

(1) *the pair consisting of  $M_{r \times s}(D_{\mathbb{Z}})$  and  $M_{s \times r}(D_{\mathbb{Z}})$  where  $D$  is an  $\mathbb{R}$ -algebra defined over  $\mathbb{Q}$  with  $D_{\mathbb{R}} = M_d(\mathbb{R})$  such that  $D_{\mathbb{Q}}$  is a central division algebra over  $\mathbb{Q}$ ,  $d = \text{Deg}_{\mathbb{Q}} D_{\mathbb{Q}}$ ,  $rd = m$ ,  $sd = k$ ,  $D_{\mathbb{Z}}$  is a  $\mathbb{Z}$ -order of the algebra  $D_{\mathbb{Q}}$ , and  $M_{r \times s}(D_{\mathbb{Z}})$  denotes the set of  $r \times s$  matrices over the ring  $D_{\mathbb{Z}}$ ;*

(2) *the pair consisting of  $\{(X_{ij}) \in M_r(D_J) \mid X_{ij} + \sigma(X_{ji}) = 0\}$  repeated twice, where  $K$  is a real quadratic extension field of  $\mathbb{Q}$ ,  $J$  is the ring of integers of  $K$ , and  $D$  is an  $\mathbb{R}$ -algebra defined over  $K$  with  $D_{\mathbb{R}} = M_d(\mathbb{R})$  such that  $D_K$  is a central division algebra with an involution of the second kind  $\sigma$ ,  $d = \text{Deg}_K D_K$ ,  $rd = m = k$ , and  $D_J$  is a  $J$ -order of the algebra  $D_K$  compatible with  $\sigma$ .*

Moreover by conjugation by the element  $(\begin{smallmatrix} g & 0 \\ 0 & h \end{smallmatrix})$ , the subgroup  $\Gamma_{F_1, F_2}$  is commensurable to either the subgroup  $(SL_{r+s}D)_{\mathbb{Z}}$  or the subgroup  $SU(h_0)_{\mathbb{Z}} = \{Y \in (SL_{2r}D)_J \mid {}^t Y^\sigma h_0 Y = h_0\}$  where  $h_0 = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ , respectively.

*Proof.* By Theorem 4.1.4, there exists a  $\mathbb{Q}$ -form of  $G$  such that  $\Gamma_{F_1, F_2}$  is commensurable to  $G(\mathbb{Z})$ . Let  $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$  be the set of simple roots such that  $\alpha_i(\text{diag}(t_1, t_2, \dots, t_n)) = t_i - t_{i+1}$  for each  $i$ . Then  $U_1 =$

$V_{\Delta - \alpha_m}$ . By the previous corollary,  $\alpha_m \in \Delta - \Delta_0$ . If the  $\mathbb{Q}$ -form  $G$  is inner,  $G(\mathbb{Q}) = SL_j(D_{\mathbb{Q}})$ , up to conjugation, where  $D$  is described as above and  $jd = n$ . Since  $\Delta - \Delta_0 = \{a_{jd} \in \Delta\}$ , we have  $m = rd$  for some  $r$  and hence  $k = (j - r)d$ . If the  $\mathbb{Q}$ -form  $G$  is outer and has a minimal horospherical subgroup defined over  $\mathbb{Q}$ , then  $n$  should be even and the standard minimal horospherical  $\mathbb{Q}$ -subgroup is  $V_{\Delta - \alpha_{n/2}}$ . Therefore  $m = n/2$ , followed by  $k = n/2$ . ■

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