FRACTAL CLOSURES OF GEODESIC PLANES IN HITCHIN MANIFOLDS

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Abstract. Ratner's theorem implies topological rigidity of immersed totally geodesic subspaces of noncompact type in finite-volume locally symmetric spaces. In higher rank and infinite volume, however, counterexamples to this rigidity have remained elusive.

We construct the first such examples using *floating geodesic planes*. Specifically, we exhibit a Zariski-dense Hitchin surface group $\Gamma < \mathrm{SL}_3(\mathbb{R})$ such that the Hitchin manifold $\Gamma\backslash\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$ contains immersed floating geodesic planes whose closures are fractal, with non-integer Hausdorff dimensions accumulating at 2. Moreover, Γ can be chosen inside $\mathrm{SL}_3(\mathbb{Z})$.

Contents

1. Introduction

The study of orbit closures for actions of subgroups generated by unipotent elements has been one of the central themes in homogeneous dynamics. A landmark result is Ratner's 1991 theorem [23], which resolved the conjecture of Raghunathan. It says the following: if G is a connected semisimple Lie group G and $\Gamma < G$ is a lattice (a discrete subgroup of finite covolume), then for any connected subgroup $H < G$ generated by unipotent elements, the closure of every H-orbit in $\Gamma\backslash G$ is itself a homogeneous subspace, namely,

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a subspace of the form xL where $L < G$ is a Lie subgroup containing H and x is a point in $\Gamma\backslash G$. This result implies the following topological rigidity of geodesic planes: in any locally symmetric space of noncompact type and finite volume, the closure of an immersed totally geodesic subspace of noncompact type[1] is an immersed submanifold.

An important special case was obtained earlier by Margulis and Dani-Margulis (1989): in $\mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R})$, any orbit of $\mathrm{SO}(2,1)$ is either closed or dense ([18], [7]). This implies that in the associated locally symmetric space $\mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$, every *irreducible* totally geodesic plane is either properly immersed or dense. The closed or dense dichotomy has a far-reaching consequence. In fact, Margulis's proof of the Oppenheim conjecture that for any irrational indefinite quadratic form $Q(x_1, \dots, x_n)$ with $n \geq 3$, the set of values $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$ was his famous application of this result [18].

In the infinite-volume setting, the geometry of the ambient space plays a decisive role. For convex cocompact acylindrical hyperbolic 3-manifolds, McMullen–Mohammadi–Oh proved that geodesic planes inside the interior of the convex core is either properly immersed or dense ([19], [20]), and Benoist–Oh extended this to geometrically finite acylindrical manifolds [3]. In higher dimensions, Lee–Oh gave a complete classification of geodesic plane closures for convex cocompact real hyperbolic manifolds with Fuchsian ends ([15]; see also the survey paper [21]).

The picture changes dramatically once we leave the acylindrical setting. Using bending deformations, McMullen–Mohammadi–Oh [19] constructed quasi-Fuchsian hyperbolic 3-manifolds that contain chaotic geodesic planes, arising from planes orthogonal to a chaotic geodesic of a closed hyperbolic surface. Here "chaotic" means that the closures of these planes have non-integer Hausdorff dimension. This stands in sharp contrast with the acylindrical case: acylindrical hyperbolic manifolds exhibit strong geometric constraints that enforce a certain k-thickness property for every circular slice of the limit set, whereas quasi-Fuchsian manifolds may support much thinner circular slices. Geometrically, thin circular slices of the limit set translate into scant recurrence for unipotent flows, obstructing the standard homogeneous dynamics approach to orbit closures.

In higher rank and infinite volume, examples demonstrating the failure of topological rigidity of orbit closures have remained elusive. In this paper, we provide the first such examples.

Let

$$G = \mathrm{SL}_3(\mathbb{R}), \quad K = \mathrm{SO}(3), \quad X = G/K,$$

[1]this means the image of a totally geodesic immersion of a locally symmetric space of noncompact type

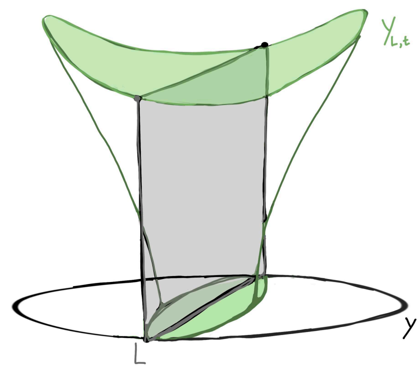

FIGURE 1. Floating geodesic plane

so that X is the Riemmanian symmetric space of unimodular positive-definite symmetric matrices. We consider

$$H = \mathrm{SO}(2,1)^\circ, \quad Y = H \cdot o \subset X, \quad \text{where } o = [K] \in X;$$

Y is an irreducible totally geodesic plane passing through the basepoint o. Our primary objects are what we call

floating geodesic planes

defined as follows: Given a complete geodesic $L \subset Y$, there exists a unique maximal flat $\mathcal{F} \subset X$ that intersects Y orthogonally along L. The plane Y can then be shifted away from itself along $\mathcal{F}$, producing ultra-parallel copies that "float" in the ambient space. More precisely, for any $t > 0$, the floating geodesic plane $Y_{L,t}$ is defined as the translate of Y along the orthogonal direction in $\mathcal{F}$ at distance t. Equivalently, if $\xi_t = a_t o$ is the unit speed geodesic in $\mathcal{F}$ orthogonal to L at o for a one-parameter diagonalizable subgroup $\{a_t : t \in \mathbb{R}\}$, then

$$Y_{L,t} = a_t Y;$$

see Figure 1. We also refer to the image of $Y_{L,t}$ in any quotient manifold $\Gamma\backslash X$ as a floating geodesic plane.

In hyperbolic spaces, there are no flats of dimension larger than one, so a geodesic plane cannot be displaced in this way. The existence of floating geodesic planes is therefore a phenomenon that only appears in higher rank.

The main result of this paper is as follows:

Theorem 1.1. *There exists a Zariski dense Hitchin surface subgroup* $\Gamma < \mathrm{SL}_3(\mathbb{R})$ *such that the locally symmetric space* $\Gamma\backslash X$ *contains a sequence of floating geodesic planes whose closures have Hausdorff dimensions strictly bigger than* 2 *and accumulating at* 2*. Moreover,* Γ *can be chosen inside* $\mathrm{SL}_3(\mathbb{Z})$.

The proof of Theorem 1.1 requires a combination of geometric and dynamical arguments that go well beyond existing rank-one techniques. Substantial difficulties already arise in the *Fuchsian case*: even for a Fuchsian surface group, relating the Hausdorff dimension of floating geodesic plane closures in the locally symmetric space $\Gamma\backslash X$ to the dynamics of the geodesic flow on the underlying hyperbolic surface requires delicate analysis beyond homogeneous space considerations.

To transport these fractal phenomena from the Fuchsian locus to Zariski-dense Hitchin subgroups, we combine Goldman's bulging deformations with a central geometric ingredient: a detailed analysis of the nearest-point projection to the reference plane. In higher rank, the fibers of this projection contain entire maximal flats, creating parallel families of geodesics and making stability under deformation substantially more subtle than in rank one.

We now explain the bulging deformations. We begin with a torsion-free cocompact Fuchsian group $\Gamma_0 < H$ and choose a simple closed geodesic β in the hyperbolic surface $S = \Gamma_0\backslash Y$. This curve β, called the bulging locus, is represented by some hyperbolic element $\delta \in \Gamma_0$. Goldman introduced the notion of bulging deformation along such a curve β ([12]; see also a recent work [4] where they use the terminology *grafting* instead of bulging). Roughly speaking, bulging is an analogue of Thurston's earthquake deformation, but in the setting of convex projective structures: one "bends" the geometry of the surface along β by a parameter from the identity component centralizer $C_G(\delta)$ of δ.

Formally, the deformation yields a representation $\rho_{\beta,\mathsf{b}} : \Gamma_0 \to G$ for any $\mathsf{b} \in C_G(\delta)^\circ$, which lies in the *Hitchin component* of the character variety of representations of Γ_0 into G. Choi–Goldman identified this Hitchin component with the space of marked convex $\mathbb{R}P^2$-structures on the surface $\Gamma_0\backslash Y$ [6]: the bulging deformation then corresponds to varying the convex projective structure by stretching along β. In particular, each representation $\rho_{\beta,\mathsf{b}}$ is discrete and faithful,

We therefore obtain a discrete subgroup of G:

$$\Gamma_{\beta,\mathsf{b}} \coloneqq \rho_{\beta,\mathsf{b}}(\Gamma_0);$$

see (8.5) for further details. Moreover $\Gamma_{\beta,\mathsf{b}}$ is Zariski dense whenever the width $\mathrm{wd}(\mathsf{b})$ of b is nonzero (see (8.4) for the definition). Subsequent developments have provided broader frameworks for understanding these groups: Labourie [13] introduced the notion of *Anosov representations*, while Fock–Goncharov [11] developed the theory of *positive representations.* Both have become central tools in the study of Hitchin representations of surface groups into split semisimple real Lie groups.

Theorem 1.1 is deduced from the following result, which shows that floating planes inherit their Hausdorff dimensions from the geodesic dynamics on the underlying hyperbolic surface $S = \Gamma_0\backslash Y$. In this paper, the notation dim always refers to the Hausdorff dimension.

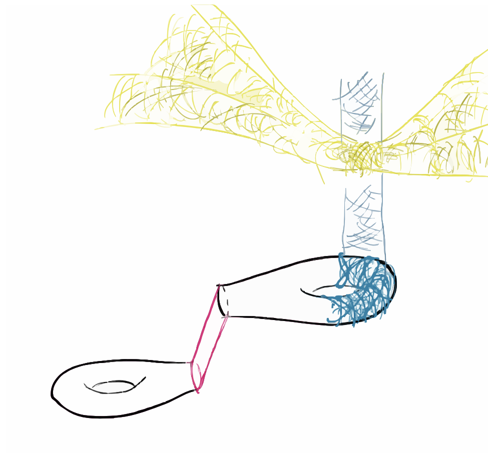

FIGURE 2. Fractal closures of floating geodesic planes

Theorem 1.2. *Let $\Gamma_0 < H$ be a torsion-free cocompact Fuchsian subgroup, and let $S = \Gamma_0\backslash Y$. Let $\beta \subset S$ be a simple closed geodesic represented by some $\delta \in \Gamma_0$.*

Let $L \subset Y$ be an admissible geodesic (Def. 7.1), and set $\ell = \Gamma_0\backslash\Gamma_0 L \subset S$. Suppose that $1 < \dim(\overline{\ell}) < 2$ and let $r := d(\ell, \beta) > 0$. Then there exists $t_0 > 0$, depending only on L and r, such that for all $t > t_0$ and any $\mathsf{b} \in C_G(\delta)^\circ$ with width $\mathrm{wd}(\mathsf{b})$ smaller than $r/2$, the Hausdorff dimension of the closure of the floating geodesic plane in $\Gamma_{\beta,\mathsf{b}}\backslash X$ satisfies

$$\tfrac{1}{2}\left(\dim\overline{\ell} + 3\right) \leq \dim\left(\overline{\Gamma_{\beta,\mathsf{b}}\backslash\Gamma_{\beta,\mathsf{b}} Y_{L,t}}\right) \leq \dim\overline{\ell} + 1.$$

Moreover, there exists a sequence of admissible geodesics $L_i \subset Y$ such that

- $r = \inf_i d(\ell_i, \beta) > 0$,
- $\dim\overline{\ell}_i > 1$ *for all $i \in \mathbb{N}$, and*
- $\dim\overline{\ell}_i \to 1$ *as $i \to \infty$.*

Hence for all sufficiently large t,

$$\dim\left(\overline{\Gamma_{\beta,\mathsf{b}}\backslash\Gamma_{\beta,\mathsf{b}} Y_{L_i,t}}\right) \to 2 \quad \text{as } i \to \infty.$$

Remark 1.3. Chaotic behavior can also be produced for geodesic planes orthogonal to a fixed irreducible geodesic plane, via their intersection locus (see Theorem A.3). Nevertheless, we find the geometry of the floating geodesic planes both more compelling and novel, and hence focus on them in this paper.

For a concrete analysis, we realize H as the identity component of the special orthogonal group associated to the quadratic form $F(x, y, z) = 2xz - y^2$. Let $A < G$ be the subgroup of positive diagonal matrices and set $A_0 = H \cap A$. We can identify H with the unit tangent bundle of Y, and the right translation action of A_0 on H with the geodesic flow. Every geodesic $L \subset Y$ is of the form hA_0o for some $h \in H$. For

$$a_t = \mathrm{diag}(e^t, e^{-2t}, e^t),$$

the floating geodesic plane over L at height t is

$$Y_{L,t} = ha_tH(o).$$

Theorem 1.2 is deduced from the following statement about the closure of the projection of $Y_{L,t}$ in $\Gamma_{\beta,\mathsf{b}}\backslash G$:

Theorem 1.4. *Under the same hypotheses as Theorem 1.2, we have*

$$\dim \overline{\Gamma_{\beta,\mathsf{b}}\backslash\Gamma_{\beta,\mathsf{b}} ha_t H} = \dim \overline{\Gamma_0\backslash\Gamma_0 hA_0} + 2;$$

$$\tfrac{1}{2}\left(3 + \dim \overline{\Gamma_0 hA_0}\right) \le \dim \overline{\Gamma_{\beta,\mathsf{b}} Y_{L,t}} \le 1 + \dim \overline{\Gamma_0 hA_0}.$$

Moreover, there exists a sequence $h_i \in H$ such that $\dim \overline{\Gamma_0\backslash\Gamma_0 h_i A_0}$ *accumulates at* 1 *as $i \to \infty$.*

On the proof ideas. Roughly speaking, the proofs of Theorems 1.2 and 1.4 proceed in four steps. A guiding principle is that the essential fractal phenomena already appear in the undeformed Fuchsian setting, while additional geometric analysis is required to transport these phenomena through bulging deformations.

1. The Fuchsian case and ambient dimension. We begin with the undeformed (Fuchsian) setting. Consider the decomposition

$$H = A_0 k_0 A_0 K_0,$$

where $k_0 \in K_0$ is the quarter turn, $A_0 = H \cap A$, and $K_0 = H \cap K$. For a cocompact Fuchsian subgroup $\Gamma_0 < H$, we show that if the orbit $\Gamma_0\backslash\Gamma_0 hA_0$ is admissible (see Theorem 7.2), then

$$\overline{\Gamma_0\backslash\Gamma_0 ha_t H} = \overline{\Gamma_0\backslash\Gamma_0 hA_0}\, a_t k_0 A_0 K_0,$$

and moreover

$$\frac{1}{2}\left(\dim(\overline{\Gamma_0\backslash\Gamma_0 hA_0}) + 3\right) \;\le\; \dim \overline{\Gamma_0\backslash\Gamma_0 ha_t H(o)} \;\le\; \dim(\overline{\Gamma_0\backslash\Gamma_0 hA_0}) + 1.$$

This provides an explicit description of orbit closures and their Hausdorff dimensions in the Fuchsian case. The main difficulty is dimension-theoretic and arises when passing from $\Gamma_0\backslash G$ to $\Gamma_0\backslash X$: while the natural product map is locally bi-Lipschitz on $\Gamma_0\backslash G$, the corresponding parametrization in $\Gamma_0\backslash X$ fails to be locally injective. Consequently, Hausdorff dimension in $\Gamma_0\backslash X$ does not follow formally from the geodesic-flow closure. To obtain the above Hausdorff dimension statements, we therefore impose an admissibility condition on the underlying geodesic orbit, ensuring sufficient regularity of its local projections. This Fuchsian analysis forms the baseline for the entire argument.

2. Nearest-point projection and higher-rank geometry. To transport the Fuchsian dimension statements to Zariski-dense deformations, a central geometric ingredient of the paper is a detailed analysis of the nearest-point projection

$$\pi : X \to Y.$$

We prove that for any complete geodesic $L \subset Y$,

$$d_{\mathrm{H}}(\pi(Y_{L,t}), L) \longrightarrow 0 \quad \text{as } t \to \infty.$$

In higher rank, however, the fibers of π contain entire maximal flats, giving rise to parallel families of geodesics. This substantially complicates the analysis of projection dynamics and boundary behavior. The argument relies on a careful study of Busemann functions $\beta_\xi|_Y$ for accumulation points ξ of $Y_{L,t}$ in $\partial_\infty X$, together with precise control of how sequences approach the visual boundary.

3. Stability under bulging. Let $F_{\mathbf{b}} : \Gamma_0\backslash X \to \Gamma_{\beta,\mathbf{b}}\backslash X$ denote the map induced by bulging. Fix r_0 strictly larger than the width $\mathrm{wd}(\mathbf{b})$ of $\mathbf{b}$, and set

$$X_{\mathbf{b}} := \{x \in X : \; d(\pi(x), \Gamma_0\beta) \geq r_0\}.$$

We prove that the restriction of $F_{\mathbf{b}}$ to $\Gamma_0\backslash X_{\mathbf{b}}$ is a proper local isometric embedding into $\Gamma_{\beta,\mathbf{b}}\backslash X$.

The neartest-point projection estimates from Step 2 imply that for large t, the projection of $Y_{L,t}$ remains close to its reference geodesic L, and hence uniformly away from the bulging locus. Thus $\Gamma_0\backslash\Gamma_0 Y_{L,t}$ lies in $\Gamma_0\backslash X_{\mathbf{b}}$ for $\mathbf{b}$ of sufficiently small width. Because fibers of π contain maximal flats, bulging can in principle cause different fibers to overlap. Showing that they remain disjoint under sufficiently small deformations is therefore a genuinely higher-rank phenomenon. A careful analysis of how bulging modifies these fibers yields the required stability and allows us to transfer the orbit-closure and ambient dimension statements from the Fuchsian case to all $\mathbf{b} \in C_G(\delta)^\circ$ with sufficiently small width.

4. Fractal geodesic closures away from the bulging locus. Finally, we construct *admissible* geodesics in closed hyperbolic surfaces whose closures have Hausdorff dimension arbitrarily close to 1, while remaining uniformly separated from the bulging locus. This construction uses Sullivan's ergodicity theorem for the Bowen–Margulis–Sullivan measure on convex cocompact surfaces [25].

Combining this with the existence of Zariski-dense Hitchin subgroups $\Gamma < \mathrm{SL}_3(\mathbb{Z})$ due to Long–Thistlethwaite [17], and applying the above deformation procedure, produces the arithmetic examples required in Theorem 1.2.

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2. Limit sets in G/P

Let $G = \mathrm{SL}_3(\mathbb{R})$. Let

$$\Theta : G \to G \tag{2.1}$$

be the Cartan involution given by $\Theta(g) = (g^T)^{-1}$ for $g \in G$. Let $K < G$ be the maximal compact subgroup:

$$K = \mathrm{SO}(3) = \{g \in G : \Theta(g) = g\}.$$

Let $A < G$ be the diagonal subgroup:

$$A = \{\mathrm{diag}(a_1, a_2, a_3) \in G : a_1, a_2, a_3 > 0\}.$$

Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R}) = \{x \in \mathrm{M}_3(\mathbb{R}) : \mathrm{Tr}\, x = 0\}$, $\mathfrak{k} = \mathfrak{so}_3 = \{x \in \mathfrak{sl}_3(\mathbb{R}) : x = -x^T\}$ and $\mathfrak{a}$ denote the Lie algebra of G, K and A, respectively. We may identify $\mathfrak{a}$ with the hyperplane

$$\mathfrak{a} = \{u = (u_1, u_2, u_3) \in \mathbb{R}^3 : u_1 + u_2 + u_3 = 0\}.$$

Let α_1 and α_2 be the simple roots of $(\mathfrak{g}, \mathfrak{a})$ given by

$$\alpha_i(u_1, u_2, u_3) = u_i - u_{i+1} \quad i = 1, 2.$$

Let $\mathfrak{a}^+$ denote the positive Weyl chamber

$$\mathfrak{a}^+ = \{(u_1, u_2, u_3) \in \mathfrak{a} : u_1 \geq u_2 \geq u_3\}.$$

and set

$$A^+ = \exp \mathfrak{a}^+.$$

The Killing form on $\mathfrak{g}$ is

$$\mathrm{B}(x, y) = 6\, \mathrm{Tr}(xy), \quad x, y \in \mathfrak{sl}_3(\mathbb{R}),$$

which induces the inner product on $\mathfrak{a}$: for $u, v \in \mathfrak{a}$,

$$\langle u, v\rangle = 6(u_1v_1 + u_2v_2 + u_3v_3).$$

We denote by $\|\cdot\|$ the corresponding norm on $\mathfrak{a}$.

Let $X = G/K$ be the Riemannian symmetric space equipped with the metric d_X induced by the Killing form. It is a nonpositively curved CAT (0)-space [5]. We also consider a left G-invariant and right K-invariant metric d_G on G compatible with d_X. We will omit subscripts and write d for both metrics. Let $o = [K] \in X$. We have the Cartan decomposition

$$G = KA^+K,$$

which says that for any $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$ such that $g \in K \exp \mu(g)\, K$. The map

$$\mu : G \to \mathfrak{a}^+$$

is called the Cartan projection.

Let P be the upper triangular subgroup of G

$$P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix},$$

which is a minimal parabolic subgroup of G. Then $P = MAN$ where

$$M = \{\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in G : \varepsilon_i \in \{1, -1\}, i = 1, 2, 3\}$$

is the centralizer of A in K and N is the strictly upper triangular subgroup.

There are two maximal parabolic subgroups of G containing P:

$$P_1 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}.$$

Denote by e_1, e_2, e_3 the standard column vectors of $\mathbb{R}^3$. The group G acts transitively on the projective space $\mathbb{R}P^2$, and the stabilizer of the point $[e_1] \in \mathbb{R}P^2$ is P_1. Therefore G/P_1 can be identified with $\mathbb{R}P^2$. Similarly, the stabilizer of the line $[e_1 \wedge e_2]$ in $\mathbb{R}P^2$ is P_2 and G/P_2 can be identified with the space of lines in $\mathbb{R}P^2$.

Since $P = P_1 \cap P_2$, the map $gP \mapsto (gP_1, gP_2)$ defines an embedding

$$G/P \hookrightarrow G/P_1 \times G/P_2, \tag{2.2}$$

via which we identify G/P with its image. Thus

$$G/P$$

is the full flag variety in $\mathbb{R}^3$: a point $\xi \in G/P$ is a pair (p, ℓ) where $p \in \mathbb{R}P^2$ is a point and ℓ is a line in $\mathbb{R}P^2$ containing p.

We now define convergence to points in G/P for sequences in G and X.

Definition 2.1. Let $g_n \in G$ be a sequence.

- We say $g_n \to \infty$ regularly if $\alpha_i(\mu(g_n)) \to \infty$ for both $i = 1, 2$.
- We say $g_n \to \xi \in G/P$ if $g_n \to \infty$ regularly and $\xi = \lim_{n\to\infty} k_n P$ where $k_n \in G$ is a sequence such that $g_n \in k_n A^+ K$.
- We say $g_n \to \infty$ *uniformly regularly* if $g_n \to \infty$ regularly and there exists $c > 0$ such that for each $i = 1, 2$,

$$\alpha_i(\mu(g_n)) \geq c\|\mu(g_n)\| \quad \text{for all } n \in \mathbb{N}.$$

- For a sequence $x_n = g_n o \in X$, we say $x_n \to \infty$ regularly (resp. uniformly regularly) if $g_n \to \infty$ regularly (resp. uniformly regularly) and that $x_n \to \xi \in G/P$ if $g_n \to \xi$.

By the Cartan decomposition $G = KA^+K$, these notions are all well-defined.

Definition 2.2 (Limit sets)**.** For $Z \subset G$, the limit set Λ_Z is the set of all accumulation points of sequences from Z in G/P. Similarly, if $Z \subset X$, the limit set Λ_Z is the set of all accumulation points of sequences from Z in

G/P. The projection of Λ_Z to G/P_i via (2.2) will be referred to as the limit set of Z on G/P_i for $i = 1, 2$.

Lemma 2.3. *If $Z_1, Z_2 \subset X$ have bounded Hausdorff distance, then $\Lambda_{Z_1} = \Lambda_{Z_2}$ in G/P.*

Proof. Suppose that the Hausdorff distance between Z_1 and Z_2 is at most R. Let $\xi \in \Lambda_{Z_1}$. Then some sequence $g_i o \in Z_1$ converges to ξ. There exists a sequence $q_i \in G$ with $d(q_i o, o) \leq R$ such that $g_i q_i o \in Z_2$. By [16, Lemma 2.10], $g_i q_i o \to \xi$. Hence $\xi \in \Lambda_{Z_2}$. Reversing the role of Z_1 and Z_2 gives the claim. ☐

3. Nearest projection to Y

Fix the quadratic form F in $\mathbb{R}^3$:

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2xz - y^2.$$

Let H be the identity component of the special linear group

$$\mathrm{SO}(F) = \left\{ g \in G : F\left(g(v)\right) = F(v) \text{ for all } v \in \mathbb{R}^3 \right\}.$$

For the symmetric matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.1}$$

we have

$$H = \{g \in G : g^T J g = J\}^\circ.$$

In particular, $H \simeq \mathrm{SO}(1,2)^\circ$.

The Lie algebra of H is

$$\mathfrak{h} = \left\{ \begin{pmatrix} s & x & 0 \\ y & 0 & x \\ 0 & y & -s \end{pmatrix} : x, y, s \in \mathbb{R} \right\}. \tag{3.2}$$

Let $\mathcal{S}$ be the space of all symmetric matrices of signature $(1, 2)$ with determinant one. The action of G on $\mathcal{S}$ by $g.J_0 = gJ_0g^T$ identifies $G.J \simeq G/\mathrm{SO}(F) \simeq \mathcal{S}$. A non-degenerate quadric in $\mathbb{R}P^2$ is a projective circle (ellipses, hyperbolas, paraboloids). The map

$$g\,\mathrm{SO}(F) \mapsto \{F \circ g = 0\} \tag{3.3}$$

identifies $G/\mathrm{SO}(F)$ with the space of non-degenerate quadrics in $\mathbb{R}P^2$ [12].

For $s \in \mathbb{R}$, set

$$h_s = \begin{pmatrix} e^s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-s} \end{pmatrix} \in H.$$

Note that h_s is a regular semisimple element, i.e., its centralizer is the diagonal subgroup, and that $h_s \to \infty$ regularly as $|s| \to \infty$. These are important facts which will be used often. We set

$$A_0 := H \cap A = \{h_s : s \in \mathbb{R}\} \quad \text{and} \quad K_0 := H \cap K.$$

The Lie algebra of K_0 is

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

Setting

$$k(\theta) := \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sqrt{2}\sin\theta}{2} & \frac{1-\cos\theta}{2} \\ \frac{\sqrt{2}\sin\theta}{2} & \cos\theta & -\frac{\sqrt{2}\sin\theta}{2} \\ \frac{1-\cos\theta}{2} & \frac{\sqrt{2}\sin\theta}{2} & \frac{1+\cos\theta}{2} \end{pmatrix}, \tag{3.4}$$

we get the parametrization

$$K_0 = \{k(\theta) : \theta \in \mathbb{R}\}.$$

The subgroup K_0 is a maximal compact subgroup of H and we have

$$H = K_0 A_0^+ K_0 \tag{3.5}$$

where $A_0^+ = \{h_s : s \geq 0\}$.

The limit set of Y. The quotient space

$$Y := H/K_0 = K_0 A_0 o$$

is a totally geodesic subspace of X. The quadric $\{F = 0\} = \{y^2 = 2xz\}$ divides $\mathbb{R}P^2$ into two H-invariant connected components: an open disk

$$D = \{[x : y : z] \in \mathbb{R}P^2 : y^2 < 2xz\} \tag{3.6}$$

and an open Möbius band

$$\{[x : y : z] \in \mathbb{R}P^2 : y^2 > 2xz\}. \tag{3.7}$$

Lemma 3.1. *We have*

$$\Lambda_Y = K_0 P/P \quad \textit{and}$$

$$\Lambda_Y = \{(p, \ell) \in G/P_1 \times G/P_2 : p \in \partial D,\ \ell \textit{ is tangent to } \partial D \textit{ at } p\}.$$

Proof. Since $h_s \to \infty$ regularly as $|s| \to \infty$, any unbounded sequence in Y has a subsequence which converges to a point in G/P. Moreover, in view of (3.5), any limit of an infinite sequence in H in G/P belongs to $K_0 P/P$. Hence $\Lambda_Y = K_0 P/P$. Note that P corresponds to the pair $(p_0, \ell_0) \in G/P_1 \times G/P_2$, where $p_0 = [e_1]$ and $\ell_0 = [e_1 \wedge e_2] = \{[x : y : z] : \ z = 0\}$. Moreover, $\ell_0 \cap \partial D = \{p_0\}$ and hence ℓ_0 is tangent to ∂D at p_0. Since ∂D is a single K_0-orbit, the claim follows. ∎

We fix the element

$$k_0 := \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \in K_0. \tag{3.8}$$

The geodesics $k_0 A_0 o$ and $A_0 o$ in Y are orthogonal. Therefore Y is swept out by the family of orthogonal geodesics to $A_0 o$; indeed,

$$Y = A_0 k_0 A_0 o.$$

It follows that

$$H = A_0 k_0 A_0 K_0.$$

Generalized Cartan decomposition. Recalling the symmetric matrix J from (3.1), consider the following involution $\sigma : G \to G$:

$$\sigma(g) = J\Theta(g)J.$$

We then have

$$H = \{g \in G : \sigma(g) = g\}^\circ;$$

therefore H is an affine symmetric subgroup of G. The generalized Cartan decomposition of G with respect to H is described in [24], as we recall below.

Observe that the differential $d\sigma : \mathfrak{g} \to \mathfrak{g}$ commutes with $d\Theta$, and we have

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$$

which are decompositions into $\pm$ eigenspaces for $d\Theta$ and σ, respectively. The subspace

$$\mathfrak{b} := \left\{ \begin{pmatrix} t & 0 & s \\ 0 & -2t & 0 \\ s & 0 & t \end{pmatrix} : t, s \in \mathbb{R} \right\} \tag{3.9}$$

is a maximal abelian subalgebra of $\mathfrak{p} \cap \mathfrak{q}$. It is also a maximal abelian subalgebra of of $\mathfrak{p}$, since the rank of G is 2. The maximal split torus $B := \exp \mathfrak{b}$ of G is

$$B = \left\{ \begin{pmatrix} e^t \cosh s & 0 & e^t \sinh s \\ 0 & e^{-2t} & 0 \\ e^t \sinh s & 0 & e^t \cosh s \end{pmatrix} : t, s \in \mathbb{R} \right\}.$$

We set

$$k_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \in K. \tag{3.10}$$

We then have

$$\mathfrak{b} = k_1 \mathfrak{a} k_1^{-1} \quad \text{and} \quad B = k_1 A k_1^{-1}.$$

Letting $\mathcal{W}_\sigma = N_K(\mathfrak{b})/C_K(\mathfrak{b})$ and $\mathcal{W}_{\sigma,\theta} = N_{K_0}(\mathfrak{b})/C_{K_0}(\mathfrak{b})$, choose a finite subset

$$\mathcal{W} \subset N_K(\mathfrak{b}) \tag{3.11}$$

of representatives for $\mathcal{W}_{\sigma,\theta} \backslash \mathcal{W}_\sigma$. Set $B^+ = k_1 A^+ k_1^{-1}$.

Theorem 3.2. *We have*

$$G = HBK = H\mathcal{W}B^+K$$

in the sense that for any $g \in G$, there exist unique elements $b \in B^+$ and $w \in \mathcal{W}$ such that

$$g \in HwbK.$$

Nearest point projection to Y. Since $X = G/K$ is non-positively curved and $Y \subset X$ is totally geodesic, for any $x \in X$, there exists a unique $y \in Y$ such that

$$d(x, y) = \inf_{y' \in Y} d(x, y').$$

This y is called the *nearest point projection* of x to Y. See [2, p. 8] for further details.

Let

$$\pi : X \to Y \tag{3.12}$$

be the nearest projection map. Since $X = HBo$, by Theorem 3.2, we have the following description of π:

Proposition 3.3. *For any $h \in H$ and $b \in B$,*

$$\pi(hbo) = ho. \tag{3.13}$$

Proof. Since π is H-equivariant, it suffices to show $\pi(z) = o$ for all $z \in Bo$. Since $\mathrm{Tr}(xy) = 0$ for all $x \in \mathfrak{h}$ and $y \in \mathfrak{b}$, the subspaces $\mathfrak{h}$ and $\mathfrak{b}$ are orthogonal to each other. Moreover, Bo and Y are totally geodesic. Thus, for $z \in Bo$, the geodesic segment connecting z to o lies in Bo and is orthogonal to Y, $\pi(z) = o$. □

Corollary 3.4. *We have*

$$\pi^{-1}(o) = K_0Bo = \bigcup_{w \in \mathcal{W}} K_0k_1wA^+o$$

and

$$\Lambda_{\pi^{-1}(o)} = \bigcup_{w \in \mathcal{W}} K_0k_1wP$$

where k_1 and $\mathcal{W}$ are given in (3.10) and (3.11) respectively.

Proof. The first part follows from Proposition 3.3 and Theorem 3.2. The second follows from the definition of the limit set and the fact that $K_0k_1\mathcal{W} \subset K$. □

4. Floating geodesic planes

As in the last section, consider the totally geodesic plane

$$Y = Ho \subset X.$$

Since the centralizer of A_0 is equal to the diagonal subgroup of G, it follows that Ao is the unique maximal flat in X containing the geodesic A_0o.

Lemma 4.1. *The geodesic plane Y is perpendicular to the flat Ao and $Y \cap Ao = A_0 o$.*

Proof. The intersection of Y and Ao is a totally geodesic submanifold of X, which contains $A_0 o$. Since neither Y nor Ao contains the other, dimensional considerations imply that the intersection is precisely $A_0 o$.

The Lie algebra $\mathfrak{a}$ splits orthogonally as a direct sum $\mathfrak{a}_0 \oplus \mathfrak{a}'$ where $\mathfrak{a}_0 = \mathrm{Lie}(A_0)$ and $\mathfrak{a}' = \{\mathrm{diag}(t, -2t, t) : t \in \mathbb{R}\}$. Since $\mathfrak{a}_0 \subset \mathfrak{h}$ and $\mathfrak{a}' \perp \mathfrak{h}$, the claim about orthogonality thus follows. $\square$

For $t \in \mathbb{R}$, set

$$a_t = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Definition 4.2. Given a complete geodesic L in Y and $t \in \mathbb{R}$, define the *floating* geodesic plane

$$Y_{L,t} := h a_t Y,$$

where $h \in H$ is chosen so that $L = hA_0 o$; this is well-defined since h is unique modulo the action of A_0, which commutes with a_t.

For the geodesic $L = A_0 o$, we simply write

$$Y_t := Y_{L,t} = a_t Y.$$

Lemma 4.3. *For $L = hA_0 o$ and $t \in \mathbb{R}$, let $L_t := h a_t A_0 o \subset Y_{L,t}$. We have $\pi(L_t) = L$ and $d(hh_r o, h a_t h_r o) = |t| = d(L, L_t)$ for all $r \in \mathbb{R}$.*

Proof. By the H-equivariance of π, it suffices to consider the case when $L = A_0 o$. Since $a_t A_0 = A_0 a_t$ and $\pi(a_t o) = o$, we get $\pi(a_t A_0 o) = A_0 \pi(a_t o) = A_0 o$. Similarly, $d(h_r o, a_t h_r o) = d(h_r o, h_r a_t o) = d(o, a_t o) = |t|$. Since L and L_t lies in the same flat Ao, and L and $\{a_t o : t \in \mathbb{R}\}$ are orthogonal at o, $d(L, L_t) = d(o, a_t o) = |t|$. $\square$

The geodesic plane $Y_{L,t}$ is "ultra-parallel" to Y at distance t:

Lemma 4.4. *Given a geodesic $L = hA_0 o \subset Y$ and $t \in \mathbb{R}$, we have*

$$d(Y, Y_{L,t}) = \min\{d(y,z) : \ y \in Y,\ z \in Y_{L,t}\} = |t|.$$

Moreover, for $t \neq 0$, the locus where the distance is minimized is precisely $\{(hh_r o, h a_t h_r o) : \ hh_r o \in L\}$.

Proof. By the H-equivariance, it suffices to prove the claim when $L = A_0 o$ and $Y_{L,t} = a_t Y$. Let $y \in Y$ and $z \in Y_t$. Let $y_0 \in L$ (resp. $z_0 \in L_t = a_t L$) denote the nearest point projection of y (resp. z) to L (resp. $a_t L$). Let L_0 be the geodesic segment connecting y_0 to z_0. Since the flat Ao is orthogonal to Y and $a_t \in A$, Ao is also orthogonal to Y_t. It follows that the points y and z lie on two geodesics perpendicular to L_0 and passing through its endpoints.

By [1, Ch. I, Prop. 5.4], in a non-positively curved space, for any geodesic segment $[a, b]$ and for any perpendicular complete geodesics $L_1(t)$ and $L_2(t)$ to $[a, b]$ with $L_1(0) = a$ and $L_2(0) = b$, we have

$$\inf\{d(L_1(t_1), L_2(t_2)) : \ t_1 \in \mathbb{R}, t_2 \in \mathbb{R}\} = d(a, b).$$

Since the geodesic segment $[y, y_0]$ lies in Y and is perpendicular to L, it is perpendicular to the whole maximal flat Ao. Similarly, $[z, z_0]$ is also perpendicular to Ao. So

$$|t| \leq d(y_0, z_0) \leq d(y, z). \tag{4.1}$$

This proves the first claim.

For the second claim, without loss of generality, assume that $t > 0$ and suppose that $d(y, z) = t$. It suffices to show that $y = y_0$, $z = z_0$ and $z_0 = a_t y_0$. The inequality (4.1) forces $t = d(y_0, z_0)$. Hence by Lemma 4.3, $d(L, a_t L) = t = d(y_0, z_0)$. Therefore the geodesic segment $[y_0, z_0]$ is perpendicular to L and $a_t L$. Since a_t translates Ao orthogonal to L, it implies that $z_0 = a_t y_0$.

We now claim that

$$y = y_0 \quad \text{and} \quad z = z_0. \tag{4.2}$$

First suppose that $\{y_0, z_0\} \cap \{y, z\} = \emptyset$. Two complete geodesics in X are either parallel (they have finite Hausdorff distance) or the minimum distance between them is realized by a unique pair of points or the minimum distance is not realized. (This follows from [1, Ch. I, Prop. 5.4].) Hence in the setting at hand, the complete geodesics $\mathcal{G}_1$ and $\mathcal{G}_2$ in X passing through y, y_0 and z, z_0 respectively must have a finite Hausdorff distance. Since $A_0\mathcal{G}_1 = Y$ and $A_0\mathcal{G}_2 = Y_t$, Y and Y_t are at a finite Hausdorff distance. Thus, the limit sets of Y and Y_t in $G/P_1 = \mathbb{R}P^2$ are the same by Lemma 2.3. Since $G/\operatorname{SO}(F)$ is in bijection with the space of non-degenerate quadrics via the map in (3.3) it follows that $Y_t = Y$. Hence $t = 0$, a contradiction.

Now suppose that $\{y_0, z_0\} \cap \{y, z\}$ is a singleton. Without loss of generality, we may assume that $y = y_0$ and $z \neq z_0$. Since the distance function is strictly convex in a Hadamard manifold [2, Sec. 1.4], any point z_1 in the geodesic segment $[z, z_0]$ other than the endpoints satisfies $d(y, z_1) < d(y, z) = t$, which is a contradiction to $t = d(\mathcal{G}_1, \mathcal{G}_2)$. Therefore $\{y_0, z_0\} = \{y, z\}$, and consequently $y_0 = y$ and $z_0 = z$, proving (4.2). □

Remark 4.5. Up to an isometry, any totally geodesic plane in X is given by Y, or $\mathrm{SL}_2(\mathbb{R})o$, where $\mathrm{SL}_2(\mathbb{R})$ is embedded as the left upper corner of $\mathrm{SL}_3(\mathbb{R})$, or a maximal flat $A(o)$. It is natural to call first type of geodesic planes as *irreducible* geodesic planes.

The limit set of the floating planes. For each $t \in \mathbb{R}$, consider the quadric

$$Q_t = \{[x : y : z] \in \mathbb{R}P^2 : e^{4t} y^2 = 2e^{-2t} xz\}$$

passing through $[e_1]$ and $[e_3]$. This is a projective circle.

Since a_t sends the boundary of the disk $D = \{[x : y : z] \in \mathbb{R}P^2 : y^2 < 2xz\}$ to Q_t, Lemma 3.1 implies:

Lemma 4.6. *For $t \in \mathbb{R}$, the limit set of $a_t Y$ in G/P is given by*

$$\Lambda_{a_t Y} = \{(p, \ell) \in G/P : p \in Q_t, \ell \text{ is a line tangent to } Q_t \text{ at } p\}.$$

5. Limits of the sequence $\gamma_t(s) = a_t k_0 h_s o$

Since $Y = A_0 k_0 A_0 o$, where $k_0 \in K_0$ is as in (3.8), and the nearest projection map π defined in (3.12) is H-equivariant, we have

$$\pi(a_t Y) = A_0 \pi(a_t k_0 A_0 o) \quad \text{for any } t \in \mathbb{R}.$$

Therefore to understand the image $\pi(a_t Y)$, it suffices to analyze the sequence

$$\gamma_t(s) := a_t k_0 h_s o. \tag{5.1}$$

In this section we determine all accumulation points of $\gamma_t(s)$ in G/P, according to the relative rates at which t and s tend to ∞.

The main goal in this section is to show:

Proposition 5.1. *Any accumulation of the sequence $\gamma_t(s)$ in G/P as $t, |s| \to \infty$ belongs to $\Lambda_{\pi^{-1}(o)}$. Moreover, we have $\liminf |s_n|/t_n > 0$ if and only if $\gamma_{t_n}(s_n) \to \infty$ uniformly regularly.*

We begin with calculating the Cartan projection of such a sequence up to a uniform bounded subset:

Lemma 5.2. *There is a compact subset $C \subset \mathfrak{a}$ such that for any $t > 0$ and $s \in \mathbb{R}$, the Cartan projection $\mu(a_t k_0 h_s)$ satisfies*

$$\mu(a_t k_0 h_s) \in \begin{pmatrix} t + |s| & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & -2t - |s| \end{pmatrix} + C.$$

Proof. For simplicity, set $g = a_t k_0 h_s$. Write $g = ka\ell \in KA^+K$ in Cartan decomposition so that $\mu(g) = \log a$, where $a = \operatorname{diag}(a_1, a_2, a_3)$. We estimate the a_i up to a uniform multiplicative constant.

Since $gg^T = ka^2k^{-1}$, the eigenvalues of gg^T determine the a_i^2. Let $c_s = e^s + e^{-s}$ and $d_s = e^s - e^{-s}$. A direct computation gives

$$gg^T = \frac{1}{4} \begin{pmatrix} e^{2t} c_s^2 & -\sqrt{2} e^{-t} c_s d_s & e^{2t} d_s^2 \\ -\sqrt{2} e^{-t} c_s d_s & 2e^{-4t}(e^{2s} + e^{-2s}) & -\sqrt{2} e^{-t} c_s d_s \\ e^{2t} d_s^2 & -\sqrt{2} e^{-t} c_s d_s & e^{2t} c_s^2 \end{pmatrix}.$$

Since $t > 0$ and $|d_s| \le c_s$, the Frobenius norm of gg^T satisfies

$$\|gg^T\|^2 \asymp e^{4t} c_s^4.$$

Here, $\asymp$ denotes equality up to a uniform multiplicative constant. As $\|gg^T\|^2 = a_1^4$ and $c_s \asymp e^{|s|}$, we get

$$a_1 \asymp e^t e^{|s|}. \tag{5.2}$$

For a_2, we now compute $\wedge^2(gg^T)$ with respect to the ordered basis $e_2 \wedge e_3, e_1 \wedge e_3, e_1 \wedge e_2$:

$$\wedge^2(gg^T) = \frac{1}{4}\begin{pmatrix} e^{-2t}c_s^2 & -\sqrt{2}e^t c_s d_s & e^{-2t}d_s^2 \\ -\sqrt{2}e^t c_s d_s & 2e^{4t}(e^{2s}+e^{-2s}) & -\sqrt{2}e^t c_s d_s \\ e^{-2t}d_s^2 & -\sqrt{2}e^t c_s d_s & e^{-2t}c_s^2 \end{pmatrix}.$$

Hence

$$\| \wedge^2 (gg^T)\|^2 \asymp e^{8t}e^{4|s|}.$$

Since the exponential of the Cartan projection of $\wedge^2(gg^T)$ has entries $a_1^2a_2^2, a_2^2a_3^2, a_1^2a_3^2$ with the largest one being $a_1^2a_2^2$, we have

$$\| \wedge^2 (gg^T)\|^2 \asymp a_1^4a_2^4 = e^{8t}e^{4|s|},$$

which with (5.2) implies

$$a_2 \asymp e^t.$$

This proves the claim. □

A regularity criterion. A consequence of Lemma 5.2 is as follows:

Corollary 5.3. *Let $\gamma_{t_n}(s_n) = a_{t_n}k_0h_{s_n}o$ with $t_n > 0$ and $s_n \in \mathbb{R}$. Then we have*

- *$\gamma_{t_n}(s_n) \to \infty$ regularly if and only if $|s_n| \to \infty$;*
- *$\gamma_{t_n}(s_n) \to \infty$ uniformly regularly if and only if $|s_n| \to \infty$ and*

$$\liminf_n |s_n|/t_n > 0.$$

Recall that $G/P = KP/P$, identified with

$$\{([ke_1], [k(e_1 \wedge e_2)]) : k \in K\} \subset G/P_1 \times G/P_2.$$

Proposition 5.1 follows from the following together with Corollary 5.3:

Proposition 5.4 (Limits of $\gamma_t(s)$)**.** *If $t_n \to \infty$ and $|s_n| \to \infty$, then any limit of $\gamma_{t_n}(s_n)$ in G/P belongs to K_0k_1P/P, where k_1 is as in (3.10). In particular, if $\zeta = k^*P$ is such a limit for some $k^* \in K$, then k^*e_1 is proportional to $e_1 + e_3$ and*

$$\zeta \in \Lambda_{\pi^{-1}(o)}.$$

Proof. Write $g_n = a_{t_n}k_0h_{s_n} = k_na_nl_n \in KA^+K$ and $a_n = \operatorname{diag}(a_{n,1}, a_{n,2}, a_{n,3})$. By Lemma 5.2, as $|s_n| \to \infty$, we have $xg_n \to \infty$ regularly. Suppose $k_n \to k^*$. Using the notation $k(\theta)$ from (3.4), in order to prove that $k^* \in K_0k_1P = \{k(\theta)k_1 : \theta \in \mathbb{R}\}P$, it suffices to show that the first two columns of k^* are proportional to $(1,0,1)^T$ and $(-\sin\theta, \sqrt{2}\cos\theta, \sin\theta)^T$ for some $\theta \in \mathbb{R}$.

Let

$$w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then $w^T g_n g_n^T w = w^T k_n a_n^2 (w^T k_n)^T$, so the columns of $w^T k_n$ are eigenvectors of $w^T g_n g_n^T w$ in decreasing order of eigenvalue. For each $i = 1, 2, 3$, write the i-th column vector

$$u_{n,i} = w^T k_n e_i = x_{n,i} e_1 + y_{n,i} e_2 + z_{n,i} e_3.$$

We show that the first column of $w^T k^*$ is parallel to $(-1, 1, 0)^T$ and the second column of $w^T k^*$ is parallel to $(p, p, q)^T$ for some $p, q \in \mathbb{R}$. This implies the required structure of k^*.

Since $u_{n,i}$ are unit vectors, all $|x_{n,i}|, |y_{n,i}|, |z_{n,i}|$ are at most 1. Let $c_n = e^{s_n} + e^{-s_n}$ and $d_n = e^{s_n} - e^{-s_n}$. A direct computation gives

$$Q_n := w^T g_n g_n^T w = \frac{1}{4} \begin{pmatrix} e^{2t_n} c_n^2 & -e^{2t_n} d_n^2 & -\sqrt{2} e^{-t_n} c_s d_n \\ -e^{2t_n} d_n^2 & e^{2t_n} c_n^2 & \sqrt{2} e^{-t_n} c_n d_n \\ -\sqrt{2} e^{-t_n} c_n d_n & \sqrt{2} e^{-t_n} c_n d_n & 2e^{-4t_n}(e^{2s_n} + e^{-2s_n}) \end{pmatrix}$$

Setting $f_n = d_n / c_n$, we compute

$$\begin{aligned} Q_n u_{n,1} &= (w^T k_n) a_n^2 (w^T k_n)^T u_{n,1} = (w^T k_n) a_n^2 e_1 \\ &= a_{n,1}^2 u_{n,1} = a_{n,1}^2 (x_{n,1} e_1 + y_{n,1} e_2 + z_{n,1} e_3) \end{aligned}$$

On the other hand,

$$Q_n u_{n,1} = x_{n,1} Q_n e_1 + y_{n,1} Q_n e_2 + z_{n,1} Q_n e_3 =$$

$$\frac{e^{2t_n} c_n^2}{4} \left(x_{n,1} \begin{pmatrix} 1 \\ -f_n^2 \\ \frac{-\sqrt{2} f_n}{e^{3t_n}} \end{pmatrix} + y_{n,1} \begin{pmatrix} -f_n^2 \\ 1 \\ \frac{\sqrt{2} f_n}{e^{3t_n}} \end{pmatrix} + z_{n,1} \begin{pmatrix} \frac{-\sqrt{2} f_n}{e^{3t_n}} \\ \frac{\sqrt{2} f_n}{e^{3t_n}} \\ \frac{2(e^{2s_n} + e^{-2s_n})}{e^{6t_n} c_n^2} \end{pmatrix} \right).$$

Hence

$$\frac{4a_{n,1}^2}{e^{2t_n} c_n^2} (x_{n,1} e_1 + y_{n,1} e_2 + z_{n,1} e_3) =$$

$$x_{n,1} \begin{pmatrix} 1 \\ -f_n^2 \\ \frac{-\sqrt{2} f_n}{e^{3t_n}} \end{pmatrix} + y_{n,1} \begin{pmatrix} -f_n^2 \\ 1 \\ \frac{\sqrt{2} f_n}{e^{3t_n}} \end{pmatrix} + z_{n,1} \begin{pmatrix} \frac{-\sqrt{2} f_n}{e^{3t_n}} \\ \frac{\sqrt{2} f_n}{e^{3t_n}} \\ \frac{2(e^{2s_n} + e^{-2s_n})}{e^{6t_n} c_n^2} \end{pmatrix}$$

By Lemma 5.2, we may assume $r_0 := \lim_{t,s \to +\infty} \frac{4a_{n,1}^2}{e^{2t_n} c_n^2} > 0$ exists, after passing to a subsequence. Indeed,

$$r_0 = 2$$

because $a_{n,1}^2$ is the largest eigenvalue of $w^T g_n g_n^T w$. Since $\lim_{n \to \infty} f_n = 1$, taking the limit of the above equation yields

$$\begin{aligned} &2(\lim x_{n,1} e_1 + \lim y_{n,1} e_2 + \lim z_{n,1} e_3) \\ &\qquad = \lim x_{n,1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lim y_{n,1} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lim z_{n,1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Comparing the e_3-components gives

$$\lim z_{n,1} = 0.$$

Comparing the e_1 and e_2-components gives

$$\lim y_{n,1} = -\lim x_{n,1}.$$

Thus $\lim u_{n,1}$ is parallel to $(1,-1,0)^T$, that is, the first column of $w^T k^*$ is parallel to $(-1,1,0)^T$. Since each $u_{n,2}$ is orthogonal to $u_{n,1}$, the limit $\lim u_{n,2}$ must be orthogonal to $(-1,1,0)^T$ and hence of the form $(p,p,q)^T$. This proves the claim about the second column of $w^T k^*$. □

6. Nearest projection of floating planes to Y

Let $\pi : X \to Y$ be the nearest projection map. The main result of this section is as follows: Fix a complete geodesic $L \subset Y$ and $t \in \mathbb{R}$. Let $Y_{L,t}$ be the associated floating plane. Then:

Theorem 6.1. *The Hausdorff distance between $\pi(Y_{L,t})$ and L tends to 0 as $|t| \to \infty$.*

By the H-equivariance of π, we may assume without loss of generality that

$$L = A_0(o) \quad \text{and hence } Y_{L,t} = a_t Y.$$

For simplicity, we set $Y_t := Y_{A_0 o,t}$.

Busemann functions. The visual boundary $\partial_\infty X$ consists of equivalence classes of asymptotic geodesic rays. (Recall that two geodesic rays in X are *asymptotic* if they are within a finite Hausdorff distance). We equip the visual boundary with the *cone topology.*

Definition 6.2. For $\xi \in \partial_\infty X$, the *Busemann function* $b_\xi : X \to \mathbb{R}$ is

$$b_\xi(x) = \lim_{t\to\infty} (d(x,\xi_t) - t)$$

where $\{\xi_t : t \geq 0\}$ is the unit speed geodesic ray in the class ξ such that $\xi_0 = o$. Since X is nonpositively curved, this is well-defined: there exists a unique unit speed geodesic ray from o representing each class.

The horofunction compactification of X is obtained by attaching the visual boundary $\partial_\infty X$. More precisely, for $x \in X$, define $d_x : X \to \mathbb{R}$ be given by

$$d_x(y) = d(y,x) - d(o,x).$$

If x_n is a sequence in X converging to $\xi \in \partial_\infty X$ (with ξ represented by a ray from o), then

$$d_{x_n} \to b_\xi$$

uniformly on compact sets of X [5, Chapter II.8].

The next lemma reduces the proof of Theorem 6.1 to controlling the Busemann functions $b_\xi|_Y$ at every accumulation point $\xi \in \partial_\infty X$ of the sequence $\gamma_t(s)$.

Lemma 6.3. *Let $\xi \in \partial_\infty X$ and $x_n \in X$ be a sequence converging to ξ in the visual topology. Suppose $Z \subset X$ is a closed convex set and $b_\xi|_Z$ has a unique minimum at $z_0 \in Z$. Then the nearest point projection map* $\mathrm{pr} : X \to Z$ *satisfies* $\mathrm{pr}(x_n) \to z_0$ *as* $n \to \infty$.

Proof. Without loss of generality, assume that $b_\xi|_Z(z_0) = 0$. For $r > 0$, let $B_r(z_0)$ be the closed ball of radius r. Let

$$\delta \coloneqq \min\{b_\xi(z) : \ z \in \partial B_r(z_0)\} > 0.$$

Since $d_{x_n} \to b_\gamma$ uniformly on compacts, for all large enough n, we have

$$d_{x_n}(z_0) \leq \delta/3, \quad d_{x_n}(z) \geq 2\delta/3 \text{ for all } z \in \partial B_r(z).$$

Since d_{x_n} is strictly convex along geodesics in X, for all $n \geq n_0$, d_{x_n} must achieve its unique minima in $B_r(z_0)$. Taking $r \to 0$, we finish the proof. □

Properness via relative position in G/P. Any unit-speed geodesic ray in X has the form $t \mapsto g\exp(tv)o$ for some $g \in G$ and a unit vector $v \in \mathfrak{a}^+$. The ray is called regular if $v \in \operatorname{int}\mathfrak{a}^+$ and singular otherwise. A point $\xi \in \partial_\infty X$ is called regular and asymptotic to $f \in G/P$ if it is represented by a regular geodesic ray $g\exp(tv)o$, $t \in [0,\infty)$ with $f = gP$. Denote the set of all regular points in $\partial_\infty X$ by $\partial_\infty^{\mathrm{reg}} X$. There is a well-defined map

$$f : \partial_\infty^{\mathrm{reg}} X \to G/P, \tag{6.1}$$

sending ξ to $f_\xi \coloneqq gP$.

The Weyl group $W = \{e, w_1, w_1', w_2, w_2', w_0\}$ of G is depicted in Figure 3. Here w_0 denotes the longest Weyl element so that $w_0 P w_0^{-1}$ is the lower

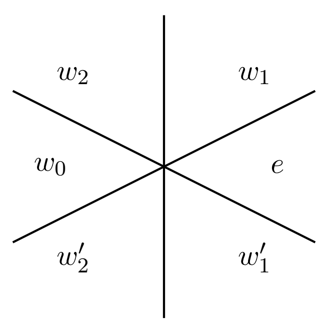

FIGURE 3. The Weyl group.

triangular subgroup. The Schubert cell decomposition of G/P:

$$G/P = \bigsqcup_{w \in W} PwP,$$

with the unique open P-orbit Pw_0P (a 3-cell); two 2-cells Pw_2P and $Pw_2'P$; two 1-cells Pw_1P and $Pw_1'P$; and the zero cell is P. Identifying G/P with the full flag variety $\{(p,\ell) : p \in \mathbb{R}P^2, \ell \subset \mathbb{R}P^2 \text{ a line }, p \in \ell\}$, the Schubert cells relative to a given $(p,\ell) \in G/P$ can be characterized as follows:

(1) 0-cell: $\{(p,\ell)\}$.
(2) 1-cells: $\{((p',\ell'):\ p=p' \text{ and } \ell\neq\ell'\}$, $\{((p',\ell'):\ p\neq p' \text{ and } \ell=\ell'\}$.
(3) 2-cells: $\{(p',\ell'):\ p\in\ell'-p'\}$, $\{(p',\ell'):\ p'\in\ell-p\}$.
(4) 3-cell: $\{(p',\ell'):\ p\notin\ell' \text{ and } p'\notin\ell\}$.

Definition 6.4. For $1\le k\le 3$, two points $\xi=(p,\ell),\xi'=(p',\ell')\in G/P$ are in *relative position* k if (p',ℓ') lies in a k-cell in the Schubert cell decomposition of G/P with respect to (p,ℓ), or equivalently, if there exists $g\in G$ such that $\xi=gP$ and $\xi=gwP$ for $w\in\{w_k,w_k'\}$ where we have put $w_3=w_3'=w_0$.

Since any geodesic ray in Y is regular, we have $\partial_\infty Y\subset\partial_\infty^{\text{reg}}X$, and $f(\partial_\infty Y)=\Lambda_Y=K_0P/P$.

Lemma 6.5. *Let $\xi\in\partial_\infty^{\text{reg}}X$. If f_ξ has relative position 2 or 3 with respect to every point of Λ_Y, then $b_\xi|_Y$ is proper and bounded below, and it attains a unique minimum in Y.*

Proof. Busemann functions are convex, so to prove properness and boundedness below, it suffices to show that $b_\xi\to\infty$ along every ray in Y issuing from o. Let $\{\xi_t : t\ge 0\}$ be a ray from o representing $\xi\in\partial_\infty X$, and let $\{\rho_t : t\ge 0\}$ be a ray in Y from o. Let $\rho\in\partial_\infty Y$ represented by it. By the hypothesis, there exists $g\in G$ such that we have $f_\rho=gP$ and $f_\xi=gwP$ for some $w\in\{w_2,w_2',w_0\}$. Consider the maximal flat $F:=gAo$ and its two Weyl chambers $W_-:=gwA^+o$ and $W_+:=gA^+o$ asymptotic to f_ξ and f_ρ, respectively. Set $x=go$. Let ξ' (resp. ρ') be the central ray in W_- (resp. W_+) from x, asymptotic to ξ (resp. ρ). Since ξ' and ξ are asymptotic, their Busemann functions $b_{\xi'}$ and b_ξ differ only by an additive constant. Moreover, since Busemann functions are 1-Lipschitz, showing that b_ξ goes to infinity along ρ is equivalent to showing that $b_{\xi'}$ goes to infinity along ρ'.

Note that the restriction $b_{\xi'}|_F$ is the Busemann function on the Euclidean plane F corresponding to the ray ξ', which must go to infinity along any ray which makes an angle strictly more that $\pi/2$ with ξ'. Since $w\in\{w_2,w_2',w_0\}$, W_- and W_+ are not adjacent and hence the angle between ρ' and ξ' is strictly bigger than $\pi/2$. Therefore b_ξ goes to infinity along ρ.

The second part of theclaim that $b_\xi|_Y$ has a unique minimum follows from the first part. Indeed, by the first part, $b_\xi|_Y$ has a minimum. If $b_\xi|_Y$ had two distinct minima $y_1,y_2\in Y$, then convexity would imply that b_ξ is constant along the geodesic segment in X joining y_1 and y_2. Since Busemann functions are real analytic, it would then follow that $b_\xi|_Y$ is constant along the complete bi-infinite geodesic extension of that segment, contradicting properness. □

Uniform regularity and properness.

Lemma 6.6. *If $t_n\to\infty$, $\gamma_{t_n}(s_n)=a_{t_n}k_0h_{s_n}o\to\infty$ uniformly regularly as in Def.* (2.1)*, then*
$$\pi(\gamma_{t_n}(s_n))\to o\quad \textit{as } n\to\infty.$$

Proof. Since $\gamma_{t_n(s_n)} \to \infty$ regularly, $|s_n| \to \infty$ by Corollary 5.3. After passing to a subsequence, we may assume that $\gamma_{t_n(s_n)}$ converges to some $\zeta \in G/P$ in the sense of Definition 2.1. By Proposition 5.4, $\zeta = (p_0, \ell_0)$, where $p_0 = [e_1 + e_3]$ and $\zeta \in \Lambda_{\pi^{-1}(o)}$.

The quadric $y^2 = 2xz$ splits $\mathbb{R}P^2$ into the disk $D = \{[x : y : z] : y^2 < 2xz\}$ and the Möbius strip $\{[x : y : z] : y^2 > 2xz\}$. Since by Lemma 3.1,

$$\Lambda_Y = \{(p, \ell) : p \in \partial D, \ell \text{ is tangent to } \partial D \text{ at } p\}$$

and $p \in D$, it follows that ζ has relative position 2 or 3 with respect to any point in Λ_Y. Since $\gamma_{t_n}(s_n) \to \infty$ uniformly regularly, the accumulation set of $\gamma_{t_n}(s_n)$ is a compact subset $C \subset \partial_\infty^{\text{reg}} X$ with $f(C) = \zeta$ (see (6.1)). By Lemma 6.5, for each $\xi \in C$, $b_\xi|_Y$ has a unique minimum in Y.

We claim that this minimum is achieved at o, which would finish the proof by Lemma 6.3. By Corollary 3.4,

$$\pi^{-1}(o) = \bigcup_{k \in K_0, w \in \mathcal{W}} k k_1 w A^+ o$$

is a union of Weyl chambers emanating from o. Let Δ be the Weyl chamber emanating from o and asymptotic to ζ. Since $\zeta \in \Lambda_{\pi^{-1}(o)}$, we have $\Delta \subset \pi^{-1}(o)$. After extraction, $\gamma_{t_n}(s_n)$ is asymptotic to a ray γ in the Weyl chamber Δ in $\pi^{-1}(o)$ emanating from o (i.e., $\gamma_{t_n}(s_n) \in X$ converges to $[\xi] \in \partial_\infty X$ in the compactification $X \sqcup \partial_\infty X$). This ray γ must be perpendicular to Y at o. Thus the Busemann function b_γ attains its minimum at o. ☐

Lemma 6.7. *If $t_n \to \infty$ and $\gamma_{t_n}(s_n)$ has no subsequence which tends to ∞ uniformly regularly, then*

$$d(L, \pi(\gamma_{t_n}(s_n))) \to 0 \quad \text{as } n \to \infty.$$

Proof. By Corollary 5.3, we have $|s_n|/t_n \to 0$ as $n \to \infty$. In this case, $\gamma_{t_n}(s_n)$ converges to the ray $\xi := \{a_t o : t \geq 0\}$ in the visual topology: to see this, we consider the right-angled triangle $\triangle(o, a_{t_n} o, \gamma_{t_n}(s_n))$. Since X is a CAT(0)-space, this triangle is thinner than a euclidean triangle with the same side lengths,

$$\angle_o(a_{t_n} o, \gamma_{t_n}(s_n)) \leq \tan^{-1} \frac{|s_n|}{t_n} \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\gamma_{t_n}(s_n)$ converges to the ray $\xi := \{a_t o : t \geq 0\}$ in $\partial_\infty X$ and $d_{\gamma_{t_n}(s_n)} \to b_\xi$.

Since $\pi(a_{t_n} o) = o$ and π is 1-Lipschitz, we have

$$d(\pi(\gamma_{t_n}(s_n)), o) \leq d(a_{t_n} k_0 h_{s_n} o, a_{t_n} o) = |s_n|.$$

Since $Y = (H \cap A) k_0 L$, there exists $s'_n \in \mathbb{R}$ such that

$$h_{s'_n} \pi(\gamma_{t_n}(s_n)) \in k_0 L. \tag{6.2}$$

Since $h_{-s'_n} k_0 L$ is orthogonal to L at $h_{-s'_n} o$, $h_{-s'_n} o$ is the nearest projection of $\pi(\gamma_{t_n}(s_n))$ to L, and hence

$$d(\pi(\gamma_{t_n}(s_n)), h_{-s'_n}(o)) \leq d(\pi(\gamma_{t_n}(s_n)), o) \leq |s_n|.$$

Hence by the triangle inequality,

$$|s_n'| = d(o, h_{-s_n'}(o)) \le d(o, \pi(\gamma_{t_n}(s_n))) + d(\pi(\gamma_{t_n}(s_n)), h_{-s_n'}(o)) \le 2|s_n|.$$

(In fact, since the geodesic triangle in Y with vertices $h_{-s_n'}(o)$, $\pi(\gamma_{t_n}(s_n)), o$ has the right angle at $h_{-s_n'}(o)$, we even get $|s_n'| \le |s_n|$).

Since $2|s_n|/t_n \to 0$ as $n \to \infty$, we conclude again that

$$d_{h_{s_n'}\gamma_{t_n}(s_n)} \to b_\xi,$$

uniformly on compacts.

Since the ray $\xi = \{a_t o : \ t \ge 0\}$ is perpendicular to k_0L at o, the point o is a minimum of b_ξ on k_0L. We claim that this minimum is unique: Suppose that $y \in k_0L$ is another point such that $b_\xi(o) = b_\xi(y)$. Let $\{\rho_t : \ t \ge 0\}$ be the ray in X emanating from y and asymptotic to ξ and consider the Busemann function

$$\tilde{b}_\rho : X \to \mathbb{R} : \ \tilde{b}_\rho(x) = \lim_{t\to\infty} d(\rho_t, x) - t.$$

Then $\tilde{b}_\rho - b_\xi$ is a constant function. Since y is a minimum of b_ξ on k_0L, it must be a minimum of $\tilde{b}_\rho$. Thus ρ must be orthogonal to k_0L at y. Since ξ and ρ are perpendicular to the segment $[o, y]$ at its endpoints and are asymptotic, it follows from the Flat Strip Theorem (see [1, Ch. I, Cor. 5.8(i)]) that ξ and ρ bounds a flat half strip; in particular, ξ and ρ (and hence the segment oy) must lie in a 2-flat $F \subset X$. Thus F contains the ray $\xi = \{a_t o : \ t \ge 0\}$ as well as the geodesic $k_0L = \{k_0 h_t o : t \in \mathbb{R}\}$ (since it contains the segment $oy \subset k_0L$). Hence for all $t_0 \in \mathbb{R}$, $k_0 h_{t_0} k_0^{-1}$ lies in the centralizer of $\{a_t : \ t \in \mathbb{R}\}$, which can be verified to be false by a straightforward computation. So, we arrive at a contradiction.

Since o is the unique minimum of $b_\xi|_{k_0L}$, Lemma 6.3 implies that the nearest projection of $h_{s_n'}\gamma_{t_n}(s_n)$ to k_0L converges to o. By (6.2), we have

$$d(\pi(h_{s_n'}\gamma_{t_n}(s_n)), o) \to 0 \quad \text{as } n \to \infty.$$

Since π is H-invariant, we finally have

$$d(\pi(\gamma_{t_n}(s_n)), L) \to 0 \quad \text{as } n \to \infty$$

as asserted. □

Projection is bounded.

Theorem 6.8. *For any $\varepsilon > 0$, the union $\bigcup_{|t|>\varepsilon} \pi(Y_t)$ lies within a bounded distance from L.*

Since $\pi(Y_t) = A_0\pi(a_t k_0 A_0 o)$, the result follows from the following result:

Lemma 6.9. *For $\varepsilon > 0$, the set $\{\pi(\gamma_t(s)) : |t| > \varepsilon, s \in \mathbb{R}\}$ lies within a bounded distance from L. Moreover, if either t_n or $|s_n|$ is bounded, then $\pi(\gamma_{t_n}(s_n))$ is bounded.*

For the proof of this lemma, we assume that the parameter t is positive. More precisely, we show:

Lemma 6.10. *For all* $\varepsilon > 0$*, the set* $\{\pi(\gamma_t(s)) : t > \varepsilon,\ s \in \mathbb{R}\}$ *lies within a bounded distance from* L*. Moreover, if either* t_n *or* $|s_n|$ *is bounded, then* $\pi(\gamma_{t_n}(s_n))$ *is bounded.*

This suffices, as can be seen as follows: the Cartan involution $\Theta : G \to G$ given by (2.1) induces an isometry $\iota : X \to X$ defined by $\iota([gK]) = [(g^T)^{-1}K]$ (a *point reflection* about the basepoint o). This isometry preserves all totally geodesic subspaces passing through o, such as Y, L, and Ao. Note that $\Theta(a_t) = a_{-t}$. Since the nearest point projection $\pi : X \to Y$, viewed as a map from $X \to X$, commutes with ι, Lemma 6.10 also implies that the set $\{\pi(\gamma_t(s)) : -t < -\varepsilon,\ s \in \mathbb{R}\}$ lies within a bounded distance from L, thereby concluding the proof of Lemma 6.9.

Proof of Lemma 6.10. Suppose that the claim is not true. Then there exist $\varepsilon > 0$ and sequences s_n and $t_n > \varepsilon$ such that the sequence $\pi(\gamma_{t_n}(s_n))$ diverges away from L. After extraction, we have the following cases.

Case 0: Both s_n and t_n are bounded. Then $\pi(\gamma_{t_n}(s_n))$ is bounded, leading to a contradiction.

Case 1: t_n is bounded and $|s_n| \to \infty$. Passing to a subsequence, we can assume that $t_n \to t$ and $s_n \to \infty$.

Let $\xi \in \partial_\infty X$ denote the equivalence class of the ray $\{\gamma_t(s) : \ s > 0\}$. Then, $\gamma_{t_n}(s_n) \to \xi$ as $n \to \infty$. Thus, we have

$$d_{\gamma_{t_n}(s_n)} \to b_\xi$$

uniformly on compacts.

We observe that f_ξ (cf. (6.1)) has position either 2 or 3 relative to any point in Λ_Y: To see this, note that if we write $f_\xi = (p, \ell)$ (as described in the paragraph before Definition 6.4), then

$$p = a_t k_0[e_1] = [e^t e_1 - \sqrt{2}e^{-2t}e_2 + e^t e_3].$$

Thus, for $t > 0$, p lies in the interior of the disk $D \subset \mathbb{R}\mathrm{P}^2$ bounded by the conic $y^2 = 2xz$, which implies the assertion.

Therefore, by Lemma 6.5, $b_\xi|_Y$ is proper and bounded below, and has a unique minimum in Y. By Lemma 6.3, it follows that $(\pi(\gamma_{t_n}(s_n)))$ converges to this unique minimum and thus we again have a contradiction.

Case 2: If the sequence s_n is bounded, then the sequence $\gamma_{t_n}(s_n)$ lies in a bounded neighborhood of the singular ray $\{a_t o : t > 0\}$, and thus must have a bounded projection to Y. This is a contradiction.

Case 3: Finally, if $t_n \to \infty$ and $|s_n| \to \infty$ as $n \to \infty$, then it follows from Lemma 6.6 and Lemma 6.7 that $\pi(\gamma_{t_n}(s_n))$ is bounded. Again, this is a contradiction.

The "moreover" part follows from Cases 1 and 2 discussed above. □

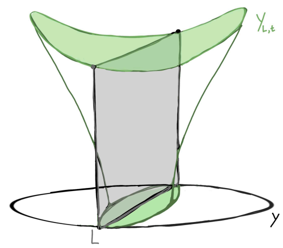

FIGURE 4. shadow of the floating plane

Projection is narrow.

Theorem 6.11. *The Hausdorff distance between* $\pi(Y_t)$ *and* L *goes to* 0 *as* $|t| \to \infty$.

Proof. Again, we make the assumption that $t > 0$, as the case when $t < 0$ is analogous.

Suppose the claim is not true. Since $L \subset \pi(Y_t)$, there exist $\delta > 0$, $t_n \to \infty$, and $x_n \in Y_{t_n}$ with $d(L, \pi(x_n)) \geq \delta$. As in the proof of Theorem 6.8, write $x_n = h_{s_n'} a_{t_n} k_0 h_{s_n} o$ and use the A_0-equivariance to take $s_n' = 0$. Lemmas 6.6 and 6.7 then give a contradiction. □

7. Closures of floating planes: The Fuchsian case

In this section, we consider the closure of an H-orbit $[h]a_t H$ in $\Gamma\backslash G$ where $h \in H$ and Γ is a cocompact lattice in H. Let N^- and N^+ be the strictly upper and lower triangular subgroups of G and set $N_0^\pm = H \cap N^\pm$.

For any $h \in H$, the product map $hN_0^+ \times N_0^- \times A_0 \to H$ is a diffeomorphism onto its image which is Zariski open and dense. Any right A_0-invariant open neighborhood of $h \in H$ contains a subset of the form $\mathcal{O} = hU^+U^-A_0$ for some open neighborhood $U^\pm \subset N_0^\pm$. We will call a subset of this type a basic open subset and let $\pi_\pm = \pi_{\mathcal{O},\pm} : \mathcal{O} \to N_0^\pm$ be the Bruhat projections

$$\pi_+(hn^+n^-a) = n^+ \quad \text{and} \quad \pi_-(hn^+n^-a) = n^-. \tag{7.1}$$

Definition 7.1. We say that an A_0-invariant subset $Z \subset H$ is admissible if the box dimension of $\pi_\pm(\overline{Z} \cap \mathcal{O})$ is equal to its Hausdorff dimension for all sufficiently small basic open subsets $\mathcal{O}$. If $L = hA_0 o$ is a geodesic in Y, we say L or $\ell = \Gamma\backslash L$ is admissible if $\Gamma h A_0$ is admissible.

The following theorem says that when the floating height t is non-zero, it can be as chaotic as the closure of its reference geodesic $\Gamma\backslash\Gamma h A_0$ in $\Gamma\backslash H$.

Theorem 7.2. *Let* $\Gamma < H$ *be a discrete subgroup and* $h \in H$. *Let* $t \neq 0$.

(1) *We have*

$$\dim \overline{\Gamma h a_t H} = 2 + \dim \overline{\Gamma h A_0}.$$

(2) *Let* $Y_{L,t} = ha_tHo$ *for* $L = hA_0o$. *Suppose that* L *is admissible. Then*

$$\frac{1}{2}\left(3 + \dim \overline{\Gamma h A_0}\right) \le \dim \overline{\Gamma Y_{L,t}} \le 1 + \dim \overline{\Gamma h A_0}.$$

(3) *In particular, if* $1 < \dim \overline{\Gamma\backslash\Gamma h A_0} < 2$, *then*

$$\dim \overline{\Gamma Y_{L,t}} \in (2,3).$$

The rest of this section is devoted to proving Theorem 7.2. We begin by showing that the closure of $[h]a_tH$ in $\Gamma\backslash G$ is governed by the closure of $[h]A_0 \subset \Gamma\backslash H$. Note that for $\Gamma < H$, the quotient $\Gamma\backslash H$ is a closed subset of $\Gamma\backslash G$.

Proposition 7.3. *Let* $\Gamma < H$ *be a discrete subgroup and* $h \in H$. *For any* $t \in \mathbb{R}$, *we have*

$$\overline{\Gamma h a_t H} = \overline{\Gamma h A_0} a_t k_0 A_0 K_0.$$

In particular, for $L = hA_0o$,

$$\overline{\Gamma Y_{L,t}} = \overline{\Gamma h A_0} a_t k_0 A_0 o.$$

Proof. Since $H = A_0k_0A_0K_0$, we have

$$\Gamma h a_t H = \Gamma h A_0 a_t k_0 A_0 K_0.$$

Let $g \in \overline{\Gamma h a_t H}$, i.e., $\gamma_i h a_t h_i \to g$ for an infinite sequence $\gamma_i \in \Gamma, h_i \in H$. We write $h_i = c_i k_0 p_i k_i \in A_0k_0A_0K_0$, so $a_th_i = c_i a_t k_0 p_i k_i$. It suffices to show that $a_tk_0p_i$ is bounded, so that its limit lies in $a_tk_0A_0K_0$. Write

$$a_tk_0p_i = h_i' b_i k_i' \in HBK$$

using Theorem 3.2. As t is fixed,

$$\pi(a_tk_0p_io) = h_i'o$$

is bounded by Lemma 6.9 and hence $h_i' \in H$ is bounded. Using Theorem 3.2, we can write $g = h^*b^*k^* \in HBK$. Since

$$\gamma_i h a_t h_i = \gamma_i h c_i (a_t k_0 p_i) k_i = (\gamma_i h c_i h_i') b_i (k_i' k_i) \in HBK$$

and the B-component in the $G = HBK$ decomposition is uniquely determined modulo a compact subset, we must have $b_i \to b^*$ modulo a compact subset, and hence b_i must be a bounded sequence. Since both h_i' and b_i are bounded sequences and $a_tk_0p_i = h_i'b_ik_i' \in HBK$, it follows that $a_tk_0p_i$ is bounded, as desired. ☐

To relate the Hausdorff dimension of $\overline{\Gamma h A_0}$ with that of $\overline{\Gamma h a_t H}$, we use the local product structure in G. The following lemma shows that $H \times a_tk_0A_0K_0$ maps locally diffeomorphically into G, allowing us to invoke Proposition 7.3 in the proof of Theorem 7.2(1).

Lemma 7.4. *Let* $t \neq 0$. *The product map*

$$m : H \times a_tk_0A_0K_0 \to G, \quad (h,s) \mapsto hs$$

with $h \in H$ and $s \in a_t k_0 A_0 K_0$ is a local diffeomorphism onto its image at every point, i.e., for any $p = (h, s)$, there exists an open neighborhood U of p such that $m(U)$ is a submanifold of G and $m|_U : U \to m(U)$ is a diffeomorphism.

Proof. Write $g = a_t k_0$ and set $S_t = gA_0K_0g^{-1}$. To prove the claim, it suffices to show that for the product map $m : H \times S_t \to G$ given by $(h, s) \mapsto hs$, $dm_{(h,s)}$ is injective at every $(h, s) \in H \times S_t$. Then by the constant rank theorem, the claim would follow.

So, let $h \in H$ and $s = g(ak)g^{-1} \in S_t$ where $a \in A_0$ and $k \in K_0$. For any $U \in \mathrm{T}_h H$ and $V \in \mathrm{T}_s S_t$, we get

$$(L_{(hs)^{-1}})_* dm_{(h,s)}(U, V) = \mathrm{Ad}_{s^{-1}}(L_{h^{-1}})_* U + (L_{s^{-1}})_* V$$

where $L_x : G \to G$ denotes the left translation by $x \in G$ and $(L_x)_* : \mathrm{T}_y G \to \mathrm{T}_{xy} G$ denotes the differential at $y \in G$. Therefore

$$\ker dm_{(h,s)} \simeq \mathrm{Ad}_{s^{-1}} \mathfrak{h} \cap (L_{s^{-1}})_* \mathrm{T}_s(S_t).$$

Now

$$(L_{(ak)^{-1}})_* \mathrm{T}_{ak}(A_0K_0) = \mathrm{Ad}_{k^{-1}} \mathfrak{a}_0 \oplus \mathfrak{k}_0$$

and conjugating by g gives

$$(L_{s^{-1}})_* \mathrm{T}_s S_t = \mathrm{Ad}_g \left(\mathrm{Ad}_{k^{-1}} \mathfrak{a}_0 \oplus \mathfrak{k}_0\right).$$

Setting $\mathfrak{h}' = \mathrm{Ad}_{g^{-1}} \mathfrak{h}$, we can write

$$\mathrm{Ad}_{s^{-1}} \mathfrak{h} = \mathrm{Ad}_g(\mathrm{Ad}_{a^{-1}k^{-1}} \mathfrak{h}').$$

It remains to show that

$$\mathrm{Ad}_{a^{-1}k^{-1}} \mathfrak{h}' \cap (\mathrm{Ad}_{k^{-1}} \mathfrak{a}_0 \oplus \mathfrak{k}_0) = \{0\}.$$

Since $k = k(\theta)$ as in (3.4), we can compute that any matrix in $\mathrm{Ad}_{k^{-1}} \mathfrak{a}_0 \oplus \mathfrak{k}_0$ is of the form

$$\begin{pmatrix} b & p & 0 \\ q & 0 & p \\ 0 & q & -b \end{pmatrix} \tag{7.2}$$

for some $b, p, q \in \mathbb{R}$. Moreover, if $k = e$, then $p = q$.

Now, an element in $\mathfrak{h}'$ is of the form

$$g^{-1} \begin{pmatrix} s & x & 0 \\ y & 0 & x \\ 0 & y & -s \end{pmatrix} g = \frac{1}{2} \begin{pmatrix} X & Y & Z \\ W & 0 & Y \\ Z & W & -X \end{pmatrix} \tag{7.3}$$

for some $x, y, s \in \mathbb{R}$, where

- $X = -\sqrt{2}(x + y)(e^{3t} + e^{-3t})$,
- $Y = 2\sqrt{2}s + 2e^{3t}(x - y)$,
- $Z = \sqrt{2}e^{-3t}(x + y)$,
- $W = 2\sqrt{2}s - 2e^{-3t}(x - y)$.

Let $k = k_\theta$, $\mathsf{c} = \cos\theta$, $\mathsf{s} = \sin\theta$, and $a = h_r$. Then a matrix $I \in \mathrm{Ad}_{a^{-1}k^{-1}}\mathfrak{h}'$ is, up to a uniform constant multiple, of the form

$$I_{11} = 2\mathsf{c}X + \sqrt{2}\mathsf{s}Y + \sqrt{2}\mathsf{s}W, \quad I_{22} = -\sqrt{2}\mathsf{cs}W, \quad I_{33} = 2\mathsf{c}Z + \sqrt{\mathsf{s}}(1-\mathsf{c})W$$

$$I_{12} = (-\sqrt{2}\mathsf{s}X + (\mathsf{c}+1)Y + \sqrt{2}\mathsf{s}Z - \mathsf{s}^2W)e^{-3r},$$

$$I_{23} = (-\sqrt{2}\mathsf{s}X + (\mathsf{c}+1)Y - \sqrt{2}\mathsf{s}Z + \mathsf{c}(1-\mathsf{c})W)e^{3r},$$

$$I_{21} = (-\sqrt{2}\mathsf{s}X + (\mathsf{c}-1)Y - \sqrt{2}\mathsf{s}Z + (\mathsf{c}+1)W)e^{3r},$$

$$I_{32} = (-\sqrt{2}\mathsf{s}X + (\mathsf{c}-1)Y + \sqrt{2}\mathsf{s}Z)e^{-3r},$$

$$I_{13} = 2\mathsf{c}Z + \sqrt{2}^{-1}(1-\mathsf{c})\mathsf{s}W, \quad I_{31} = -2\mathsf{c}Z.$$

Suppose that $I \in \mathrm{Ad}_{a^{-1}k^{-1}}\mathfrak{h}' \cap (\mathrm{Ad}_{k^{-1}}\mathfrak{a}_0 \oplus \mathfrak{k}_0)$. To show that $I = 0$, we consider the following three cases separately.

Case I: $\mathsf{c}\,\mathsf{s} \neq 0$**.** Since $I_{13} = I_{22} = I_{31} = 0$, we must have $Z = W = 0 = I_{33}$. Since $Z = 0$, $x = -y$. Together with $W = 0$, this gives $\sqrt{2}s = 2e^{-3t}x$. Since $I_{33} = 0$, $I_{11} = 0$, i.e, $\sqrt{2}\mathsf{c}X + \mathsf{s}Y = 0$, and this means that $\sqrt{2}s = -2e^{3t}x$. Since $t \neq 0$, we get $x = 0$, which also means that $s = y = 0$. Therefore $I = 0$.

Case II: $\mathsf{c} = 0$**.** Then from $I_{13} = 0$, we get $W = 0$, which implies $I_{33} = 0$. From $I_{11} = -I_{33}$, we get $Y = 0$. This implies that $x = y = s = 0$. Hence $I = 0$.

Case III: $\mathsf{s} = 0$**.** In this case, $k = e$. So $I_{12} = I_{21}$. Then from $I_{13} = 0$, we get $x + y = 0$. Since $I_{32} = 0$, we get $I_{21} = 0$, which gives us $W = 0$, and hence $Y = 0$. This implies $x = y = s = 0$; so $I = 0$.

This finishes the proof. □

Proof of Theorem 7.2(1) By Lemma 7.4, the product map $f : H \times a_t k_0 A_0 K_0 \to G$ is locally bi-Lipschitz on a countable cover. Since Hausdorff dimension is countably stable, it follows that for any subset Σ of $H \times a_t k_0 A_0 K_0$, the Hausdorff dimension of Σ is equal to that of its image under f.

Lemma 7.3 gives

$$\overline{\Gamma h a_t H} = \overline{\Gamma h A_0} a_t k_0 A_0 K_0,$$

and $\overline{\Gamma h A_0} \subset H$ as $\Gamma \subset H$, Thus the Hausdorff dimension of $\overline{\Gamma a_t H}$ is equal to that of the product $\overline{\Gamma h A_0} \times a_t k_0 A_0 K_0$. Since $a_t k_0 A_0 K_0$ is a 2-dimensional smooth submanifold, the claim follows.

Floating geodesic planes. For Part (2) of Theorem 7.2(2), we need an analogue of Lemma 7.4 at the level of the symmetric space X. Unfortunately, the product map $H \times a_t K_0 A_0 o \to X$ is in general not locally injective, and this is precisely why we cannot conclude $\dim \overline{\Gamma Y_{L,t}} = 1 + \dim \overline{\Gamma h A_0}$ in Theorem 7.2(2). The next lemma shows that replacing H by $hN_0^{\pm} A_0$ restores local injectivity in X, and this will allow us to prove the dimension estimates in Theorem 7.2(2).

Lemma 7.5. *Let $h \in H$. The multiplication map*

$$hN_0^{\pm} A_0 \times a_t k_0 A_0 o \to X, \quad (hh', so) \mapsto hh'so$$

with $h' \in N_0^{\pm} A_0$ and $s \in a_t k_0 A_0$ is a local diffeomorphism onto its image everywhere.

Proof. Recall the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Set $H^{\pm} = H \cap N^{\pm} A$ and $\mathfrak{h}^{\pm} := \mathrm{Lie}(H^{\pm})$. Let $g = a_t k_0$. It suffices to show that the multiplication map $\Phi : g^{-1} H^{\pm} g \times A_0 o \to X$ is a local diffeomorphism at $(e, h_r o)$ for any $r \in \mathbb{R}$, since the left-translation by an element of $g^{-1} H^{\pm} g$ is an isometry. The image of $d\Phi_{(e,h_r o)}$ is given by $dL_{h_r}(\mathrm{Ad}_{h_r^{-1} g^{-1}}(\mathfrak{h}^{\pm})_{\mathfrak{p}} + \mathfrak{a}_0)$. Thus it is enough to show

$$(h_r^{-1} g^{-1} \mathfrak{h}^{\pm} g h_r)_{\mathfrak{p}} \cap \mathfrak{a}_0 = \{0\} \tag{7.4}$$

where $(\cdot)_{\mathfrak{p}}$ denotes the projection to $\mathfrak{p}$.

Since the projection $\mathfrak{g} \to \mathfrak{p}$ is given by $u \mapsto (u + u^T)/2$, an element of $(g^{-1} \mathfrak{h}^+ g)_{\mathfrak{p}}$ is of the form as given in (7.3) with

$$h_r^{-1} g^{-1} \begin{pmatrix} s & 0 & 0 \\ y & 0 & 0 \\ 0 & y & -s \end{pmatrix} g h_r = \frac{1}{4} \begin{pmatrix} 2X & Y+W & 2e^{-2r} Z \\ e^r (Y+W) & 0 & e^{-r}(Y+W) \\ 2e^{2r} Z & Y+W & -2X \end{pmatrix} \tag{7.5}$$

where

- $X = -\sqrt{2} y (e^{3t} + e^{-3t})$,
- $Y = 2\sqrt{2} s - 2e^{3t} y$,
- $Z = \sqrt{2} e^{-3t} y$,
- $W = 2\sqrt{2} s + 2e^{-3t} y$.

Hence $2e^{-2r} Z = 0$ and $Y + W = 0$ implies that $y = 0 = s$, proving (7.4) for $\mathfrak{h}^+$. The computation for $\mathfrak{h}^-$ is analogous. ☐

Lemma 7.6. *For $\mathcal{G} = \Gamma h A_0$ with $h \in H$, suppose that $\overline{\mathcal{G}}$ is admissible. Then*

$$\max\{\dim \pi_+ \left(\overline{\mathcal{G}} \cap \mathcal{O}\right), \dim \pi_- \left(\overline{\mathcal{G}} \cap \mathcal{O}\right)\} \geq \frac{\dim(\overline{\mathcal{G}}) - 1}{2}$$

where the supremum is taken over all basic open subsets $\mathcal{O}$ of H.

Proof. Since H can be covered by countably many basic open sets,

$$\dim(\overline{\mathcal{G}}) = \sup_{\mathcal{O}} \dim(\overline{\mathcal{G}} \cap \mathcal{O}).$$

Hence if $\dim(\overline{\mathcal{G}}) > c_0$, then for some basic open subset $\mathcal{O}$, we must have $\dim(\overline{\mathcal{G}} \cap \mathcal{O}) > c_0$. Write $\mathcal{O} = hU_0^+ U_o^- A_0$ and $\overline{\mathcal{G}} \cap \mathcal{O} = h \bigcup_\alpha (n_\alpha^+ n_\alpha^-) A_0$ with $n_\alpha^\pm \in U_0^\pm$. Hence $\pi_\pm \left(\overline{\mathcal{G}} \cap \mathcal{O}\right) = \cup_\alpha n_\alpha^\pm$. Since $\overline{\mathcal{G}}$ is admissible, we have

$$c_0 < \dim(\overline{\mathcal{G}} \cap \mathcal{O}) \leq \dim\left(\bigcup (n_\alpha^+ n_\alpha^-)\right) + 1 \leq \dim \bigcup n_\alpha^+ + \dim \bigcup n_\alpha^- + 1.$$

This inequality forces at least one of $\dim \bigcup n_\alpha^+$ or $\dim \bigcup n_\alpha^-$ to be bigger than $(c_0 - 1)/2$, giving the claim. ☐

Proof of Theorem 7.2(2) By Lemma 7.3,

$$\overline{\Gamma h a_t H o} = \overline{\Gamma h A_0} a_t k_0 A_0 o.$$

Hence the upper bound is immediate. For the lower bound, by Lemma 7.6, it suffices to show that for any basic open subset $\mathcal{O} \subset H$, we have

$$\max\{\dim \pi_+ \left(\overline{\mathcal{G}} \cap \mathcal{O}\right), \dim \pi_- \left(\overline{\mathcal{G}} \cap \mathcal{O}\right)\} + 2 \leq \dim \overline{\Gamma Y_{L,t}}.$$

write $\mathcal{O} = h_0 U^+ U^- A_0$ for some open neighborhood $U^\pm \subset N_0^\pm$ and $h_0 \in H$. Write

$$\overline{\Gamma h A_0} \cap \mathcal{O} = \bigcup_\alpha h_0 n_\alpha^+ n_\alpha^- A_0$$

which is a disjoint union of A_0-orbits with $n_\alpha^\pm \in U^\pm$. So

$$\dim \overline{\Gamma h a_t H o} \geq \dim\left(\overline{\Gamma h A_0} \cap \mathcal{O}\right) a_t k_0 A_0 o \geq \dim\left((\bigcup h_0 n_\alpha^+ A_0) a_t k_0 A_0 o\right)$$

Since $(\bigcup n_\alpha^+ A_0) \subset N_0^+ A_0$, by Lemma 7.5, we have

$$\dim\left(h_0 (\bigcup n_\alpha^+ A_0) a_t k_0 A_0 o\right) = \dim\left(h_0 \bigcup n_\alpha^+ A_0\right) + 1$$

which is equal to

$$\dim\left(\bigcup_\alpha n_\alpha^+\right) + 2 = \dim \pi_+ \left(\overline{\Gamma h A_0} \cap \mathcal{O}\right) + 2.$$

Hence

$$\dim \overline{\Gamma h a_t H o} \geq \dim \pi_+ \left(\overline{\mathcal{G}} \cap \mathcal{O}\right) + 2.$$

The statement $\dim \overline{\Gamma h a_t H o} \geq \dim \pi_- \left(\overline{\mathcal{G}} \cap \mathcal{O}\right) + 2$ can be proved similarly.

8. Bulging deformations and floating planes

Let $\Gamma < H$ be a torsion-free cocompact lattice and let

$$S = \Gamma \backslash Y$$

be the closed orientable hyperbolic surface. Let $\rho_0 : \Gamma \to H$ denote the inclusion map. Fix a diagonalizable element $\delta \in \Gamma$ representing the homotopy class of an essential simple closed curve $\beta \subset S$. We describe the notion of bulging deformations of Γ in G, introduced by Goldman [12].

Geometrically, a bulging deformation along β alters the convex $\mathbb{R}P^2$-structure on S by inserting a projective "bulge" along β. This is achieved by deforming the holonomy representation using a one-parameter subgroup of projective transformations that fix the endpoints of the holonomy of β while "stretching" transversely to it.

We give a more precise description of the holonomy representation. Suppose first that β is separating. In this case, the complement of β in S consists of two connected subsurfaces whose closures we denote by S_1 and S_2, with β as their common boundary. The inclusion maps $\beta \hookrightarrow S_i$ $(i = 1, 2)$ induce a decomposition of Γ as an amalgamated free product

$$\Gamma = \Delta_1 *_{\langle\delta\rangle} \Delta_2,$$

where $\Delta_i = \pi_1(S_i)$ for $i = 1, 2$, and $\langle\delta\rangle$ is the image of $\pi_1(\beta)$ under the inclusion maps, viewed as a common subgroup of Δ_1 and Δ_2.

Let B denote identity component of the centralizer of δ in $\mathrm{SL}_3(\mathbb{R})$, which is a maximal real split torus. For any $\mathbf{b} \in B$, we have a unique homomorphism $\rho_{\mathbf{b}} : \Gamma \to G$ extending

$$\rho_{\mathsf{b}}(\gamma) = \begin{cases} \gamma & \text{for } \gamma \in \Delta_1 \\ \mathsf{b}\gamma\mathsf{b}^{-1} & \text{for } \gamma \in \Delta_2. \end{cases} \tag{8.1}$$

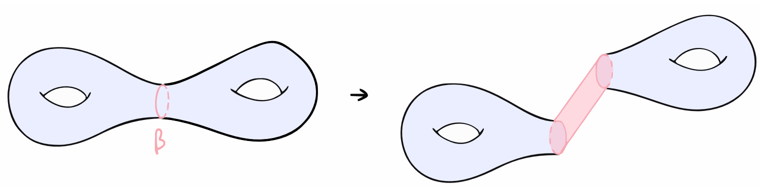

FIGURE 5. Bulging deformation

Now we discuss the case when $\beta \subset S$ is non-separating. Cutting along β gives a surface S_1 with boundary components β_1, β_2, and re-gluing by an orientation-reversing homeomorphism $f : \beta_1 \to \beta_2$ recovers S. Setting $\Delta = \pi_1(S_1)$, the group $\Gamma = \pi_1(S)$ is an HNN extension

$$\Gamma \cong \Delta *_\psi := \langle \Delta, t \mid tbt^{-1} = \psi(b),\ \delta \in \iota_1(\pi_1(\beta_1)) \rangle, \tag{8.2}$$

where $\psi = \iota_2 \circ f_* \circ \iota_1^{-1}$, with $f_* : \pi_1(\beta_1) \to \pi_1(\beta_2)$ and $\iota_i : \pi_1(\beta_i) \hookrightarrow \Delta$ the induced maps.

The group Δ naturally embeds in Γ, so we can view Δ as a subgroup of Γ. Let δ be a generator for the image of $\pi_1(\beta_1)$ in $\Delta \subset \Gamma$ and let $B = C_G(\delta)$ as above. For any $\mathbf{b} \in B$, we have a unique homomorphism $\rho_{\mathsf{b}} : \Gamma \to G$ extending

$$\rho_{\mathsf{b}}(\gamma) = \begin{cases} \gamma & \text{for } \gamma \in \Delta \\ \mathsf{b}t & \text{for } \gamma = t. \end{cases} \tag{8.3}$$

If $h \in H$ is such that $h\delta h^{-1} \in A_0$, then $hBh^{-1} = A$, and hence

$$h\mathsf{b}h^{-1} = a_{c_0} h_{d_0} \quad \text{for some } c_0, d_0 \in \mathbb{R}.$$

We will call $|c_0|$ the *width* of b, which will be denoted by

$$\mathrm{wd}(\mathsf{b}). \tag{8.4}$$

We set

$$\Gamma_{\mathsf{b}} = \rho_{\mathsf{b}}(\Gamma) < G. \tag{8.5}$$

Hitchin property and Zariski density. If $\mathsf{b} = \exp u$ for $u \in \mathfrak{g}$, then for $s \in \mathbb{R}$, set

$$\mathsf{b}_s = \exp su, \tag{8.6}$$

and consider the one-parameter family of deformations

$$\rho_s := \rho_{\mathsf{b}_s}.$$

Noting that $h\mathsf{b}_s h^{-1} = a_{c_0 s} h_{d_0 s}$ for all $s \in \mathbb{R}$, the width of b_s is $\mathrm{wd}(\mathsf{b}_s) = |s|\,\mathrm{wd}(\mathsf{b}) = |c_0 s|$.

Clearly, $\{\rho_s : s \in \mathbb{R}\}$ lies in the Hitchin component of $\mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$, which is the connected component containing ρ_0. Therefore, according to Choi-Goldman [6], ρ_{b} is discrete and faithful. Later, Labourie [13] initiated the theory of Anosov representations and showed each representations in the Hitchin component is Anosov.

Theorem 8.1. *For all $\mathsf{b} \in C_G(\delta)^\circ$, Γ_{b} is an Anosov (in particular, discrete) subgroup of G. Moreover, if $\mathrm{wd}(\mathsf{b}) \neq 0$, then Γ_{b} is Zariski dense in G.*

We justify the "Zariski dense" part in the above result only in the case when $\beta \subset S$ is separating, the non-separating case is similar: Note that Γ_{b} contains Δ_1 and $\mathsf{b}\Delta_2\mathsf{b}^{-1}$. Since each Δ_i is Zariski dense in H, the Zariski closures of Γ_{b} contains both H and $\mathsf{b}H\mathsf{b}^{-1}$. If $\mathrm{wd}(\mathsf{b}) \neq 0$, then $\mathsf{b} \notin N_G(H)$. Since H is a maximal connected Lie subgroup of G, it follows that Γ_{b} is Zariski dense in G. We refer to [8, Sec. 1.2] for a direct proof of the Anosov property of Γ_{b}.

Proper embedding away from the bulging locus. Fix a complete geodesic $\tilde{\beta}$ in Y which projects to β. Every representation $\rho_{\mathsf{b}} : \Gamma \to G$ admits a ρ_{b}-equivariant locally isometric map

$$\phi_{\mathsf{b}} : (Y - \Gamma \cdot \tilde{\beta}) \to X,$$

constructed as follows. Consider the dual graph T (actually a tree) to the lamination $\Gamma \cdot \tilde{\beta} \subset Y \cong \mathbb{H}^2$, whose each vertex uniquely correspond to a connected component of $Y - \Gamma \cdot \tilde{\beta}$, and there is an edge between two vertices if the corresponding connected components of $Y - \Gamma \cdot \tilde{\beta}$ are adjacent.

If β is separating, then there are two Γ-orbits of the connected components of $Y - \Gamma \cdot \tilde{\beta}$. So, the vertices of T is bicolored. Note that T is precisely the *Bass-Serre tree* associated with the amalgamated free product decomposition

$$\Gamma = \Delta_1 *_{\gamma_0} \Delta_2.$$

We fix an edge $e = [v_1, v_2]$ in T, i.e. a fundamental domain for the action $\Gamma \curvearrowright T$ and let Y_1 and Y_2 be the connected component of $Y - \Gamma \cdot \tilde{\beta}$ corresponding to v_1 and v_2, respectively. We may assume that Δ_1, Δ_2, and $\langle\gamma_0\rangle$ are the stabilizers in Γ of v_1, v_2, and e, respectively.

If β is non-separating, the dual graph T to $\Gamma\cdot\tilde{\beta}\subset Y$ is again the Bass–Serre tree of the HNN extension in (8.2). The connected components of $Y-\Gamma\cdot\tilde{\beta}$ (correspond to vertices of T) lie in a single Γ-orbit; we choose the component $Y_1\subset Y-\Gamma\cdot\tilde{\beta}$ stabilized by Δ. We fix the notation v_1 and v_2 to denote the (adjacent) vertices in T corresponding to Y_1 and $Y_2 \coloneqq tY_1$, respectively.

Define $\phi_{\mathsf{b}} : (Y-\Gamma\cdot\tilde{\beta}) \to X$ by

$$\phi_{\mathsf{b}}|_{Y_1} = i_{Y_1}, \quad \phi_{\mathsf{b}}|_{Y_2} = \mathsf{b}\circ i_{Y_2}$$

where $i_Z : Z \hookrightarrow X$ denotes the inclusion map.

Using the H-equivariant nearest-point projection map $\pi : X \to Y$, we extend ϕ_{b} to a ρ_{b}-equivariant local isometry

$$F_{\mathsf{b}} : (X - \pi^{-1}(\Gamma\cdot\tilde{\beta})) \to X$$

by setting F_{b} equal to the inclusion map on $\pi^{-1}(Y_1)$ and to $\mathsf{b}\circ i_{\pi^{-1}(Y_2)}$ on $\pi^{-1}(Y_2)$, and extending equivariantly. In particular, it satisfies

$$F_{\mathsf{b}}(\Gamma x) = \Gamma_{\mathsf{b}} x, \quad \text{for all } x\in X-\pi^{-1}(\Gamma\cdot\tilde{\beta}). \tag{8.7}$$

For $c>0$, let Y_c be the complement of the open c-neighborhood $\mathcal{N}_{c_0}(\Gamma\cdot\tilde{\beta})$ of $\Gamma\cdot\tilde{\beta}$ in Y. Set

$$X_c \coloneqq \pi^{-1}(Y_c). \tag{8.8}$$

Both Y_c and X_c are Γ-invariant. Thus the restriction of F_{b} to X_c descends to local isometry

$$f_{\mathsf{b},c} : \Gamma\backslash X_c \to \Gamma_{\mathsf{b}}\backslash X.$$

In this section, we prove:

Theorem 8.2. *For all $c > \mathrm{wd}(\mathsf{b})$, the map*

$$f_{\mathsf{b},c} : \Gamma\backslash X_c \to \Gamma_{\mathsf{b}}\backslash X$$

is a proper locally isometric embedding, which is given by

$$f_{\mathsf{b},c}([\Gamma x]) = [\Gamma_{\mathsf{b}} x] \quad \textit{for all } x\in X_c.$$

The above discussion has an analogue when X is replaced by G, as we now discuss: Consider the fibration $p : G\to X$ given by $g\mapsto go, \in G$, whose fibers are isomorphic to K. Define a ρ_{b}-equivariant map

$$\bar{F}_{\mathsf{b}} : G-(\pi\circ p)^{-1}(\Gamma\cdot\tilde{\beta}) \ \to\ G$$

as follows. In the separating case, set $\bar{F}_{\mathsf{b}}$ to be the identity on $(\pi\circ p)^{-1}(Y_1)$, the composition of the inclusion map with b on $(\pi\circ p)^{-1}(Y_2)$, and then extend uniquely by requiring equivariance. The construction is analogous in the non-separating case. In particular,

$$\bar{F}_{\mathsf{b}}(\Gamma g) = \Gamma_{\mathsf{b}} g \quad \text{for all } g\in G-(\pi\circ p)^{-1}(\Gamma\cdot\tilde{\beta}). \tag{8.9}$$

For $c>0$, let

$$G_c \coloneqq p^{-1}(X_c),$$

where X_c is defined by (8.8). The map $\bar{F}_{\mathsf{b}}$ descends to a local isometry

$$\bar{f}_{\mathsf{b},c} : \Gamma\backslash G_c \to \Gamma_{\mathsf{b}}\backslash G.$$

In this setting, Theorem 8.2 implies the following:

Corollary 8.3. *For all $c > \mathrm{wd}(\mathsf{b})$, the map*

$$\bar{f}_{\mathsf{b},c} : \Gamma\backslash G_c \to \Gamma_{\mathsf{b}}\backslash G$$

is a proper locally isometric embedding, which is given by

$$\bar{f}_{\mathsf{b},c}([\Gamma g]) = [\Gamma_{\mathsf{b}} g] \quad \textit{for all } g \in G_c.$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \Gamma\backslash G_c & \xrightarrow{\bar{f}_{\mathsf{b},c}} & \Gamma_{\mathsf{b}}\backslash G \\ p\big\downarrow & & \big\downarrow p \\ \Gamma\backslash X_c & \xrightarrow{f_{\mathsf{b},c}} & \Gamma_{\mathsf{b}}\backslash X \end{array}$$

The properness of $\bar{f}_{\mathsf{b},c}$ follows from that of $f_{\mathsf{b},c}$ (by Theorem 8.2) since the fibers of the vertical maps in the above diagram are compact.

We show that $\bar{f}_{\mathsf{b},c}$ is injective. Let $g_1, g_2 \in \Gamma\backslash G_c$ be distinct points. Suppose first that $p(g_1) \neq p(g_2)$. Since $f_{\mathsf{b},c}$ is injective (by Theorem 8.2), we have $f_{\mathsf{b},c}(p(g_1)) \neq f_{\mathsf{b},c}(p(g_2))$. The commutativity of the diagram then implies that $\bar{f}_{\mathsf{b},c}(g_1) \neq \bar{f}_{\mathsf{b},c}(g_2)$. If instead $p(g_1) = p(g_2)$, then the conclusion $\bar{f}_{\mathsf{b},c}(g_1) \neq \bar{f}_{\mathsf{b},c}(g_2)$ follows from the fact that $\bar{F}(s)$, and hence $\bar{f}_{\mathsf{b},c}$, maps fibers isomorphically onto fibers of p. ∎

The rest of this section is devoted to the proof Theorem 8.2. Without loss of generality, we may assume that $\mathsf{b} \in A$ by conjugating Γ.

Nearest point projection revisited. The proof of Theorem 8.2 is based on the study of fibers of the the nearest point projection map $\pi : X \to Y$. Recall that $\pi^{-1}(ho) = hK_0Bo$ for any $h \in H$.

Also, recall the notation $k(\theta) \in K_0$ from (3.4) and $k_1 \in K$ from (3.10).

Lemma 8.4. *For all $\theta, c \in \mathbb{R}$, there exists $\theta' \in \mathbb{R}$ such that*

$$k_1^{-1} k(\theta')\, a_c\, k(\theta)\, k_1 \in A.$$

Proof. Let

$$Q(\theta', \theta, c) := k_1^{-1} k(\theta') a_c k(\theta) k_1.$$

A direct multiplication yields $Q(\theta', \theta, c)$ is given by

$$\frac{1}{2}\begin{pmatrix} e^c & 0 & 0 \\ 0 & \frac{1}{e^{2c}}\left(\cos\theta\cos\theta' - e^{3c}\sin\theta\sin\theta'\right) & \frac{1}{e^{2c}}\left(\cos\theta\sin\theta' + e^{3c}\sin\theta\cos\theta'\right) \\ 0 & \frac{-1}{e^{2c}}\left(\cos\theta\sin\theta' + e^{3c}\sin\theta\cos\theta'\right) & \frac{1}{e^{2c}}\left(e^{3c}\cos\theta\cos\theta' - \sin\theta\sin\theta'\right) \end{pmatrix}.$$

If $\sin\theta = 0$ then set $\theta' = 0$. Else, if $\sin\theta \neq 0$, choose θ' by

$$\cot\theta' = -\, e^{-3c}\cot\theta. \tag{3}$$

Then $Q(\theta', \theta, c)$ is diagonal. If the diagonal entry has a negative sign, we can replace θ' by $\theta' + \pi$ to make all diagonal entries of $Q(\theta', \theta, c)$ positive. □

Lemma 8.5. *For $c \in \mathbb{R}$, the Hausdorff distance between $\pi^{-1}(o)$ and $a_c\pi^{-1}(o)$ is at most $|c|$.*

Proof. By Corollary 3.4,

$$\pi^{-1}(o) = K_0 k_1 \mathcal{W} A^+ o.$$

Let $w \in \mathcal{W}$. By Lemma 8.4, for any $k \in K_0$, there exists $k' \in K_0$ such that $a_c k k_1 \in k' k_1 A$. Since $A \subset wPw^{-1}$, we get

$$a_c k k_1 w P = k' k_1 w P.$$

In other words, the Weyl chambers $a_c k k_1 w A^+ o$ and $k' k_1 w A^+ o$ in X are asymptotic.

Therefore by [9, 1.6.6(4)],

$$d_{\text{Haus}}(a_c k k_1 w A^+ o, k' k_1 w A^+ o) = d(a_c k k_1 w o, k' k_1 w o) = d(a_c o, o) = |c|,$$

where d_{Haus} denotes the Hausdorff distance. It follows that the Hausdorff distance between $\pi^{-1}(o) = K_0 k_1 \mathcal{W} A^+ o$ and $a_c \pi^{-1}(o) = a_c K_0 k_1 \mathcal{W} A^+ o$ is at most $|c|$. □

Corollary 8.6. *Let $c \in \mathbb{R}$.*

1. *The Hausdorff distance between $\pi^{-1}(\tilde{\beta})$ and $\mathsf{b}\pi^{-1}(\tilde{\beta}) = a_c\pi^{-1}(\tilde{\beta})$ is at most $|c|$.*
2. *Suppose that $c \geq \operatorname{wd}(\mathsf{b})$. If y and y' are points in $Y - \overline{\mathcal{N}_c(\tilde{\beta})}$ lying in distinct connected components, then the fibers $\pi^{-1}(y)$ and $\mathsf{b}\pi^{-1}(y')$ are disjoint.*

Proof. (1). Using the A_0-equivariance of π, this follows from Lemma 8.5.

(2). Since the nearest point projection from X to a convex subset is 1-Lipschitz, the minimal distance between any two fibers $\pi^{-1}(y)$ and $\pi^{-1}(y')$ is precisely $d(y, y')$.

Write Y as the union of two closed half-planes Y_- and Y_+ sharing the common boundary $\tilde{\beta}$. Using the nearest point projection π, the symmetric space X can be written as the union of the following connected smooth submanifolds with boundaries:

$$X_\pm \coloneqq \pi^{-1}(Y_\pm).$$

Note that $\partial X_\pm = \pi^{-1}(\tilde{\beta})$. The interiors of X_+ and X_- are disjoint.

Let $y \in (Y_- - \overline{\mathcal{N}_c(\tilde{\beta})})$ be an arbitrary point, where $c \geq \operatorname{wd}(\mathsf{b})$. So, $d(y, \tilde{\beta}) = c + \varepsilon$ for some $\varepsilon > 0$. Then

$$d(\mathsf{b}_s\pi^{-1}(y), \partial(\mathsf{b}_s X_+)) = d(\pi^{-1}(y), \pi^{-1}(\tilde{\beta})) \geq c + \varepsilon, \quad s \in \mathbb{R},$$

where b_s is defined in (8.6). By part (1), the Hausdorff distance between $\partial X_- = \pi^{-1}(\tilde{\beta})$ and $\partial(\mathsf{b}_s X_+) = \mathsf{b}_s\pi^{-1}(\tilde{\beta})$ for $s \in [0, 1]$ is at most $\operatorname{wd}(\mathsf{b})$.

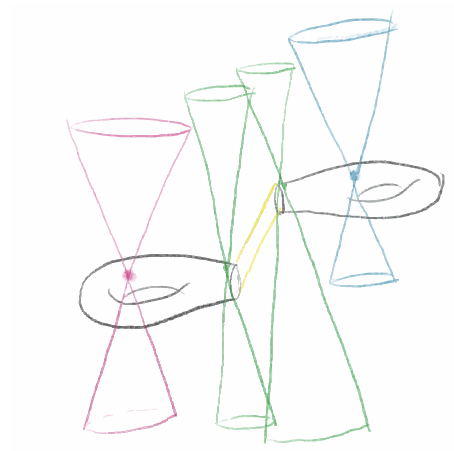

FIGURE 6. Fibers of the nearest projection map

Since $c \geq \mathrm{wd}(\mathsf{b})$, by the triangle inequality, it follows that

$$d(\mathsf{b}_s\pi^{-1}(y), \pi^{-1}(\tilde{\beta})) \geq \varepsilon, \quad s \in [0,1].$$

Thus $\{\mathsf{b}_s\pi^{-1}(y) : s \in [0,1]\}$ lies in the same connected component of $X - \pi^{-1}(\tilde{\beta})$ (since the family is disjoint from $\pi^{-1}(\tilde{\beta})$), which must be X_- as $\mathsf{b}_0\pi^{-1}(y') = \pi^{-1}(y) \subset X_-$. Thus, $\mathsf{b}_1\pi^{-1}(y) = \mathsf{b}\pi^{-1}(y)$ must be disjoint from any fiber $\pi^{-1}(y')$ contained in the interior of X_+.

The same conclusion holds for $y \in (Y_+ - \overline{\mathcal{N}_c(\tilde{\beta})})$. □

We are now in a position to finish the proof of Theorem 8.2.

Proof of Theorem 8.2. That $f_{\mathsf{b},c}$ is a local isometry follows from the definition of the map. Thus, to show that $f_{\mathsf{b},c}$ is injective, it is sufficient to show that $F_{\mathsf{b}}|_{X_c}$ is injective for all $c > \mathrm{wd}(\mathsf{b})$.

Properness of $f_{\mathsf{b},c}$ then follows: suppose, for contradiction, that there exists a divergent sequence x_n in $\Gamma\backslash X_c$ such that the sequence $f_{\mathsf{b},c}(x_n)$ converges to some point $z \in \Gamma_{\mathsf{b}}\backslash X$. Being a local isometry, $f_{\mathsf{b},c}$ is an open map in the interior of $\Gamma\backslash X_{\mathsf{b}}$ and so, by replacing c by some $c' \in (\mathrm{wd}(\mathsf{b}), c)$, we have that z belongs to the interior of $\mathrm{Im}\, f_{\mathsf{b},c'}$. As $f_{\mathsf{b},c'}$ is injective, $f_{\mathsf{b},c'}^{-1}$ is a continuous map in a sufficiently small neighborhood of z contained in $\mathrm{Im}\, f_{\mathsf{b},c'}$. This implies that x_n must converges to $f_{\mathsf{b},c'}^{-1}(z)$, contradicting the assumption that (x_n) is divergent.

Now we return to showing that $F_{\mathsf{b}}|_{X_c}$ is injective. For this, we only need to show that for any two distinct connected components $Y', Y'' \subset Y - N_{c_0}(\Gamma\cdot\tilde{\beta})$,

$$F_{\mathsf{b}_s}(\pi^{-1}(Y')) \cap F_{\mathsf{b}_s}(\pi^{-1}(Y'')) = \emptyset \quad \text{for all } s \in [0,1]. \tag{8.10}$$

Since $\mathsf{b}_1 = \mathsf{b}$, the injectivity of $F_{\mathsf{b}}|_{X_{\mathsf{b}}}$ follows from this.

We will prove (8.10) by induction on the Bass-Serre tree distance $d_T(v_{Y'}, v_{Y''})$ between the vertices corresponding to Y' and Y''.

The base case when $d_T(v_{Y'}, v_{Y''}) = 1$ follows from Corollary 8.6(2).

For the inductive step, suppose that the assertion (8.10) holds whenever $d_T(v_{Y'}, v_{Y''}) \le n_0 - 1$, for some $n_0 \ge 2$. We show that the assertion still holds if $d_T(v_{Y'}, v_{Y''}) = n_0$. By the ρ_{b_s}-equivariance, we may assume that $v_{Y'}$ is either v_1 or v_2, say v_1. Then, there is a geodesic sequence in T

$$\tilde{v}_1 = v_1, \tilde{v}_2, \dots, \tilde{v}_{n_0+1} = v_{Y''}$$

connecting v_1 to $v_{Y''}$. Let $L_2 \subset Y_{\tilde{v}_2}$ be a complete geodesic separating (in Y) $Y_1 = Y_{v_1}$ from $Y_{\tilde{v}_3}$, where Y_v denotes the connected component of $Y - N_{c_0}(\Gamma \cdot \tilde{\beta})$ corresponding to the vertex $v \in T$. Clearly, L_2 also separates Y_1 from $Y_{\tilde{v}_4} \dots Y_{\tilde{v}_{n_0+1}}$. By our induction hypothesis, for all $s \in [0, 1]$, the hypersurface $F_{\mathsf{b}_s}(\pi^{-1}(L_2))$ in X does not intersect

$$F_{\mathsf{b}_s}(\pi^{-1}(Y_{v_1})), \quad F_{\mathsf{b}_s}(\pi^{-1}(Y_{\tilde{v}_{n_0+1}})).$$

For $s = 0$, i.e., for F_{b_0}, the identity map, the hypersurface $F_{\mathsf{b}_0}(\pi^{-1}(L_2))$ in X separates $F_0(\pi^{-1}(Y_1))$ from $F_0(\pi^{-1}(Y_{\tilde{v}_{n_0+1}}))$. Therefore, by continuity, we conclude that in X, for all $s \in [0, 1]$, $F_{\mathsf{b}_s}(\pi^{-1}(L_2))$ separates $F_{\mathsf{b}_s}(\pi^{-1}(Y_{v_1}))$ from $F_{\mathsf{b}_s}(\pi^{-1}(Y_{\tilde{v}_{n_0+1}}))$. This proves (8.10), thereby concluding the proof of the result. □

Chaotic floating planes.

Theorem 8.7. *Let $L = hA_0 o \in Y$ be a geodesic so that $\Gamma\backslash\Gamma L$ is disjoint from the $2r$-neighborhood of β for some $r > 0$. There exists $t_0 > 0$, depending only on L and r such that for all $t > t_0$ and any $\mathsf{b} \in B$ with width $\mathrm{wd}(\mathsf{b}) < r$, we have*

1. $\dim \overline{\Gamma_{\mathsf{b}}\backslash\Gamma_{\mathsf{b}} h a_t H} = \dim \overline{\Gamma\backslash\Gamma h A_0} + 2$;
2. *If $\Gamma h A_0$ is admissible, then*
$$\frac{1}{2}\left(\dim \overline{\Gamma\backslash\Gamma h A_0} + 3\right) \le \overline{\Gamma_{\mathsf{b}}\backslash\Gamma_{\mathsf{b}} Y_{L,t}} \le \dim \overline{\Gamma\backslash\Gamma h A_0} + 1.$$
3. *Let L be an admissible geodesic with $1 < \dim(\overline{\Gamma\backslash\Gamma L}) < 2$. Then*
$$\frac{1}{2}\left(\dim \overline{\Gamma\backslash\Gamma L} + 3\right) \le \overline{\Gamma_{\mathsf{b}}\backslash\Gamma_{\mathsf{b}} Y_{L,t}} \le \dim \overline{\Gamma\backslash\Gamma L} + 1.$$
In particular,
$$2 < \dim \overline{\Gamma_{\mathsf{b}} Y_{L,t}} < 3.$$

Proof. Let $L := hA_0 o$. Since $\beta \subset S = \Gamma\backslash Y$ is closed, $\{x \in S : \ d(x, \beta) \le r\}$ is closed and hence $\overline{\Gamma L}$ is disjoint from $N_{r_0}(\tilde{\beta})$ for some r_0 strictly bigger than $\mathrm{wd}(\mathsf{b})$. Now consider the floating plane $Y_{L,t} = h a_t H o$. By Theorem 6.11, there exists $t_0 > 0$ depending on L and r such that for all $t \ge t_0$, the nearest projection $\pi(Y_{L,t})$ is contained in the $r/2$-neighborhood of L. By the H-equivariance, the nearest projection $\pi(\gamma Y_{L,t})$ is contained in the $r/2$-neighborhood of γL for all $\gamma \in \Gamma$. Therefore $\pi(\overline{\Gamma h a_t H o})$ is disjoint from r-neighborhood of $\tilde{\beta}$. Hence,

$$\overline{\Gamma\backslash\Gamma h a_t H} \subset \Gamma\backslash G_r.$$

Since $\mathrm{wd}(\mathsf{b}) < r$, Corollary 8.3 implies that

$$\bar{f}_{\mathsf{b},r} : \Gamma\backslash G_r \to \Gamma_{\mathsf{b}}\backslash G$$

is a proper locally isometric embedding, where $\bar{f}_{\mathsf{b},r}([\Gamma g]) = [\Gamma_{\mathsf{b}} g]$ for all $g \in G_{\mathsf{b}}$. The local isometric property everywhere implies that the Hausdorff dimension of $\overline{\Gamma\backslash\Gamma h a_t H}$ is equal to that of its image in $\Gamma_{\mathsf{b}}\backslash G$. Since $\bar{f}_{\mathsf{b},r}(\Gamma\backslash\Gamma g) = \Gamma_{\mathsf{b}}\backslash\Gamma_{\mathsf{b}} g$ and $\bar{f}_{\mathsf{b},r}$ is a proper map, the image of $\overline{\Gamma\backslash\Gamma h a_t H}$ is closed and hence

$$\bar{f}_{\mathsf{b},r}(\overline{\Gamma\backslash\Gamma h a_t H}) = \overline{\Gamma_{\mathsf{b}}\backslash\Gamma_{\mathsf{b}} h a_t H}.$$

Therefore the claim (1) follows from Theorem 7.2. Similarly, Equation (3.12) and theorem 7.2 imply (2). If $1 < \dim(\overline{\Gamma\backslash\Gamma L}) < 2$, then $\dim(\overline{\Gamma\backslash\Gamma L}) = \dim(\overline{\Gamma\backslash\Gamma h A_0})$; see Corollary 9.3 below. Hence (3) follows. ☐

Remark 8.8. Much of the discussion in this paper also applies when $\Gamma < H$ is a torsion-free nonuniform lattice. In this situation, the quotient $\Gamma\backslash Y$ is a noncompact, finite-area hyperbolic surface with finitely many cusps. Choosing a nonperipheral simple closed curve $\beta \subset S$, one may again decompose Γ over the cyclic subgroup generated by $\delta = [\beta] \in \Gamma$, as before. For each $\mathbf{b} \in B = C_G(\delta)^\circ$, we obtain a homomorphism $\rho_{\mathbf{b}} : \Gamma \to G$ defined as above.

For each $\mathbf{b} \in B$, $\rho_{\mathbf{b}}$ is discrete and faithful ([11]). In particular, Theorem 1.2 remains valid in this setting.

Remark 8.9. For a cocompact lattice $\Gamma < H$, Pavez showed that any Hausdorff dimension between 1 and 3 can occur as the dimension of the closure of some orbit $\Gamma\backslash\Gamma h A_0 \subset \Gamma\backslash H$ [22]; any number between 3 and 5 can arise as the Hausdorff dimension of the closure of a floating plane $\Gamma\backslash\Gamma h a_t H$.

It is a natural question, for a given simple closed geodesic β, which Hausdorff dimensions can be realized by geodesic flow closures $\overline{\Gamma\backslash\Gamma h A_0} \subset \Gamma\backslash H$ whose projection to the surface $\Gamma\backslash Y$ is disjoint from β. The maximum possible value in this setting is $1 + 2\delta_0$ where δ_0 denotes the maximum of the critical exponents of the components of $\Gamma\backslash Y - \{\beta\}$. Thus the precise question is whether any number between 1 and $1 + 2\delta_0$ can indeed be achieved.

In the next section, we show that we can find geodesic-flow closures that come arbitrarily close to dimension 1, yet avoid a prescribed closed simple geodesic.

9. Geodesic closures away from a given simple closed geodesic

The goal of this subsection is to prove Theorem 9.1 and finish the proofs of all theorems in the introduction.

In the whole section, let $S = \Gamma\backslash\mathbb{H}^2$ be a closed hyperbolic surface for a torsion-free cocompact lattice $\Gamma < \mathrm{PSL}_2(\mathbb{R})$. The following theorem combined with with Theorem 8.7 implies Theorem 1.2 and 1.4.

Theorem 9.1. *Let $\beta_1, \dots, \beta_m$ be pairwise disjoint simple closed geodesics in S and let $S' \subset S$ be a connected component of $S - \bigcup_i \beta_i$. There exists* *a sequence of immersed complete admissible geodesics $\ell_n \subset S$, contained in S', such that the following hold:*

(1) $\inf\{d(\ell_n, \bigcup_{k=1}^m \beta_k) : n \in \mathbb{N}\} > 0$.
(2) *For all* $n \in \mathbb{N}$, $\dim \overline{\ell_n} > 1$.
(3) $\dim \overline{\ell_n} \to 1$ *as* $n \to \infty$.

Let A_0 be the diagonal subgroup of $\mathrm{PSL}_2(\mathbb{R})$, $K_0 = \mathrm{SO}(2)$ and $o = [K_0] = \mathrm{PSL}_2(\mathbb{R})/K_0 \simeq \mathbb{H}^2$. The unit tangent bundle of $\mathbb{H}^2$ is $\mathrm{PSL}_2(\mathbb{R})$ and an orbit of the geodesic flow is of the form $hA_0 \subset \mathrm{PSL}_2(\mathbb{R})$. Let $p : \Gamma\backslash \mathrm{PSL}_2(\mathbb{R}) \to \Gamma\backslash\mathbb{H}^2$ be the basepoint projection $x \mapsto xo$.

We recall the following theorem of Ledrappier and Lindenstrauss: for a Borel measure σ on a metric space, we denote by $\underline{\dim}\,\sigma$ the lower information dimension

$$\underline{\dim}\,\sigma = \text{ess-inf}_x \left(\liminf_{\varepsilon\to 0} \frac{\log \sigma(B(x,\varepsilon))}{\log \varepsilon} \right).$$

Theorem 9.2 ([14, Theorem 1.1])**.** *Let μ be an A-invariant probability measure on $\Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$. If $\underline{\dim}\,\mu \le 2$, then*

$$\underline{\dim}\,\mu = \underline{\dim}\,p_*\mu.$$

In order to deduce the comparison between the Hausdorff dimension of an A_0-orbit and its projection to S from Theorem 9.2, we first note a general principle: for any finite Borel measure σ on a metric space, its lower information dimension satisfies

$$\underline{\dim}\,\sigma \le \dim(\text{supp }\sigma). \tag{9.1}$$

Indeed, if $\alpha < \underline{\dim}\,\sigma$, then for almost all x, there exists $r_x > 0$ such that $\sigma(B(x,r)) \le r^\alpha$ for all $0 < r < r_x$. By the mass distribution principle, this implies that the α-Hausdorff measure of $\operatorname{supp}\sigma$ is positive, and hence $\dim(\operatorname{supp}\sigma) \ge \alpha$. Letting $\alpha \to \underline{\dim}\,\sigma$ yields (9.1).

Corollary 9.3. *Let $\overline{xA_0} \subset \Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$ be the support of an A_0-invariant probability measure μ. Suppose that $\underline{\dim}\,\mu = \dim \overline{xA_0}$ and $\dim \overline{xA_0} \le 2$. Then*

$$\dim \overline{xA_0 o} = \dim \overline{xA_0}.$$

Proof. Applying (9.1) to the measure $\sigma = p_*\mu$ gives $\underline{\dim}\,p_*\mu \le \dim(\overline{xA_0 o})$. Since the projection p is 1-Lipschitz, Hausdorff dimension can only decrease under p. Hence

$$\dim \overline{xA_0 o} \le \dim \overline{xA_0} = \underline{\dim}\,\mu.$$

By Theorem 9.2, we have $\underline{\dim}\,p_*\mu = \underline{\dim}\,\mu$ and the two inequalities above therefore give $\dim \overline{xA_0 o} = \dim \overline{xA_0}$. □

For a non-elementary convex cocompact subgroup $\Gamma_0 < \mathrm{PSL}_2(\mathbb{R})$, let $\Omega_{\Gamma_0} \subset \Gamma_0\backslash \mathrm{PSL}_2(\mathbb{R}) = \mathrm{T}^1(\Gamma_0\backslash\mathbb{H}^2)$ denote the non-wandering set of the geodesic flow, which is the union of all $v \in \mathrm{T}^1(\Gamma_0\backslash\mathbb{H}^2)$ whose forward and backward end points of the geodesic determined by v belong to the limit set of Γ_0. Let $\mathsf{m} = \mathsf{m}_{\Gamma_0}$ denote the Bowen-Margulis-Sullivan measure on Ω_{Γ_0}, which is the A_0-invariant probability measure of maximal entropy [25].

Proposition 9.4. *Let $\Gamma_0 < \mathrm{PSL}_2(\mathbb{R})$ be a non-elementary convex cocompact subgroup. Then*

$$\underline{\dim}\, \mathsf{m}_{\Gamma_0} = \dim \Omega_{\Gamma_0} = 1 + 2\delta_{\Gamma_0}$$

where δ_{Γ_0} is the critical exponent of Γ_0. Moreover, Ω_{Γ_0} is admissible in the sense of Definition 7.1.

Proof. Let Λ_0 denote the limit set of Γ_0 and set $\delta_0 = \delta_{\Gamma_0}$. By Sullivan [25], the Patterson-Sullivan measure on Λ_0 is proportional to the Hausdorff measure $\mathcal{H}^{\delta_0}|_{\Lambda_0}$ and Λ_0 is Ahlfors δ_0-regular: there exists $c > 1$ such that for any $\xi \in \Lambda_0$ and $0 < r \le \mathrm{diam}(\Lambda_0)$,

$$c^{-1}\, r^{\delta_0} \le \mathcal{H}^{\delta_0}(B(\xi, r) \cap \Lambda_0) \le c\, r^{\delta_0}.$$

In particular, any nonempty open subset of Λ_0 has both Hausdorff and box dimension equal to δ_0, so (see, e.g., [10]),

$$\dim(\Lambda_0 \times \Lambda_0) = 2 \dim \Lambda_0 = 2\delta_0$$

The Hopf parametrization

$$\Phi : (\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 - \mathrm{diag}) \times \mathbb{R} \to \mathrm{T}^1(\mathbb{H}^2)$$

is locally bi-Lipschitz and $\Omega_{\Gamma_0} = \Gamma_0 \backslash \Phi((\Lambda_0 \times \Lambda_0 - \mathrm{diag}) \times \mathbb{R})$. Hence

$$\dim \Omega_{\Gamma_0} = 1 + \dim(\Lambda_0 \times \Lambda_0 - \mathrm{diag}) = 1 + 2\delta_0.$$

Since products and locally bi-Lipschitz images of Ahlfors regular sets are Ahlfors regular, it follows from the δ_0-Ahlfors regularity of Λ_0 that Ω_{Γ_0} is $1 + 2\delta_0$-Ahlfors regular. Moreover, in Hopf coordinates, the BMS measure m_{Γ_0} is locally equivalent to $\mathcal{H}^{\delta_0}|_{\Lambda_0} \times \mathcal{H}^{\delta_0}|_{\Lambda_0} \times \mathrm{Leb}$, so the push-forward to Ω_{Γ_0} is locally equivalent to $\mathcal{H}^{1+2\delta_0}|_{\Omega_{\Gamma_0}}$. Since Ω_{Γ_0} is $1+2\delta_0$-Ahlfors regular, it follows that

$$\underline{\dim}\, \mathsf{m}_{\Gamma_0} = 1 + 2\delta_0.$$

Finally, admissibility follows since, for all sufficiently small basic open subsets $\mathcal{O}$, $\pi_\pm(\Omega_{\Gamma_0} \cap \mathcal{O})$ are open subsets of Λ_0 $\square$

Before discussing the proof of Theorem 9.1, we describe a procedure for constructing immersed geodesics ℓ in a compact hyperbolic surface Σ' with nonempty geodesic boundary $\partial\Sigma'$ such that $\dim \overline{\ell} > 1$.

Suppose that Σ' is a convex compact surface of S whose boundary is the disjoint union of simple closed geodesics $\alpha_1, \ldots, \alpha_k$. Let Σ denote the double of Σ' along $\partial\Sigma'$; this is a closed hyperbolic surface. Consider the universal covering map $\mathbb{H}^2 \to \Sigma$, and identify $\pi_1(\Sigma)$ with a cocompact lattice in $\mathrm{PSL}(2, \mathbb{R})$ via the holonomy representation. For each i, let $\tilde{\alpha}_i \subset \mathbb{H}^2$ be a lift of α_i to $\mathbb{H}^2$. Let U be a connected component of $\mathbb{H}^2 - \bigcup_{i=1}^k \pi_1(\Sigma)\tilde{\alpha}_i$ so that the projection of U to Σ is precisely Σ'. Let Γ_0 be the stabilizer of U in $\pi_1(\Sigma)$. Then

$$\Sigma_0 := \Gamma_0 \backslash \mathbb{H}^2$$

is a convex cocompact hyperbolic surface whose compact convex core is isometric to $\Gamma_0\backslash\overline{U} = \Sigma'$. The identity map of $\mathbb{H}^2$ induces a covering map

$$f : \Sigma_0 \to \Sigma$$

and its restriction to the convex core of Σ_0 is an isometry onto Σ'. We refer to the critical exponent of Γ_0 as the critical exponent of Σ_0, or of Σ' by abuse of terminology.

Proposition 9.5. *Let Σ' be a convex compact subsurface of S whose boundary $\partial\Sigma'$ is a disjoint union of simple closed geodesics. If the critical exponent Σ', denoted by δ', is strictly smaller than $1/2$, then there exists a complete admissible geodesic $\ell \subset \Sigma'$ whose closure $\overline{\ell} \subset \Sigma'$ has Hausdorff dimension $1 + 2\delta'$.*

Proof. Let Σ_0 be as above, and consider its unit tangent bundle $\mathrm{T}^1(\Sigma_0) = \Gamma_0\backslash\operatorname{PSL}_2(\mathbb{R})$. By Proposition 9.4, the non-wandering set Ω_0 is admissible and

$$\dim \Omega_0 = 1 + 2\delta' = \underline{\dim}\, \mathsf{m}_{\Gamma_0} \tag{9.2}$$

where m_{Γ_0} denotes the Bowen-Margulis-Sullivan measure on $\Gamma_0\backslash\operatorname{PSL}_2(\mathbb{R})$. Since m_{Γ_0} is A_0-ergodic, m_{Γ_0}-almost all geodesic flow lines are dense in Ω_0. In particular, there exists a geodesic flow line $\mathcal{G} \subset \mathrm{T}^1(\Sigma_0)$ such that

$$\overline{\mathcal{G}} = \Omega_0 = \operatorname{supp} \mathsf{m}_{\Gamma_0}. \tag{9.3}$$

The basepoint projection $p : \mathrm{T}^1(\Sigma_0) \to \Sigma_0$ maps Ω_0 into the convex core of Σ_0 which embeds isometrically into Σ'. Hence $p(\mathcal{G}) \subset \Sigma'$, and we may regard $\mathcal{G}$ as a geodesic in $\mathrm{T}^1(\Sigma)$ with basepoints in Σ'. Since $\delta' < 1/2$, we have $\dim \overline{\mathcal{G}} < 2$. Combining (9.2) and (9.3), we may apply Corollary 9.3 to $\mathcal{G} \subset \mathrm{T}^1(\Sigma)$ and deduce

$$\dim \overline{p(\mathcal{G})} = \dim \overline{\mathcal{G}}.$$

Since $\overline{\mathcal{G}} = \Omega_0$, $\mathcal{G}$ is admissible. Setting $\ell = p(\mathcal{G})$, we obtain $\dim \overline{\ell} = 1 + 2\delta'$, completing the proof. ☐

To construct immersed geodesics with closure of dimension just above 1, we will produce convex cocompact subsurfaces with arbitrarily small critical exponent. The following elementary observation provides a convenient source of such examples.

Lemma 9.6. *Let g_1, g_2 be hyperbolic elements of $\operatorname{PSL}_2(\mathbb{R})$ which generate a Schottky subgroup and set $\Gamma_n := \langle g_1^n, g_2^n \rangle$ for $n \in \mathbb{N}$. Then the critical exponent δ_{Γ_n} tends to 0 as $n \to \infty$.*

Proof. Since the word metric on Γ_1 and the restriction of the hyperbolic metric on the orbit $\Gamma_1 o$ are quasi-isometric, there exists a constant $c > 1$ (independent of n) such that any reduced word w in $g_1^{\pm n}$ and $g_2^{\pm n}$ of length k satisfies $d(o, wo) \geq c^{-1}nk - c$. Since the number of reduced words of length

k in $\mathbb{Z} * \mathbb{Z}$ is $4 \cdot 3^{k-1}$, there exists $c' > 0$ such that the number of $w \in \Gamma_n$ with $d(o, wo) < T$ is at most $c' \cdot e^{c'T/n}$ for all $n \geq 1$. Since

$$\delta_{\Gamma_n} = \limsup_{T \to \infty} \frac{1}{T} \log \#\{w \in \Gamma_n : d(o, wo) < T\} \leq c'/n,$$

the claim follows. □

In particular, by replacing a Schottky subgroup with large powers of its generators, we obtain convex cocompact surfaces whose limit sets have dimension approaching zero. These will serve as building blocks for the admissible geodesics we construct inside a pair of pants.

We now explain how Lemma 9.6 may be combined with Proposition 9.5 to produce admissible geodesics inside a fixed pair of pants $S' \subset S$ whose closures have Hausdorff dimension just slightly larger than 1 but uniformly bounded away from the boundary of S'.

Proposition 9.7. *Let $S' \subset S$ be a pair of pant with geodesic boundary. Then there exists a sequence of admissible geodesics ℓ_n such that $\overline{\ell_n} \subset \operatorname{int} S'$, $\inf_{n \in \mathbb{N}} d(\partial S', \overline{\ell_n}) > 0$, $\dim \overline{\ell_n} > 1$ for all $n \in \mathbb{N}$, and*

$$\dim \overline{\ell_n} \to 1 \quad \textit{as } n \to \infty.$$

Proof. Let $\Gamma_0 < \Gamma$ be a convex cocompact subgroup such that the core of the surface

$$\Sigma_0 := \Gamma_0 \backslash \mathbb{H}^2$$

is isometric to S'. We will identify the core of Σ_0 with S'. Choose generators $\gamma_1, \gamma_2 \in \Gamma_0$ such that the boundary curves of S' are represented by γ_1, γ_2, $\gamma_1\gamma_2 \in \Gamma_0$.

Our goal is to find, inside S', a family of convex cocompact coverings Σ_n whose critical exponents δ_{Γ_n} tend to zero. Proposition 9.5 will then guarantee the existence of admissible geodesics on each Σ_n with closure of dimension $1 + 2\delta_{\Gamma_n}$.

Pick a pair of hyperbolic elements $g_1, g_2 \in \Gamma_0$ which generates a Schottky subgroup that contains no nontrivial powers of conjugates of the elements γ_1, γ_2, $\gamma_1\gamma_2$. To see such $g_1, g_2 \in \Gamma_0$ exist, put a hyperbolic structure on Σ_0 such that all its ends are cusps. In this case the only parabolic elements of Γ_0 are the nontrivial powers of conjugates of γ_1, γ_2, $\gamma_1\gamma_2$. But since Γ_0 is non-elementary, it contains a convex cocompact Schottky group generated by g_1 and g_2 which then cannot contain any such elements.

For each $n \in \mathbb{N}$, set

$$\Gamma_n := \langle g_1^n, g_2^n \rangle, \quad \Sigma_n := \Gamma_n \backslash \mathbb{H}^2.$$

By Lemma 9.6,

$$\delta_{\Gamma_n} \to 0 \quad \text{as } n \to \infty.$$

Consider the covering map

$$p_n : \Sigma_n \to S$$

which clearly factors through $p_1 : \Sigma_1 \to S$.

We claim that the image $p_1(\mathrm{core}(\Sigma_1))$ misses the boundary of S'. To see this, suppose for contradiction that there exists $x \in \mathrm{core}(\Sigma_1)$ such that $p_1(x) \in c'$, where $c' \subset \partial S'$ is a connected component. Since p_1 is a local isometry and $p_1(\mathrm{core}(\Sigma_1)) \subset S'$, it follows that $x \in \partial\,\mathrm{core}(\Sigma_1)$. Let $c \subset \partial\,\mathrm{core}(\Sigma_1)$ be the connected component containing x. Then $p_1(c)$ is a closed geodesic contained in S' containing $p_1(x)$. On the other hand, since $p_1(x) \in \partial S'$, c' is the only closed geodesic passing through $p_1(x)$ contained in S'. Therefore $p_1(c) = c'$. Since $c' \subset S'$ is peripheral in S', the conjugacy class in Γ_1 representing c contains a conjugate of a nontrivial power of γ_1, γ_2, or $\gamma_1\gamma_2$, contradicting the choice of g_1, g_2 made above.

Since $\delta_{\Gamma_n} \to 0$ as $n \to \infty$, Proposition 9.5 guarantees that for all large enough $n \in \mathbb{N}$, there exists a geodesic ℓ_n' lying in the interior of $\mathrm{core}(\Sigma_n)$ such that $\overline{\ell_n'}$ has Hausdorff dimension $1 + 2\delta_{\Gamma_n}$. Let

$$\ell_n = p_n(\ell_n') \subset p(\mathrm{core}(\Sigma_1)).$$

Since $\mathrm{core}(\Sigma_n)$ is compact and hence the restriction of p_n to $\mathrm{core}(\Sigma_n)$ is a proper immersion, it follows that

$$\overline{\ell_n} = p_n(\overline{\ell_n'}) \quad \text{and} \quad \dim \overline{\ell_n} = \dim \overline{\ell_n'} = 1 + 2\delta_{\Gamma_n}$$

for all large $n \in \mathbb{N}$. This finishes the proof. □

Now we are ready for the proof of Theorem 9.1.

Proof of Theorem 9.1. We can extend the set of simple closed geodesics $\{\beta_1, \ldots, \beta_m\} \subset S$ to pants decomposition of S whose boundaries are geodesics. Let $S' \subset S$ be a pair of geodesic pant in this decomposition and ℓ_n be the sequence of admissible geodesics given by Proposition 9.7. Since $\overline{\ell_n}$ lies in the interior of S', they do not intersect any of β_i, as desired. □

Proof of Theorem 1.1. It follows from Theorem 1.2 except for the integrability claim.

By [17], there exists a closed hyperbolic surface $S = \Gamma\backslash\mathbb{H}^2$, where $\Gamma < H$ is a cocompact lattice, such that the inclusion homomorphism $\varphi : \Gamma \to G$,

$$\Gamma \to H \hookrightarrow G$$

is *integral*, i.e., $\varphi(\Gamma) < \mathrm{SL}_3(\mathbb{Z})$. Let β be an oriented simple closed geodesic in S. Choosing a basepoint x_0 on β, we identify $\pi_1(S, x_0)$ with Γ. Let $\gamma = [\beta] \in \Gamma$ be the element representing β. We can further assume that $\beta \in A_0 = A \cap H$. Suppose that there also exists an element $\mathsf{a} \in (A - A_0)$ such that the bulged representation ρ_{a} is also integral. See [17] for such examples.

Pick an auxiliary oriented non-separating simple closed geodesic $\sigma \subset S$ disjoint from β and extend $\sigma \cup \beta$ it to a geodesic pants decomposition of S so that there exists a pair of pant $S_0 \subset S$ disjoint from σ. By Theorem 9.1, there exists an immersed complete geodesic $\ell = \Gamma\backslash\Gamma L \subset S$ contained in S_0 such that

$$d(\overline{\ell}, \beta \cup \sigma) > 0 \quad \text{and} \quad 1 < \dim \overline{\ell} < 2.$$

For any $r > 0$, there exists a finite Riemannian cover

$$p : S' \to S \tag{9.4}$$

such that the following holds: there are connected components β' of $p^{-1}(\beta)$ and a lift ℓ' of ℓ in S' such that

$$p|_{\beta'} \quad \text{and} \quad p|_{\overline{\ell'}}$$

are both homeomorphisms onto β and $\overline{\ell}$, respectively, and

$$d(\overline{\ell'}, \beta') > r. \tag{9.5}$$

One can construct such a cover (9.4) as follows. Let g be the genus of S. Since σ is non-separating, the abelianization map

$$\Gamma \to \Gamma^{\text{ab}} \cong \mathbb{Z}^{2\mathsf{g}}$$

maps the homotopy class $[\sigma]$ to a nontrivial primitive element $z \in \Gamma^{\text{ab}}$. Fix a homomorphism $\Gamma^{\text{ab}} \to \langle z \rangle \cong \mathbb{Z}$, extending the identity map $\langle z \rangle \to \langle z \rangle$, and consider its composition with $\Gamma \to \Gamma^{\text{ab}}$; its mod n reduction gives a surjection

$$f_n : \Gamma \to \mathbb{Z}/n\mathbb{Z}.$$

Consider the degree n regular Riemannian covering

$$p_n : S_n = \ker f_n \backslash \mathbb{H}^2 \to S.$$

Each $p_n^{-1}(\overline{\ell})$ and $p_n^{-1}(\beta)$ has n connected components, each homeomorphic to $\overline{\ell}$ and β, respectively, under the covering map p_n. Picking n large enough, one may choose appropriate connected components $\overline{\ell}'$ and β' of $p_n^{-1}(\overline{\ell})$ and $p_n^{-1}(\beta)$ so that (9.5) holds. Compare with the figure below:

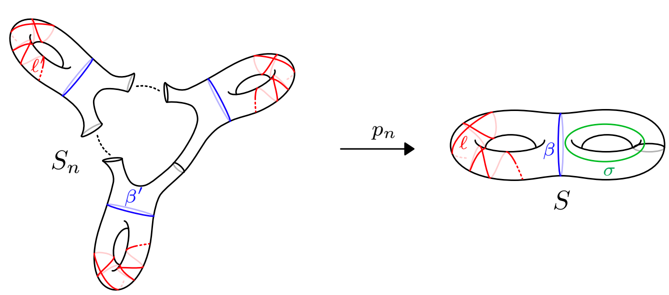

FIGURE 7

Since p is a proper immersion, one has

$$\dim \overline{\ell} = \dim \overline{\ell'}.$$

Note that $\mathsf{a} \in a_{s_0} A_0$, where $s_0 = \text{wd}(\mathsf{a})$. Choose an appropriate covering p in (9.4) such that r given by (9.5) is strictly greater than $|s_0|$. Equip S' with the basepoint $x_0' = p|_{\beta'}^{-1}(x_0)$. Then, the monomorphism

$$p_* : \pi_1(S', x_0') \longrightarrow \pi_1(S, x_0) = \Gamma$$

maps the homotopy class of β' to γ. Let $\Gamma' := \rho(\pi_1(S', x_0'))$. Clearly, the representation

$$\rho'_{\mathsf{a}} := (\rho_{\mathsf{a}})|_{\Gamma'}$$

is integral and $\rho_{\mathsf{a}}(\Gamma') < \mathrm{SL}_3(\mathbb{Z})$ satisfies the hypothesis of Theorem 1.2.

Appendix A. Orthogonal planes with fractal closures

In this appendix, we describe the closures of geodesic planes which are orthogonal to Y along a geodesic L. Using the bulging deformation, we show that closures of an orthogonal plane can be as chaotic as the closure of L (Theorem A.3). By an irreducible geodesic plane in X, we mean a totally geodesic plane of the form $gY = gHo \subset X$ for some $g \in G$.

Lemma A.1. *For every complete geodesic line $L \subset Y$, there is a unique irreducible geodesic plane Z_L in X such that $Y \cap Z_L = L$. Moreover, Y and Z_L are orthogonal to each other.*

Proof. Without loss of generality, we may assume that $L = A_0 o = \{h_t o : t \in \mathbb{R}\}$. Let $Z = gY$, $g \in G$, be an irreducible geodesic plane containing L. Since Z contains o, we may assume $g = k \in K$, replacing g by gh for some suitable $h \in H$. Now $Z = kHk^{-1}(o) \supset A_0 o$. By considering the tangent subspaces, it implies that $\mathfrak{a}_0 \subset k\mathfrak{h}k^{-1}$. Therefore $k^{-1}\mathfrak{a}_0 k$ is a one-dimensional symmetric subspace of $\mathfrak{h}$ and hence must be of the form $k_0^{-1}\mathfrak{a}_0 k_0$ for some $k_0 \in K_0$. Hence by replacing k by kk_0^{-1}, we may assume that $k \in N_K(\mathfrak{a}_0)$. By a direct computation, we can show that $N_K(\mathfrak{a}_0)$ is the subgroup generated by $A \cap K = \{g_i : 1 \le i \le 4\}$ and $w_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ where $g_1 = e$, $g_2 = \mathrm{diag}(-1, 1, -1)$, $g_3 = \mathrm{diag}(1, -1, -1)$ and $g_4 = (-1, -1, 1)$. Since g_2 and w_0 normalizes H, and $g_3 = g_4 g_2$, we may assume that $g = g_3$ up to the normalizer of H. Therefore $kY = g_3 Y$ is the only irreducible geodesic plane containing L, which is different from Y. We set $Z_L := g_3 Y$. Moreover, Y and Z_L are orthogonal to each other as can be checked using the Killing form. This proves the claim. □

The following result is an analog of Theorem 7.2 in the case of orthogonal planes:

Lemma A.2. *Let $\Gamma < H$ be a discrete subgroup and let $L \subset Y$ be a complete geodesic with $\dim \overline{\Gamma L} < 2$. Then*

$$\dim\left(\overline{\Gamma Z_L}\right) = \dim\left(\overline{\Gamma L}\right) + 1. \tag{A.1}$$

Proof. Write $\overline{\Gamma Z_L} = \bigcup_l Z_l$ where the union is taken over all complete geodesics $l \subset \overline{\Gamma L}$. Consider the normal bundle $N_Y X$ of $Y \subset X$. The exponential map

$$\exp : N_Y X \longrightarrow X$$

is a diffeomorphism. Now, $\overline{\Gamma Z_L}$ can be written as the union of a family $\mathcal{L}$ of geodesic lines in X orthogonal to Y, where any two such lines are allowed to intersect only inside Y. Correspondingly, there is a family of lines $\mathcal{L}' \subset N_Y X$ whose image under the exponential map is precisely $\mathcal{L}$. Since the exponential map is a diffeomorphism, the Hausdorff dimension of $\overline{\Gamma Z_L}$ is precisely $\dim\big(\bigcup_{l\in\mathcal{L}'} l\big)$. Denote by $\mathbb{P}(N_Y X)$ the projectivization of $N_Y X$. Then the family $\mathcal{L}'$ determines a closed subset $R \subset \mathbb{P}(N_Y X)$. Since the natural projection

$$p : \mathbb{P}(N_Y X) \longrightarrow Y$$

is a submersion and satisfies $p(R) = \overline{\Gamma L}$, we obtain $\dim R \geq \dim\big(\overline{\Gamma L}\big)$. On the other hand, the union of the lines satisfies

$$\dim \overline{\Gamma Z_L} = \dim\Big(\bigcup_{l\in\mathcal{L}'} l\Big) = \dim R + 1.$$

Therefore $\dim \overline{\Gamma Z_L} \geq \dim\big(\overline{\Gamma L}\big) + 1$. To prove the reverse inequality, note that on each basic open subset $\mathcal{O} = h_0 U^+ U^- A_0$ of G where $U^\pm \subset N_0^\pm$ and $h_0 \in H$, we have $\overline{\Gamma A_0} \cap \mathcal{O} = C_{\mathcal{O}} \times A_0$, where $C_{\mathcal{O}} \subset h_0 U^+ U^-$ is a closed subset. It follows that

$$\dim C_{\mathcal{O}} + 1 = \dim\big(\overline{\Gamma A_0} \cap \mathcal{O}\big) \leq \dim \overline{\Gamma A_0}.$$

Since

$$\overline{\Gamma L} = \overline{\Gamma A_0} o = \bigcup_{\mathcal{O}} (C_{\mathcal{O}} A_0 o) = \bigcup_{\mathcal{O}} C_{\mathcal{O}} L,$$

we have $\overline{\Gamma Z_L} = \bigcup_{\mathcal{O}} C_{\mathcal{O}} Z_L$. Hence

$$\dim \overline{\Gamma Z_L} \leq 2 + \sup \dim C_{\mathcal{O}} = 1 + \dim \overline{\Gamma A_0} = 1 + \dim \overline{\Gamma L}, \tag{A.2}$$

by Corollary 9.3. This finishes the proof. □

Therefore, we deduce the following from Theorems 8.2 and 9.1:

Theorem A.3. *Let $L = hA_0 o \subset Y$ be a geodesic so that $\ell = \Gamma\backslash\Gamma L$ is disjoint from the $2r$-neighborhood of β for some $r > 0$. Suppose that $1 < \dim \overline{\ell} < 2$. Then for all $\mathsf{b} \in B$ with width* $\mathrm{wd}(\mathsf{b})$ *smaller than r, we have*

$$\dim \overline{\Gamma_{\beta,\mathsf{b}}\backslash\Gamma_{\beta,\mathsf{b}} Z_L} = \dim \overline{\ell} + 1.$$

Moreover, there exists a sequence of geodesics $L_i \subset Y$ such that

$$\dim\left(\overline{\Gamma_{\beta,\mathsf{b}}\backslash\Gamma_{\beta,\mathsf{b}} Z_{L_i}}\right) \to 2 \quad \textit{as } i \to \infty.$$

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