LECTURE NOTES: ORBIT CLOSURES FOR THE $\mathrm{PSL}_2(\mathbb{R})$-ACTION ON HYPERBOLIC 3-MANIFOLDS

HEE OH

These notes correspond to my lectures which were delivered at the summer school on *Teichmüller theory and its connections to geometry, topology and dynamics*, held in the Fields Institute, Toronto, in August of 2018. The main goal of the lecture series was to give a self-contained proof of the closed or dense dichotomy of a geodesic plane in the interior of the convex core in any convex cocompact rigid acylindrical hyperbolic 3-manifold.

This proof amounts to the classification of orbit closures of the $\mathrm{PSL}_2(\mathbb{R})$-action on the frame bundle of the corresponding hyperbolic 3-manifold as the title of these lecture notes indicates.

The main theorem that is presented here represents joint work with C. McMullen and A. Mohammadi ([7], [8]) although the proofs are somewhat simplified in these notes. I hope these notes will be useful for those who would like to learn more about homogeneous dynamics in the setting of hyperbolic manifolds of infinite volume.

1. Lecture I

We will use the upper half-space model for the hyperbolic 3-space

$$\mathbb{H}^3 = \{(x_1, x_2, y) : y > 0\}, \quad ds = \frac{\sqrt{dx_1^2 + dx_2^2 + dy^2}}{y}.$$

In this model of the hyperbolic 3-space $(\mathbb{H}^3, ds)$, a geodesic in $\mathbb{H}^3$ is either a vertical line or a vertical semi-circle and a geodesic plane in $\mathbb{H}^3$ is either a vertical plane or a vertical hemisphere.

The geometric boundary of $\mathbb{H}^3$ is given by the Riemann sphere $S^2 = \hat{\mathbb{C}}$, when we identify the plane $(x_1, x_2, 0)$ with the complex plane $\mathbb{C}$.

The group $G := \mathrm{PSL}_2(\mathbb{C})$ acts on $\hat{\mathbb{C}}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

This action of $\mathrm{PSL}_2(\mathbb{C})$ extends to an isometric action on $\mathbb{H}^3$ as follows: each $g \in \mathrm{PSL}_2(\mathbb{C})$ can be expressed as the composition of inversions $\mathrm{Inv}_{C_1} \circ \cdots \circ \mathrm{Inv}_{C_k}$ where Inv_{C_ℓ} denotes the inversion with respect to a circle C_ℓ in $\hat{\mathbb{C}}$. If $C = \{z : |z - z_0| = r\}$, then $\mathrm{Inv}_C(z)$ is the unique point on the ray $\{tz : t > 0\}$, satisfying the equation $|z - z_0| \cdot |\,\mathrm{Inv}_C(z) - z_0| = r^2$ for all $z \neq z_0$, and $\mathrm{Inv}_C(z_0) = \infty$. If we set $\Phi(g) = \mathrm{Inv}_{\hat{C}_1} \circ \cdots \circ \mathrm{Inv}_{\hat{C}_k}$ where $\mathrm{Inv}_{\hat{C}_\ell}$

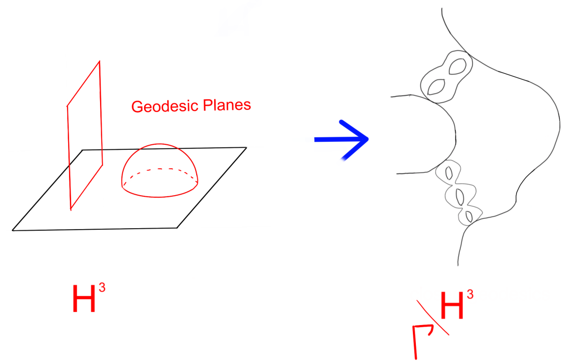

FIGURE 1. Geodesic planes

is the inversion with respect to the sphere $\hat{C}_\ell$ in $\mathbb{R}^3$ which is orthogonal to $\mathbb{C}$ and $\hat{C}_\ell \cap \mathbb{C} = C_\ell$, then $\Phi(g)$ preserves $(\mathbb{H}^3, ds)$. Moreover the Poincaré extension theorem says that Φ is an isomorphism between the two real Lie groups:

$$\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3),$$

where the group $\mathrm{PSL}_2(\mathbb{C})$ is regarded as a 6-dimensional real Lie group and $\mathrm{Isom}^+(\mathbb{H}^3)$ denotes the group of all orientation preserving isometries of $\mathbb{H}^3$ (cf. [9]).

Definition 1.1. *A torsion-free discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$ *is called a Kleinian group.*

Any complete hyperbolic 3-manifold M can be presented as a quotient

$$M = \Gamma\backslash\mathbb{H}^3$$

of the hyperbolic 3-space by a Kleinian group Γ, which we fix from now on together with the quotient map

$$\pi : \mathbb{H}^3 \to M = \Gamma\backslash\mathbb{H}^3.$$

Definition 1.2. *A geodesic plane P in M is the image of a geodesic plane in $\mathbb{H}^3$ under π. Equivalently, P is a totally geodesic immersion of a hyperbolic plane $\mathbb{H}^2$ in M.*

The question we address is that:

- can we classify all possible closures of a geodesic plane P in M?

In other words, how can a geodesic plane $P = \pi(\mathbb{H}^2)$ be sitting inside a hyperbolic 3-manifold M? In the universal cover $\mathbb{H}^3$, there is nothing blocking it, so $\mathbb{H}^2$ is sitting very comfortably in $\mathbb{H}^3$, but in general, due to the symmetries coming from Γ, $P = \pi(\mathbb{H}^2)$ has to be folded in a non-trivial way and its closure may be quite complicated.

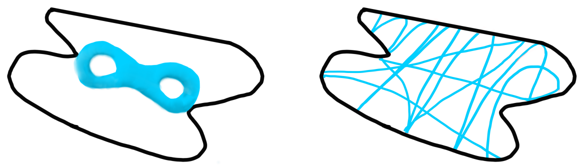

FIGURE 2. Closed or dense

Nonetheless, when M has finite volume, we have the following closed or dense dichotomy, due to Ratner [10] and Shah [11] independently.

Theorem 1.3 (Ratner, Shah). *If* $\mathrm{Vol}(M) < \infty$*, then any geodesic plane* P *is either closed or dense in* M*.*

In particular, the closure of P is always a submanifold of M.

This strong topological rigidity theorem applies only to countably many hyperbolic 3-manifolds since Mostow rigidity theorem implies that there are only countably many hyperbolic manifolds of finite volume, up to an isometry. Therefore it is quite natural to investigate this question for hyperbolic 3-manifolds of infinite volume.

The limit set of Γ and the convex core of the manifold M play important roles in this question. From now on, we assume that Γ is non-elementary, in other words, Γ has no abelian subgroup of finite index.

Via the Möbius transformation action of Γ on $\hat{\mathbb{C}}$, we define the following notion:

Definition 1.4. *The limit set* $\Lambda \subset \hat{\mathbb{C}}$ *of* Γ *is the set of all accumulation points of* $\Gamma(z)$ *for* $z \in \hat{\mathbb{C}}$*.*

It is easy to check that this definition is independent of the choice of z.

If $\mathrm{Vol}(M) < \infty$, then $\Lambda = \hat{\mathbb{C}}$. In general, Λ may be a fractal set with Hausdorff dimension strictly smaller than 2.

Definition 1.5. *The convex core of* M *is the convex submanifold of* M *given by*

$$\mathrm{core}(M) := \Gamma\backslash\, \mathrm{hull}(\Lambda) \subset M$$

where $\mathrm{hull}(\Lambda) \subset \mathbb{H}^3$ *is the smallest convex subset containing all geodesics connecting two points in* Λ*.*

Definition 1.6. *We say* M *or* Γ *is convex cocompact, if the convex core of* M *is compact.*

In the following, we assume

$$\Gamma \text{ is convex cocompact and Zariski dense.}$$

We set M^* to be the interior of $\mathrm{core}(M)$. As Γ is non-Fuchsian, $M^* \neq \emptyset$. Obviously there are two kinds of geodesic planes:

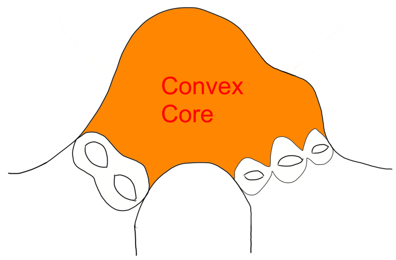

FIGURE 3. The interior of the core: M^*

- P with $P \cap M^* \neq \emptyset$,
- P with $P \cap M^* = \emptyset$.

The study of these two types of planes is expected to be different; for instance, if a plane P does not intersect M^*, then the closure of P will remain in the end component $M - M^*$ and hence such P will never be dense in M.

We will focus on the planes which intersect M^*, and study their closures in M^*.

Definition 1.7. *A geodesic plane in M^* is a non-empty intersection of a geodesic plane P of M and M^*:*

$$P^* = M^* \cap P.$$

We note that a geodesic plane P^* is always connected: if $P = \pi(\mathbb{H}^2)$, then P^* is covered by the convex subset $\mathbb{H}^2 \cap \mathrm{hull}(\Lambda)$.

Now, we ask:

- what are the possibilities for the closure of P^* in M^*?

It turns out that the answer to this question depends on the topology of the hyperbolic 3-manifold M. We are able to give a complete answer to this question for any convex cocompact acylindrical hyperbolic 3-manifold.

For a convex cocompact 3-manifold, the acylindrical condition is a topological one; its core has incompressible boundary and has no essential cylinder. Instead of defining these terminologies, I will give an equivalent definition of the acylindricality in terms of the limit set, as that is the most relevant definition to our proof.

Definition 1.8. *A convex cocompact hyperbolic 3-manifold $M = \Gamma\backslash\mathbb{H}^3$ of infinite volume is acylindrical if its limit set is a Sierpinski carpet, that is,*

$$S^2 - \Lambda = \bigcup B_i$$

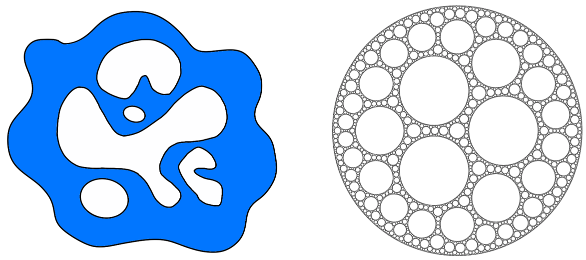

FIGURE 4. Limit sets of convex cocompact acylindrical groups

is a dense union of Jordan disks B_i's with mutually disjoint closures and with $\operatorname{diam}(B_i) \to 0$.[1]

By a theorem of Whyburn [12], any two Sierpinski carpets are homeomorphic to each other.

Example 1.1. If M is convex cocompact with $\partial(\operatorname{core}(M))$ totally geodesic, then its limit set is a Sierpinski carpet and the components of $S^2 - \Lambda$ are round disks, corresponding to the lifts of the geodesic boundary of $\operatorname{core}(M)$.

The double of the convex core of M is a closed hyperbolic 3-manifold which obeys Mostow rigidity. For this reason, we refer to such a manifold a *rigid acylindrical manifold.*

We remark that any convex cocompact acylindrical hyperbolic 3-manifold is quasi-isometric to a rigid one [6] and that the acylindricality condition on a convex cocompact hyperbolic 3-manifold M depends only on the topology of M, that is, any convex cocompact hyperbolic 3-manifold, which is homeomorphic to a convex cocompact acylindrical hyperbolic 3-manifold is also acylindrical (cf. [5] for references).

Theorem 1.9 (McMullen-Mohammadi-O.)**.** *Let M be convex cocompact and acylindrical. Any geodesic plane P^* in M^* is either closed or dense.*

If M has finite volume, then $M^* = M$; so this is a generalization of Ratner-Shah Theorem 1.3.

Theorem 1.9 is false in general without the acylindricality condition.

[1] A Jordan disk is a topological disk whose closure is homeomorphic to a closed unit disk

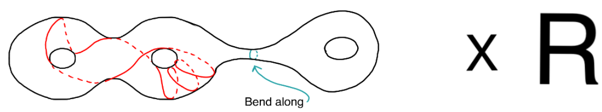

FIGURE 5. Bending deformation

Example 1.2. Consider a Fuchsian 3-manifold M which can be written as $M = S \times \mathbb{R}$ in cylindrical coordinates where S is a closed hyperbolic surface of genus at least 2. If $\gamma \subset S$ is a geodesic and P is a geodesic plane orthogonal to S with $P \cap S = \gamma$, then $\overline{P} \simeq \overline{\gamma} \times \mathbb{R}$. Therefore if we take a geodesic γ whose closure $\overline{\gamma}$ is wild, then $\overline{P}$ is very far from being a submanifold. To be fair, we have $M^* = \emptyset$ in this case as M is Fuchsian. However we can use a small bending deformation of M to obtain a quasi-Fuchsian manifold in which the same phenomenon persists. We realize S by $\Gamma\backslash\mathbb{H}^2$ where $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ is the fundamental group of S, and M by $\Gamma\backslash\mathbb{H}^3$. Let $\gamma_0 \in \Gamma$ be a primitive hyperbolic element representing a separating simple closed geodesic β in S. If S_1 and S_2 are components of $S - \beta$, then Γ can be presented as the amalgamated free product $\Gamma_1 *_{\langle\gamma_0\rangle} \Gamma_2$ where $\pi_1(S_i) \simeq \Gamma_i$ for $i = 1, 2$.

The centralizer of γ_0 in $\mathrm{PSL}_2(\mathbb{C})$ is the rotation group $\mathrm{PO}(2) = \{m_\theta : \theta \in S^1\}$. For each $m_\theta \in \mathrm{PO}(2)$, $\Gamma_1 \cap m_\theta^{-1}\Gamma_2 m_\theta = \langle\gamma_0\rangle$ and Γ is isomorphic to the group $\Gamma_\theta := \Gamma_1 *_{\langle\gamma_0\rangle} m_\theta^{-1}\Gamma_2 m_\theta$.

If θ is sufficiently small, then Γ_θ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$, $M_\theta := \Gamma_\theta\backslash\mathbb{H}^3$ is a quasi-Fuchsian manifold and there is a path isometric embedding of S into $\partial\,\mathrm{core}(M_\theta)$ which is identity on S_1,

Now let $\gamma \subset S_1$ be a geodesic whose closure $\overline{\gamma}$ is disjoint from an ϵ-neighborhood of β. Now if we set $S_1(\epsilon) := S_1 - \{\epsilon\text{-neighborhood of } \beta\}$, then for $\epsilon > 0$ small enough, $S_1(\epsilon) \times \mathbb{R}$ imbeds isometrically into M_θ and $P := \gamma \times \mathbb{R} \subset S_1(\epsilon) \times \mathbb{R}$ is a geodesic plane in M_θ such that $\overline{P} \simeq \overline{\gamma} \times \mathbb{R}$. Therefore by choosing γ whose closure is wild, we can obtain a geodesic plane P meeting M_θ^* with wild closure (cf. [7] for more details).

This example demonstrates that we cannot expect any reasonable classification for the closure of a plane in a completely general hyperbolic 3-manifold.

In order to study the closure of a plane in M, we lift this problem to the frame bundle of M which is a homogeneous space $\Gamma\backslash G$, admitting the frame flow as well as the horocyclic flow.

Note that $G/\mathrm{PSU}(2)$ can be identified with $\mathbb{H}^3$, $G/\mathrm{PSO}(2)$ with the unit tangent bundle $\mathrm{T}^1(\mathbb{H}^3)$ and G with the oriented frame bundle $\mathrm{F}(\mathbb{H}^3)$ (see

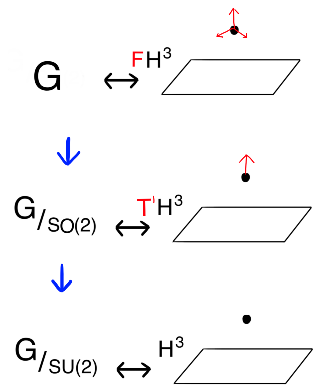

FIGURE 6. Frame bundle

Figure 7). For each oriented plane P, the set of frames (e_1, e_2, e_3) based in P such that e_1 and e_2 are tangent to P and e_3 is given by the orientation of P is an $\mathrm{PSL}_2(\mathbb{R})$-orbit. Conversely, for any frame $g = (e_1, e_2, e_3) \in G$, the orbit $g\,\mathrm{PSL}_2(\mathbb{R})$ consists of such frames lying on the unique oriented plane P given by e_3.

As $\mathrm{F}(M) = \Gamma\backslash G$, the classification of $\mathrm{PSL}_2(\mathbb{R})$-orbit closures in $\Gamma\backslash G$ will give us the desired classification of closures of planes. By a slight abuse of notation, we will denote by π both the base point projection maps $F(\mathbb{H}^3) \to \mathbb{H}^3$ and $F(M) \to M$ in the following.

As we are interested in planes in M^*, we define the following $H := \mathrm{PSL}_2(\mathbb{R})$-invariant subset in $\Gamma\backslash G$ lying above M^*:

$$F^* := \bigcup \{xH \subset \Gamma\backslash G : \pi(xH) \cap M^* \neq \emptyset\}.$$

When Γ is a lattice, we have $F^* = \Gamma\backslash G$. In general, F^* is an open subset of $\Gamma\backslash G$ foliated by H-orbits, but not admitting any transitive action of a subgroup of G.

Theorem 1.10 (H-orbit closure theorem)**.** *Let Γ be a convex cocompact acylindrical group. For any $x \in F^*$, xH is either closed or dense in F^*.*

Since M^* is an open subset of $\pi(F^*)$, it follows that any $P^* \subset M^*$ is closed or dense (Theorem 1.9).

We will state an equivalent version to Theorem 1.10 based on the following observation:

$$\text{Description of } H\text{-orbit closures in } \Gamma\backslash G$$
$$= \text{Description of } \Gamma\text{-orbit closures in } G/H$$

We also note that

$$G/H = \mathcal{C} := \text{the space of all oriented circles in } S^2$$

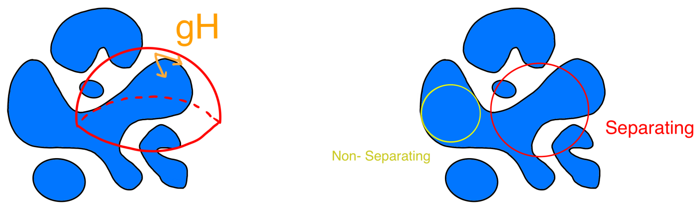

FIGURE 7. F^* and $\mathcal{C}^*$

Now the circles which correspond to H-orbits in F^* are the so-called separating ones:

$$\mathcal{C}^* := \{C \in \mathcal{C} : C \text{ separates } \Lambda\}$$

where the meaning of C separating Λ is that both open disks in $S^2 - C$ intersect Λ non-trivially.

We note that

$$\Gamma \backslash \mathcal{C}^* = F^*/H$$

where we regard $\mathcal{C}^*$ as a subset of G/H.

Theorem 1.11 (Γ-orbit closure theorem)**.** *For any $C \in \mathcal{C}^*$, the orbit ΓC is either closed or dense in $\mathcal{C}^*$.*

2. Lecture II

We set $G = \mathrm{PSL}_2(\mathbb{C})$ and let $\Gamma < G$ be a convex cocompact Zariski dense subgroup of G. Let $H = \mathrm{PSL}_2(\mathbb{R})$. In studying the action of H on $\Gamma \backslash G$, the following subgroups play crucial roles.

Let

$$A := \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}$$

and

$$U := \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

which are subgroups of H.

The right translation action of A on $\Gamma \backslash G$ is the frame flow: if $g = (e_1, e_2, e_3)$, then ga_t for $t > 0$ is the frame given by translation in direction of e_1 by hyperbolic distance t. We define

$$g^+ = g(\infty) \in \hat{\mathbb{C}} \quad \text{and} \quad g^- = g(0) \in \hat{\mathbb{C}};$$

they are the forward and backward end points of the geodesic given by e_1 respectively.

The right translation action of U on $\Gamma \backslash G$ is the horocyclic action: if $g = (e_1, e_2, e_3)$, then gu_t for $t > 0$ is the frame given by translation in the direction of e_2 by Euclidean distance t. Note that both gA and gU have their

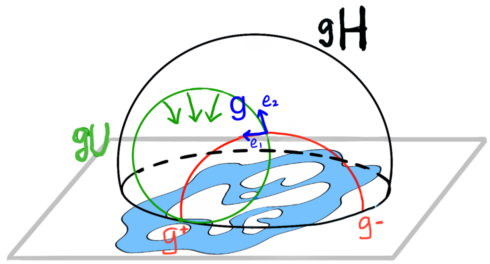

FIGURE 8. Orbits under A,U and H

trajectories on the plane $P = \pi(gH)$. In particular, $\pi(gU)$ is a Euclidean circle lying on P tangent at g^+ (see Figure 9).

We also define the 2-dimensional horospherical subgroup

$$N = \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C} \right\}$$

and a 1-dimensional

$$V = \{u_t : t \in i\mathbb{R}\}.$$

The trajectory of gN in $\mathbb{H}^3$ is a Euclidean sphere tangent to $\hat{\mathbb{C}}$ at g^+ and gN consists of frames (e_1, e_2, e_3) whose last two vectors e_2, e_3 are tangent to $\pi(gN)$.

Note that

$$N = \{g \in G : a_{-t} g a_t \to e \text{ as } t \to +\infty\}$$

that is, N is the contracting horospherical subgroup. Geometrically this means that $\pi(gNa_t)$ for $t > 0$ is a Euclidean sphere based at g^+ but shrunk toward g^+ by the hyperbolic distance t. And the normalizer of U is equal to NA, and the centralizer of U is N.

We would now like to explain the main strategy of our proof. Recall that Ratner and Shah proved the following independently:

Theorem 2.1. *If $\Gamma\backslash G$ is compact, then for any $x \in \Gamma\backslash G$, xH is either closed or dense in $\Gamma\backslash G$.*

Ratner's proof [10] relies on her measure classification theorem for measures invariant under U, whereas Shah's proof [11] is purely topological. In the case when $\Gamma\backslash G$ has infinite volume, first of all the measure classification of U invariant Radon measures is far from being known. Secondly, even if we had such a classification, it is not clear at all how the measure classification would be helpful in our orbit closure classification problem. The main issue is that it does not seem possible to define a U-invariant measure on the closure of an H-orbit, as the usual averaging argument along a U orbit will produce only the zero measure unless the H-orbit in concern is bounded.

Henceforth, we follow the strategy of Shah's topological proof, most of whose ingredients can be traced back to Margulis' proof of Oppenheim conjecture [4]. The proof breaks into the following two steps:

(1) Every N-orbit is dense in $\Gamma\backslash G$, that is, the N-action on $\Gamma\backslash G$ is minimal.
(2) If xH is not closed, then its closure $\overline{xH}$ contains an N orbit.

The first step is an easy consequence of a topological mixing of the A-action: for any open subsets O_1, O_2 in $\Gamma\backslash G$, $O_1 a_t \cap O_2 \neq \emptyset$ for all sufficiently large $t > 1$. The second step uses U-minimal sets and unipotent dynamics.

Now, unless $\Gamma < G$ is a lattice, the N-action is not minimal in $\Gamma\backslash G$. However there is a canonical closed N-invariant subset in $\Gamma\backslash G$ in which N acts minimally. It is given as

$$\mathrm{RF}_+ M = \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\}.$$

Since Λ is Γ-invariant, the condition $g^+ \in \Lambda$ for $[g]$ is well-defined.

Since $(gAMN)^+ = g^+$, the set $\mathrm{RF}_+ M$ is AMN-invariant. The Γ-minimality of Λ implies that $\mathrm{RF}_+ M$ is an AMN-minimal set.

The N-minimality of $\mathrm{RF}_+ M$. The following theorem is originally due to Ferte [2]:

Theorem 2.2. *For Γ convex cocompact and Zariski dense, the N-action is minimal on $\mathrm{RF}_+ M$.*

The following set is called the renormalized frame bundle:

$$\mathrm{RFM} = \{[g] \in \Gamma\backslash G : g^\pm \in \Lambda\}.$$

As RFM projects into the convex core of M, RFM is a compact A-invariant subset.

Theorem 2.2 can also be deduced from the topological mixing of the A-action on RFM, which was obtained in [13].

Finding an N-orbit inside the closure $\overline{xH}$

Define

$$F_\Lambda = \{[g] \in \Gamma\backslash G : \partial(gH) \cap \Lambda \neq \emptyset\}$$

and

$$F^* = \{[g] \in F_\Lambda : \partial(gH) \text{ separates } \Lambda\}$$

where $\partial(gH)$ denotes the boundary circle of the geodesic plane $\pi(gH) \subset \mathbb{H}^2$. Note that F^* is a dense open subset of F_Λ.

When the limit set Λ is connected, for instance, when Γ is acylindrical, F^* is equal to the following:

$$F^* = \{[g] \in \Gamma\backslash G : \pi([g]H) \cap M^* \neq \emptyset\}$$

and consequently F^* is open in $\Gamma\backslash G$ (as M^* is open). We observe that

$$(\mathrm{RF}_+ M)H = F_\Lambda.$$

Therefore, we may break our proof of Theorem 1.10 into the following two steps of which the first step has already been given:

(1) Every N orbit is dense in $\mathrm{RF}_+ M$.
(2) If xH is not closed in F^*, $\overline{xH}$ contains an N-orbit in $\mathrm{RF}_+ M$.

We now aim to prove the following:

Proposition 2.3. *Let Γ be acylindrical and $x \in F^*$. If xH is not closed in F^*, then*

$$\overline{xH} \supset x_0 N$$

for some $x_0 \in \mathrm{RF}_+ M$, and consequently $\overline{xH} = F_\Lambda$.

Unipotent blowup. The proof of Proposition 2.3 uses the following proposition which we call *unipotent blowup*. In order to motivate its formulation, here are a few words on how we will be using it: if the orbit xH is not closed, there has to be an accumulation of an infinite sequence $xh_n \in xH$ on some point $z \in \overline{xH} - xH$. Writing $xh_n = zg_n$, we have $g_n \to e$ in $G - H$. We then apply the u_t-flow to two nearby points z and zg_n and study how the orbits zu_t and zg_nu_t diverge from each other. Since $zg_nu_t = zu_t(u_{-t}g_nu_t)$, $u_{-t}g_nu_t$ is the difference between zu_t and zg_nu_t.

Provided g_n is not in the centralizer of U, that is, the subgroup N, the elements $u_{-t}g_nu_t$ will grow bigger and bigger as $t \to \infty$, for each fixed n. Hence, by passing to a subsequence, we can find t_n so that $u_{-t_n}g_nu_{t_n}$ converges to a non-trivial element, say, g, which necessarily lies in the normalizer of U. Provided zu_{t_n} converges to some element, say, z_0, then we get $z_0gU = z_0Ug \subset \overline{xH}$. As long as g does not belong to U, this information that we get some non-trivial translate of a U orbit inside the closure of xH is a useful one. However the limiting element g may lie in U in general. The first part of the following proposition says that we can do the time change in the parameter t to get g outside of U, that is, there is a sequence s_n (depending on t_n) such that $u_{s_n}g_nu_{t_n}$ converges to $g \in N(U) \setminus U$. Noting that $zu_{t_n} = (zu_{-s_n})u_{s_n}g_nu_{t_n}$, we will get $(\lim zu_{-s_n})g$ inside X, provided that $\lim zu_{-s_n}$ exists.

When $\Gamma\backslash G$ is compact, the limit $\lim zu_{-s_n}$ always exists, up to passing to a subsequence. But in the infinite volume case, $\lim zu_{-s_n}$ may not exist. Indeed, for any compact subset Ω of $\Gamma\backslash G$ and for *almost all* $z \in \Gamma\backslash G$ (with respect to any reasonable measure on $\Gamma\backslash G$), the Lebesque measure of the return time $\{t \in [-T, T] : zu_t \text{ lies in } \Omega\}$ is $o(T)$. Note that since the scale of s_n is dictated by the element g_n, which we won't have a control over, we would need a recurrence of zu_t to a compact subset for every time scale of t, up to a fixed multiplicative constant.

More precisely, we will need the return of zu_t into a fixed compact subset for a K-thick subset of $\mathbb{R}$:

Definition 2.4. *For a fixed $K > 1$, a subset $T \subset \mathbb{R}$ is K-thick if for any $s > 0$,*

$$(\pm[s, Ks]) \cap T \neq \emptyset.$$

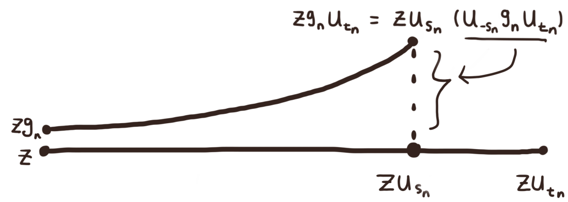

FIGURE 9. Divergence of U-orbits of two nearby points

Note that if T_n is K-thick, so is $\limsup T_n$. If T is K-thick, so is $-T$.

Proposition 2.5 (Unipotent blowup). (1) *If $g_n \to e$ in $G - AN$, then for any neighborhood O of e, there exist $t_n, s_n \in \mathbb{R}$ such that*

$$u_{t_n} g_n u_{s_n} \to g \in (AV - \{e\}) \cap O.$$

(2) *If $g_n \to e$ in $G - VH$, then for any neighborhood O of e, there exist $t_n \in \mathbb{R}$ and $h_n \in H$ such that*

$$u_{t_n} g_n h_n \to g \in (V - \{e\}) \cap O.$$

Moreover, for any fixed $K > 1$, given a sequence $T_n \subset \mathbb{R}$ of K-thick sets, we can arrange $t_n \in T_n$ in both statements.

Proof. The polynomial behavior of the u_t-action is used in the proof. We will provide a proof of the statement (2), and refer to [7] for the proof of (1), which is similar to that of (2), but slightly more technical due to the time change map. The proof of (2) involves only the conjugation by u_t. It is because we are allowed to use h_n in the statement, which implies that we can assume $\log g_n$ lies in the orthogonal complement $\mathfrak{h}^\perp$ of the Lie algebra $\mathfrak{h}$ of H. Then the conjugation of g_n by u_t remains in $\exp(\mathfrak{h}^\perp)$. Since $\exp(\mathfrak{h}^\perp) \cap N(U) \subset V$, we will get $g \in V$.

More precisely, we have $\mathrm{Lie}(G) = \mathfrak{h} \oplus i\mathfrak{h}$ where $\mathrm{Lie}(H) = \mathfrak{h}$ consists of trace zero matrices. Every element $g \in G$ sufficiently close to the identity e can be written as rh where $r \in \exp(i\mathfrak{h})$ and $h \in H$. By replacing g_n by $g_n h_n$ for a suitable $h_n \in H$, we may assume without loss of generality that $g_n = \exp(iq_n)$ for $q_n = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix} \to 0$.

Since $g_n \notin VH$, it follows that $c_n \neq 0$ or $c_n = 0, a_n \neq 0$. Now $u_t g_n u_{-t} = \exp(i u_t q_n u_{-t})$ and

$$u_t q_n u_{-t} = \begin{pmatrix} a_n + c_n t & b_n - 2a_n t + c_n t^2 \\ c_n & -a_n - c_n t \end{pmatrix}.$$

Since $c_n \neq 0$ or $c_n = 0, a_n \neq 0$, the function $P_n(t) := b_n - 2a_n t + c_n t^2$ is a non-constant polynomial of t of degree at most 2 and $P_n(0) \to 0$. Fix $\epsilon > 0$ so that the ϵ-neighborhood of e is contained in O. Let $t_n \in \mathbb{R}$ be such that $t_n := \sup\{t > 0 : |P_n[-t,t]| \leq \epsilon\}$. This means

$$\max |P_n(\pm t_n)| = \epsilon = \max_{s \in [-|t_n|,|t_n|]} |P_n(s)|.$$

Assume $|P_n(t_n)| = \epsilon$. (The case $|P_n(-t_n)| = \epsilon$ can be treated similarly). Since $q_n \to 0$, it follows that $t_n \to \infty$. Since $P_n(t_n) = t_n(c_n t_n - 2a_n) + b_n$ is bounded, we must have $c_n t_n \to 0$. Therefore $u_{t_n} q_n u_{-t_n} \to \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$ for $\beta \in \{\pm\epsilon\}$, by passing to a subsequence. Hence $u_{t_n} g_n u_{-t_n}$ converges to some $g = \exp(iq) \in V \cap O$.

Now suppose that a sequence T_n of K-thick subsets is given. Then by Lemma 2.6 below, there exist $c = c(K) > 0$ and $\tilde{t}_n \in [-|t_n|, |t_n|] \cap T_n$ such that

$$c\epsilon \leq |P_n(\tilde{t}_n)| = \max_{s \in [-|t_n|,|t_n|] \cap T_n} |P_n(s)| \leq \epsilon.$$

Then the above argument shows that $u_{\tilde{t}_n} q_n u_{-\tilde{t}_n}$ converges to some non-trivial $g \in V \cap O$, as desired. □

Lemma 2.6. *Let $K > 1$ and $d \in \mathbb{N}$ be given. There exists $c = c(K,d) > 0$ such that for any symmetric interval $I = [-s,s]$, any K-thick set T and any polynomial P of degree at most d,*

$$c \cdot \max_{t \in I} |P(t)| \leq \max_{t \in T \cap I} |P(t)|.$$

Proof. Note that the statement is invariant under rescaling of t as well as a multiplication of P by a real number. Therefore it suffices to prove the lemma for $I = [-1,1]$ and for the family $\mathcal{P}$ of polynomials P of degree at most d satisfying $\max_{x \in I} |P(x)| = 1$. We will use the fact that $\mathcal{P}$ is compact.

Suppose that the claim does not hold. Then there exist a sequence of polynomials $P_m \in \mathcal{P}$ and a sequence of K-thick sets T_m such that as $m \to \infty$,

$$\max_{T_m \cap I} |P_m(x)| \to 0.$$

Now a sequence of K-thick sets T_m converge to a K-thick set, say, T_∞ and P_m converges to a polynomial $P_\infty \in \mathcal{P}$, by passing to a subsequence. Then we get $\max_{T_\infty \cap I} |P_\infty(x)| = 0$. Since $T_\infty \cap I$ is an infinite set, this means $P_\infty = 0$. This is a contradiction to $\max_I |P_\infty(s)| = 1$. □

3. Lecture III

We continue the notation $G = \mathrm{PSL}_2(\mathbb{C})$ and $H = \mathrm{PSL}_2(\mathbb{R})$ and let $\Gamma < G$ a convex cocompact Zariski dense subgroup.

I plan to give a complete proof of Proposition 2.3 for the rigid acylindrical case.

As mentioned before, the main difficulty in carrying out unipotent dynamics in an infinite volume space $\Gamma\backslash G$ is the lack of the recurrence of U-orbits to a compact subset. And this is precisely where the topology of the manifold $M = \Gamma\backslash\mathbb{H}^3$ comes into the picture. When M is rigid acylindrical, we show that for any $x \in \mathrm{RFM}$, the return time $t \in \mathbb{R}$ such that $xu_t \in \mathrm{RFM}$ is K-thick for some uniform $K > 1$, depending only on M.

We will use the following geometric fact for a rigid acylindrical manifold M: if we write $S^2 - \Lambda = \bigcup B_i$ where B_i's are connected components which are round disks, then

$$\inf_{i \neq j} d(\mathrm{hull}(B_i), \mathrm{hull}(B_j)) \geq \epsilon_0 \tag{3.1}$$

where $2\epsilon_0$ is the systol of the double of the convex core of M. This follows because a geodesic in $\mathbb{H}^3$ which realizes the distance $d_{ij} = d(\mathrm{hull}(B_i), \mathrm{hull}(B_j))$ becomes the half of a closed geodesic in the double of $\mathrm{core}(M)$.

Proposition 3.1. *Let M be rigid acylindrical. There exists $K > 1$ such that for all $x \in \mathrm{RFM}$,*

$$T(x) := \{t \in \mathbb{R} : xu_t \in \mathrm{RFM}\}$$

is K-thick.

Proof. For $\epsilon_0 > 0$ given by (3.1), we set $K > 1$ so that

$$d_{\mathbb{H}^2}(\mathrm{hull}(-K, -1), \mathrm{hull}(1, K)) = \epsilon_0/2;$$

since $\lim_{s\to\infty} d_{\mathbb{H}^2}(\mathrm{hull}(-s, -1), \mathrm{hull}(1, s)) \to 0$, such $K > 1$ exists.

Since $z \mapsto tz$ is a hyperbolic isometry in $\mathbb{H}^2$ for any $t > 0$, we have

$$d_{\mathbb{H}^2}(\mathrm{hull}(-Kt, -t), \mathrm{hull}(t, Kt)) = \epsilon_0/2$$

for any $t > 0$.

We now show $T(x)$ is K-thick for $x \in \mathrm{RFM}$. It suffices to show the claim for $x = [g]$ where $g = (e_1, e_2, e_3)$ is based at $(0, 0, 1)$ with e_2 in the direction of the positive real axis and $g^+ = 0, g^- = \infty \in \Lambda$. Note that $gu_t \in \mathrm{RFM}$ if and only if $t = (gu_t)^- \in \Lambda$ and hence

$$T(x) = \mathbb{R} \cap \Lambda.$$

If $T(x)$ does not intersect $[-Kt, -t] \cup [t, Kt]$ for some $t > 0$, then the intervals $[-Kt, -t]$ and $[t, Kt]$ must lie in different B_i's, since $0 \in \Lambda$ separates them and B_i's are convex. Hence

$$d(\mathrm{hull}(-Kt, -t), \mathrm{hull}(t, Kt)) = \epsilon_0/2 \geq d(\mathrm{hull}(B_i), \mathrm{hull}(B_j)) \geq \epsilon_0.$$

This contradicts the choice of K.

□

Relative U-minimal sets. In the rest of this lecture, we just assume that Γ is convex cocompact and Zariski dense.

We suppose that we have a compact A-invariant subset

$$R \subset \mathrm{RFM}$$

such that for any $x \in R$,

$$T(x) := \{t \in \mathbb{R} : xu_t \in R\}$$

is K-thick for some fixed $K > 1$ independent of $x \in R$.

Let X be a closed H-invariant subset intersecting R. For instance $X = \overline{xH}$ for some $x \in R$.

Definition 3.2. *A closed U-invariant subset $Y \subset X$ is called U-minimal with respect to R, if $Y \cap R \neq \emptyset$ and yU is dense in Y for every $y \in Y \cap R$.*

By Zorn's lemma, X always contains a U-minimal subset with respect to R.

By a one-parameter semisubgroup of G, we mean a semigroup of the form $\{\exp(t\xi) : t \geq 0\}$ for some $\xi \in \mathrm{Lie}(G)$.

Lemma 3.3 (Translates of Y inside of Y)**.** *Let $Y \subset X$ be a U-minimal set with respect to R. Then*

$$YL \subset Y$$

for some one-parameter semigroup $L < AV$.

Proof. It suffices to find $q_n \to e$ in AV such that $Yq_n \subset Y$. It is an exercise to show the following: if $q_n = \exp(\xi_n)$ for $\xi_n \in \mathrm{Lie}(G)$ and ξ_∞ is the limit of $\|\xi_n\|^{-1}\xi_n$, then $YL \subset Y$ where $L = \{\exp(t\xi_\infty) : t \geq 0\}$.

Step 1: There exists $g_n \to e$ in $G - U$ such that $y_0 g_n \in Y$ for some $y_0 \in Y \cap R$. We will use the fact that there is no periodic U-orbit in $\Gamma\backslash G$ due to our assumption that Γ is convex cocompact. Take any $y \in Y \cap R$ and any $t_n \to \infty$ in $T(y_0)$ so that yu_{t_n} converges to some $y_0 \in Y \cap R$. Write $yu_{t_n} = y_0 g_n$ for $g_n \in G$. Then $g_n \to e$ in $G - U$, because if g_n were in U, then $y_0 \in yU$, and hence $yu = y$ for some non-trivial $u \in U$. This yields a contradiction.

Step 2(i): If $g_n \in AN$, by modifying g_n with elements of U, we may assume $g_n \in AV$, and get $y_0 g_n \in Y$. Since $g_n \in N(U)$ and $y_0 \in Y \cap R$, we get $y_0 U g_n = y_0 g_n U \subset Y$ and hence $\overline{y_0 U} g_n = Y g_n \subset Y$.

Step 2(ii): If $g_n \notin AN$, then by the unipotent blowup Proposition 2.5(1), for any neighborhood O of e, there exist $t_n \in -T(y_0)$ and $s_n \in \mathbb{R}$ such that $u_{t_n} g_n u_{s_n}$ converges to some $q \in (AV - \{e\}) \cap O$. Since $y_0 u_{-t_n} \in R$ and R is compact, $y_0 u_{-t_n}$ converges to $y_1 \in Y \cap R$, by passing to a subsequence. Therefore

$$y_0 u_{s_n} = (y_0 u_{-t_n})(u_{t_n} g_n u_{s_n}) \to y_1 q \in Y.$$

As $y_1 \in Y \cap R$, it follows $Yq \subset Y$. Since such q can be found in any neighborhood of e, this finishes the proof. $\square$

We leave it as an exercise to show the following:

Lemma 3.4. *A one-parameter semi-subgroup $L < AV$ is one of the following form:*

$$V_+, \quad vA_+v^{-1}, \quad A_+$$

for some non-trivial $v \in V$, where V_+ and A_+ is a one-parameter semisubgroup of V and A respectively.

When $L = V_+$ or vA_+v^{-1} for a non-trivial $v \in V$, Lemma 3.3 produces translates YL in Y in directions transversal to H, which is a new information! When $L = A_+$, $YA_+ \subset Y \subset X$ does not appear to be a useful information as $A \subset H$. However we will be able to combine that information with the following lemma to make it useful.

Lemma 3.5 (One translate of Y inside of X)**.** *Let $Y \subset X$ be a U-minimal set with respect to R such that*

$$y_0 g_n \in X$$

for some $y_0 \in Y \cap R$ and for some $g_n \to e$ in $G - H$. Then

$$Yv \subset X$$

for some $v \in V - \{e\}$.

Proof. If $g_n = v_n h_n \in VH$, then $v_n \neq e$ as $g_n \notin H$. Hence $y_0 v_n \in X$, implying that $Yv_n \subset X$ as desired.

If $g_n \notin VH$, we use the unipotent blowup Proposition 2.5(2) to get $t_n \in -T(y_0)$ and $h_n \in H$ so that $u_{t_n} g_n h_n$ converges to some non-trivial $v \in V$. Hence by passing to a subsequence, $y_0 g_n h_n = y_0 u_{-t_n}(u_{t_n} g_n h_n)$ converges to an element of the form $y_1 v$ for some $y_1 \in Y \cap R$, as $y_0 u_{-t_n} \in Y \cap R$ and $v \in V - \{e\}$. Therefore $Yv \subset Y$ as desired. □

4. Lecture IV

In this last lecture, we will now prove the following Proposition 2.3, and hence Theorem 1.10, for the rigid acylindrical case:

Proposition 4.1. *Let Γ be rigid acylindrical and $x \in F^*$. If xH is not closed in F^*, then*

$$\overline{xH} \supset x_0 N$$

for some $x_0 \in \mathrm{RF}_+ M$.

Since $U \subset H$, it suffices to find an orbit of V inside the closure of xH. The following observation shows that it suffices to find an orbit of a point in $F^* \cap \mathrm{RF}_+ M$ under an interval of V containing 0. We write $V_I = \{u_{it} : t \in I\}$ for any subset $I \subset \mathbb{R}$.

Lemma 4.2. *Let $X = \overline{xH}$ for $x \in \mathrm{RFM}$. If*

$$X \supset x_0 V_I$$

for $x_0 \in F^* \cap \mathrm{RF}_+ M$ and an interval I containing 0, then

$$X \supset z_0 N$$

for some $z_0 \in \mathrm{RFM}$*, and hence* $X = F_\Lambda$*.*

Proof. We will use the following two facts:

(1) For any $z \in F^* \cap \mathrm{RF}_+ M$, $zU \cap \mathrm{RFM} \neq \emptyset$.
(2) F^* is open in F_Λ.

The first fact is equivalent to the statement that for any circle C separating Λ, $C \cap \Lambda$ contains at least 2 points.

Without loss of generality, we may assume $I = [0, s]$ for some $s > 0$. We write $v_t := u_{it}$. Since $x_0 \in F^* \cap \mathrm{RF}_+ M$, $x_0 v_\epsilon \in F^*$ for some small $0 < \epsilon < s$ by (1) above.

By (2), there exists $x_1 \in x_0 v_\epsilon U \cap \mathrm{RFM}$. Hence $x_1 V_{[-\epsilon, s-\epsilon]} \subset X$. Since $a_t v_s a_{-t} = v_{e^t s}$, and RFM is a compact A-invariant subset, we can now take a sequence $a_{t_n} \to \infty$ in A so that $a_{t_n}^{-1} V_{[-\epsilon, s-\epsilon]} a_{t_n} \to V$ and $x_1 a_{t_n} \to z_0 \in \mathrm{RFM}$. As

$$x_1 V_{[-\epsilon, s-\epsilon]} a_{t_n} = x_1 a_{t_n} (a_{t_n}^{-1} V_{[-\epsilon, s-\epsilon]} a_{t_n}) \subset X$$

we obtain

$$z_0 V \subset X$$

as desired. □

Proof of Proposition 4.1 We now begin the proof of Proposition 4.1. Let $x \in F^*$ be such that xH is not closed in F^*. We set

$$X := \overline{xH}.$$

We break the proof of Proposition 4.1 into two cases depending on the compactness of the following set

$$R := X \cap \mathrm{RFM} \cap F^*.$$

We will show that R is always non-compact by showing the following proposition (note that $X = F_\Lambda$ implies $R = F^* \cap \mathrm{RFM}$, which is not compact):

Proposition 4.3. *If* $R := X \cap F^* \cap \mathrm{RFM}$ *is compact, then* $X = F_\Lambda$*.*

Proof. Observe that R is then a compact A-invariant subset of RFM such that for any $x \in R$,

$$T(x) = \{t : xu_t \in R\} = \{t : xu_t \in \mathrm{RFM}\}$$

is K-thick.

Step 1: We claim that X contains a U-minimal subset Y with respect to R such that $y_0 g_n \in X$ for some $y_0 \in Y \cap R$ and $g_n \to e$ in $G - H$. Note that the claim about y_0 is equivalent to the non-closedness of $X - y_0 H$. We divide our proof into two cases:

Case (a). Suppose that xH is not locally closed, i.e., $X - xH$ is not closed. In this case, any U-minimal subset $Y \subset X$ with respect to R works.

If $Y \cap R \subset xH$, then choose any $y_0 \in Y \cap R$; then $\overline{xH} - y_0 H = \overline{xH} - xH$ is not closed, which implies the claim. If $Y \cap R \not\subset xH$, choose $y_0 \in (Y \cap R) - xH$. Then $\overline{xH} - y_0 H$ contains xH, and hence cannot be closed.

Case (b). Suppose that xH is locally closed. Then $X - xH$ is a closed H-invariant subset and intersects R non-trivially, since $X - xH$ contains an H-orbit inside F^* (since xH is not closed in F^*) and any H-orbit in F^* contains an RFM point. Therefore $X - xH$ contains a U-minimal set Y with respect to R. Then any $y_0 \in Y \cap R$ has the desired property; since $y_0 \in X - xH$, there exists $h_n \in H$ such that $xh_n \to y$. If we write $xh_n = yg_n$, then $g_n \to e$ in $G - H$, since $y \notin xH$.

Step 2: We claim that X contains $x_0 V_I$ for some $x_0 \in F^* \cap \mathrm{RF}_+ M$ and for an interval I containing 0; this finishes the proof by Lemma 4.2.

Let $Y \subset X$ be given by Step (1). Then by Lemmas 3.3 and 3.5, we have

$$YL \subset Y \quad \text{and} \quad Yv_0 \subset X$$

where L is one of the following: V_+, vA_+v^{-1} or A_+ for some $v \in V$, and $v_0 \in V - \{e\}$ (here V_+ and A_+ are one-parameter semisubgroups of V and A respectively).

Case (a). When $L = V_+$, the claim follows from Lemma 4.2.

Case (b). If $L = vA_+v^{-1}$ for a non-trivial $v \in V$, then

$$X \supset Y(vA_+v^{-1})A.$$

Since $vA_+v^{-1}A$ contains V_I for some interval I containing 0, the claim follows from Lemma 4.2.

Case (c). If $L = A_+$, we first note that $YA \subset Y$; take any sequence $a_n \to \infty$ in A_+, and $y_0 \in Y \cap R$. Then $y_0 a_n \in Y \cap R$ converges to some $y_1 \in Y \cap R$. Now $\limsup a_n^{-1} A_+ = A$. Therefore $Y \supset y_1 A$, and hence $Y \supset YA$, using $\overline{y_1 U} = Y$.

Therefore $Y \supset YA$ and hence

$$X \supset Yv_0A \supset YAv_0A \subset YV_+$$

as desired. □

It now remains to deal with the case when $R = X \cap \mathrm{RFM} \cap F^*$ is not compact. Since $X \cap \mathrm{RFM}$ is compact, this situation arises when some points in R accumulate on the boundary of F^*. In order to understand the situation, we need to understand the structure of the boundary of F^*, and it is mainly in this step where the geometric structure of a rigid acylindrical manifold plays an important role.

We will call $z = [g]$ a boundary frame if $\partial(zH)$, more precisely, the boundary of the plane $\pi(gH)$, is equal to the circle $\partial(B_i)$ for a component B_i of $S^2 - \Lambda$. We will denote by BFM the collection of all boundary frames. For $z = [g] \in \mathrm{BFM}$, zH is compact, as $\pi(gH)$ lies in the boundary of the core of M.

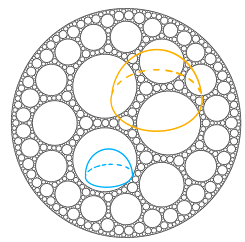

FIGURE 10. Planes in rigid acylindrical manifolds

We observe:

(1) $F_\Lambda - F^* \subset \mathrm{BFM} \cdot V$; and
(2) $(F_\Lambda - F^*) \cap \mathrm{RFM} \subset \mathrm{BFM}$.

It is only in the following lemma where we use the main feature of the rigid structure of Γ, which is that the boundary of the core of M is a compact geodesic surface;

Note that the stabilizer of B_i in G is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ and the stabilizer of B_i in Γ is cocompact in $\mathrm{PSL}_2(\mathbb{R})$ (first note that $\Gamma(B)$ is locally finite, i.e., ΓH is closed. As $(\Gamma \cap H)\backslash H$ lies above the boundary of the core of M, it is compact.

Let $\tilde{Z} = \cup z_i H V_+ H$ where V_+ is so that $\pi(Z)$ does not meet the interior of the core of M. We note that $\tilde{Z} \cap \mathrm{RFM} = \cup z_i H$.

In the following lemma, we use the following classical theorem of Hedlund [3]: if xH is compact, then every U orbit is dense in xH.

Lemma 4.4. *If $X \cap F^*$ contains $z_0 v$ for some $z_0 \in \mathrm{BFM}$ and $v \in V - \{e\}$, then $X = F_\Lambda$.*

Proof. Since $z_0 U v \subset X$, it follows from the aforementioned theorem of Hedlund that $\overline{z_0 U} \supset z_0 A$. So we get $z_0 A v A \subset X$ and hence

$$z_0 V_+ \subset X$$

for a semigroup V_+ of V containing v. Now $z_0 V = (z_0 v)(v^{-1} V_+)$.

Since $z_0 v \in F^* \cap \mathrm{RF}_+ M$ by the assumption, and $v^{-1} V_+$ contains V_I for some interval I containing 0, the claim follows from Lemma 4.2. $\square$

Proposition 4.5. *Suppose that $R = X \cap F^* \cap \mathrm{RFM}$ is non-compact. Then*

$$X = F_\Lambda.$$

Proof. By the assumption there exists $xh'_n \in R$ converging to $z \in X \cap \mathrm{RFM} \cap (F_\Lambda - F^*)$. It follows that $z \in \mathrm{BFM}$. Writing $xh'_n = zg_n$, we have $g_n \to e$ in $G - H$.

If $g_n = h_n v_n \in HV$ for some n, then $v_n \neq e$ and $zh_n \in$ BFM. Since $(zh_n)v_n \in X \cap F^*$, the condition of Lemma 4.4 has been satisfied.

Now suppose $g_n \notin HV$ for all n. By applying the unipotent blowup Proposition (2) to g_n^{-1}, we get sequences $t_n \in T(xh'_n)$ and $h_n \in H$ such that $h_n g_n u_{t_n} \to v \in V - \{e\}$.

Hence

$$xh'_n u_{t_n} = z g_n u_{t_n} = (zh_n^{-1})(h_n g_n u_{t_n}) \to z_1 v$$

for some $z_1 \in$ BFM.

On the other hand, $xh'_n u_{t_n} \in$ RFM by the choice of t_n. Since $v \neq e$ and $z_1 v \in$ RFM for $z_1 \in$ BFM,

$$z_1 v \in F^* \cap X$$

as desired.

This implies the claim by Lemma 4.4. □

Propositions 4.3 and 4.5 imply Proposition 4.1.

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