EFFECTIVE CIRCLE COUNT FOR APOLLONIAN PACKINGS AND CLOSED HOROSPHERES

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ABSTRACT. The main result of this paper is an effective count for Apollonian circle packings that are either bounded or contain two parallel lines. We obtain this by proving an effective equidistribution of closed horospheres in the unit tangent bundle of a geometrically finite hyperbolic 3-manifold, whose fundamental group has critical exponent bigger than 1. We also discuss applications to Affine sieves. Analogous results for surfaces are treated as well.

1. INTRODUCTION

1.1. **Apollonian circle packings.** An Apollonian circle packing is an ancient Greek construction which is made by repeatedly inscribing circles into the triangular interstices of four mutually tangent circles in the plane. In recent years, there have been many new and exciting developments in the study of Apollonian circle packings; for instance, see [15], [16], [32], [6], [20], [3], [4], [26], [27], [25], etc.

The main goal of this paper is to obtain an effective version of the counting theorem for circles in an Apollonian packing with bounded curvature.

Let $\mathcal{P}$ be an Apollonian circle packing, that is either bounded or lies between two parallel lines (i.e., congruent to the packing in Figure 2). For $T > 0$ and $\mathcal{P}$ bounded, we define the circle counting function as follows:

$$N_T(\mathcal{P}) := \#\{C \in \mathcal{P} : \mathrm{Curv}(C) < T\}$$

where $\mathrm{Curv}(C)$ denotes the curvature of C, i.e., the reciprocal of the radius of C. For $\mathcal{P}$ unbounded between two parallel lines, we adjust the definition of $N_T(\mathcal{P})$ to count circles only in a fixed period.

The main term in the asymptotic for $N_T(\mathcal{P})$ will be described in terms of the residual set of $\mathcal{P}$ (=the closure of the union of all circles in $\mathcal{P}$), denoted by $\mathrm{Res}(\mathcal{P})$. We denote by α the Hausdorff dimension of $\mathrm{Res}(\mathcal{P})$; α is independent of $\mathcal{P}$ and known to be approximately

1.30568(8) [24]. Let $\mathcal{H}^\alpha(\mathrm{Res}(\mathcal{P}))$ be the α-dimensional Hausdorff measure of $\mathrm{Res}(\mathcal{P})$ for bounded $\mathcal{P}$. For $\mathcal{P}$ between two parallel lines, we let $\mathcal{H}^\alpha(\mathrm{Res}(\mathcal{P}))$ be the measure of $\mathrm{Res}(\mathcal{P})$ in a fixed period.

The error term in our asymptotic formula depends directly on the L^2-spectral gap of the complete hyperbolic 3 manifold whose fundamental group is the symmetry group of $\mathcal{P}$. The group $\mathrm{PSL}_2(\mathbb{C})$ acts on the extended plane by linear fractional transformations. Set

$$\mathcal{A}_\mathcal{P} := \{g \in \mathrm{PSL}_2(\mathbb{C}) : g(\mathcal{P}) = \mathcal{P}\}.$$

It is known that $\mathcal{A}_P$ is a geometrically finite discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ with critical exponent equal to α (cf. [20]). The fact α is strictly bigger than 1 yields that $\alpha(2 - \alpha)$ is the smallest eigenvalue of the Laplacian Δ on the L^2-spectrum of the hyperbolic manifold $\mathcal{A}_\mathcal{P}\backslash\mathbb{H}^3$ by Sullivan [37] and is also isolated by Lax and Phillips [22]. Hence there exists $1 < s_1 < \alpha$ such that there is no eigenvalue of Δ in $L^2(\mathcal{A}_\mathcal{P}\backslash\mathbb{H}^3)$ between $\alpha(2 - \alpha)$ and $s_1(2 - s_1)$. Since all $\mathcal{A}_\mathcal{P}$'s are conjugate to each other by elements of $\mathrm{PSL}_2(\mathbb{C})$, s_1 is independent of $\mathcal{P}$.

Our effective counting result, which is a special case of our more general theorem (Theorem 1.8), can be stated as follows:

Theorem 1.1. *As $T \to \infty$,*

$$N_T(\mathcal{P}) = c_A \cdot \mathcal{H}^\alpha(\mathrm{Res}(\mathcal{P})) \cdot T^\alpha + O(T^{\alpha - \frac{2(\alpha - s_1)}{63}})$$

where $c_A > 0$ is a constant independent of $\mathcal{P}$.

Remark 1.2. (1) In [20], the asymptotic $N_T(\mathcal{P}) \sim c_\mathcal{P} \cdot T^\alpha$ was obtained with less clear interpretation of the constant $c_\mathcal{P}$.

(2) A similar type of asymptotic formula was obtained in [27] for all Apollonian packings (whether bounded or not) by counting circles in a bounded region, but with no error term.

(3) There are several different ways of understanding the constant $c_A \cdot \mathcal{H}^\alpha(\mathrm{Res}(\mathcal{P}))$ in front of the main term, due to different approaches to the counting problem. One description is given in our paper (see (8.8)). The aforementioned paper [27] gives another expression as well.

(4) An Apollonian packing $\mathcal{P}$ is called integral if the curvatures of all circles in $\mathcal{P}$ are integers. Any integral Apollonian packing is known to be either bounded or lies between two parallel lines. Therefore Theorem 1.1 applies to all integral Apollonian packings.
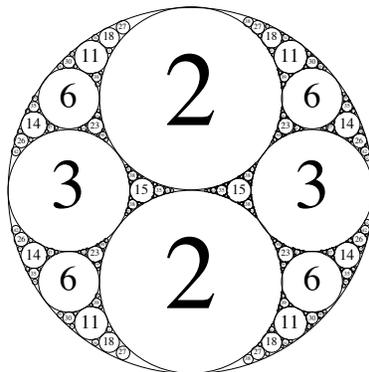
FIGURE 1. A bounded Apollonian circle packing.

FIGURE 2. An unbounded Apollonian circle packing bounded by two parallel line.

Based on the Descartes circle theorem [9], the approach in [20] was to relate the circle counting problem with the equidistribution of closed horospheres in the unit tangent bundle of the hyperbolic manifold $\mathcal{A}_{\mathcal{P}} \backslash \mathbb{H}^3$.

The new achievement of this paper is an *effective* equidistribution of closed horospheres (Theorem 1.3). Besides its application to counting problems, such equidistribution result is of independent interest in homogeneous dynamics.

1.2. **Effective equidistribution of closed horospheres.** We obtain an effective equidistribution for closed horospheres in the unit tangent bundle of hyperbolic n-manifolds for $n = 2$ or 3. Consider the upper half space $\mathbb{H}^n = \{(x, y) : x \in \mathbb{R}^{n-1}, y > 0\}$ with the metric given by $ds^2 = \frac{\sum_{i=1}^{n-1} dx_i^2 + dy^2}{y^2}$ and let $G = \text{Isom}^+(\mathbb{H}^n)$ denote the group of orientation preserving isometries of $\mathbb{H}^n$. That is, $G = \text{PSL}_2(\mathbb{R})$ for $n = 2$ and $G = \text{PSL}_2(\mathbb{C})$ for $n = 3$.

Let $\Gamma < G$ be a torsion-free discrete subgroup, which is not virtually abelian. We assume that Γ is geometrically finite, that is, it admits a finite sided fundamental domain in $\mathbb{H}^n$. The limit set $\Lambda(\Gamma)$ is the subset

of the boundary $\partial(\mathbb{H}^n) = \mathbb{R}^n \cup \{\infty\}$ consisting of all accumulation points in an orbit $\Gamma(z)$, $z \in \mathbb{H}^n$. We denote by $0 < \delta \leq n-1$ the critical exponent of Γ; it is equal to the Hausdorff dimension of $\Lambda(\Gamma)$ [38].

For $G = \mathrm{PSL}_2(\mathbb{R})$, set $K := \mathrm{PSO}(2)$, and for $G = \mathrm{PSL}_2(\mathbb{C})$, set $K := \mathrm{PSU}(2)$. In both cases, set

$$A := \left\{ a_y := \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} : y > 0 \right\},$$

and let M be the centralizer of A in K.

The hyperbolic manifold $\Gamma \backslash \mathbb{H}^n$ and its unit tangent bundle $\mathrm{T}^1(\Gamma \backslash \mathbb{H}^n)$ can be identified with the double quotient spaces $\Gamma \backslash G / K$ and $\Gamma \backslash G / M$ respectively. Accordingly, functions on $\Gamma \backslash \mathbb{H}^n$ (resp. $\mathrm{T}^1(\Gamma \backslash \mathbb{H}^n)$) can be considered as right K-invariant (resp. M-invariant) functions on $\Gamma \backslash G$. Since a_y commutes with M, a_y acts on $\Gamma \backslash G / M$ by the multiplication from the right and this action corresponds to the geodesic flow in the unit tangent bundle.

Set $N = \{g \in G : a_y^{-1} g a_y \to e \text{ as } y \to \infty\}$; the contracting horospherical subgroup under the action of a_y. Setting

$$n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

we have $N = \{n_x : x \in \mathbb{R}\}$ for $G = \mathrm{PSL}_2(\mathbb{R})$, and $N = \{n_x : x \in \mathbb{C}\}$ for $G = \mathrm{PSL}_2(\mathbb{C})$. Even though there is no action of N on $\Gamma \backslash G / M$, the N-orbits $\{[g]N := \Gamma \backslash \Gamma g M N / M : g \in G\}$ are well-defined since N is normalized by M; these orbits give rise to the stable horospherical foliation of $\mathrm{T}^1(\Gamma \backslash \mathbb{H}^n)$.

In the rest of the introduction, we assume that $(n-1)/2 < \delta < n-1$ and that $\Gamma \backslash \Gamma N$ is closed in $\Gamma \backslash G$. In particular, Γ has infinite covolume in G. By the torsion-free assumption on Γ, $\Gamma \cap NM = \Gamma \cap N$ and we can identify $\Gamma \backslash \Gamma NM / M$ with $(\Gamma \cap N) \backslash N$. Note that the quotient $(\Gamma \cap N) \backslash N$ can be naturally identified with $(\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-1-k}$ where $0 \leq k \leq n-1$ denotes the rank of the free abelian subgroup $\Gamma \cap N$.

1.3. **Equidistribution in spectral terms.** We describe the effective equidistribution of $\Gamma \backslash \Gamma N a_y$ as $y \to 0$ in $\mathrm{T}^1(\Gamma \backslash \mathbb{H}^n)$ in terms of the M-invariant spectrum of $L^2(\Gamma \backslash G)$ for a Casimir element of G.

By Lax and Phillips [22] and Sullivan [37], the Laplacian Δ on $L^2(\Gamma \backslash \mathbb{H}^n)$ has only finitely many eigenvalues

$$0 < \alpha_0 = \delta(n-1-\delta) < \alpha_1 \leq \cdots \leq \alpha_k < \tfrac{(n-1)^2}{4}$$

lying below the continuous spectrum $[\frac{(n-1)^2}{4}, \infty)$. The existence of a point eigenvalue is the precise reason that our main theorem requires the condition $\delta > (n-1)/2$. Writing $\alpha_1 = s_1(n-1-s_1)$, any positive number

$$0 < \mathbf{s}_\Gamma < \delta - s_1$$

will be referred to as a *spectral gap* of Γ.

Let $\mathcal{C}$ denote a Casimir element of $\text{Lie}(G)_\mathbb{C}$, which we normalize so that it acts on K-invariant smooth functions as the negative Laplacian $-\Delta$. Then $L^2(\Gamma\backslash G)$ contains the unique irreducible infinite dimensional subrepresentation V (a complementary series representation) on which $\mathcal{C}$ acts by the scalar $\delta(\delta - n + 1)$.

Let $\hat{K}$ denote the unitary dual of K, that is, the equivalence classes of all irreducible unitary representations of K. For $n = 2$, $\hat{K}$ can be parametrized by $\mathbb{Z}$ so that $\ell \in \hat{K}$ corresponds to the one-dimensional space V_ℓ on which $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ acts by $e^{2\ell i\theta}$. For $n = 3$, $\hat{K}$ can be parametrized by $\mathbb{Z}_{\geq 0}$ so that $\ell \in \hat{K}$ corresponds to the irreducible $2\ell + 1$ dimensional representation V_ℓ.

As a K-representation, V is decomposed into the orthogonal sum $\oplus_{\ell \in \hat{K}} V_\ell$ with the subspace V_ℓ^M of M-invariant vectors being one dimensional. Let $\phi_\ell \in C^\infty(\Gamma\backslash G) \cap L^2(\Gamma\backslash G)$ be a unit vector in V_ℓ^M for each $\ell \in \hat{K}$. We show that there exists $c_n(\ell) \neq 0$ such that for all $y > 0$,

$$\int_{n_x \in (N \cap \Gamma)\backslash N} \phi_\ell(n_x a_y)\, dx = c_n(\ell) \cdot y^{n-1-\delta}.$$

The inner product $\langle \psi_1, \psi_2 \rangle$ in $L^2(\Gamma\backslash G)$ is given by

$$\langle \psi_1, \psi_2 \rangle = \int_{\Gamma\backslash G} \psi_1(g)\overline{\psi_2(g)}\, dg$$

where dg denotes a G-invariant measure on $\Gamma\backslash G$.

The following is our main theorem on the effective equidistribution:

Theorem 1.3. *Let $n = 2$ or 3. Let $(n-1)/2 < \delta < n - 1$. For any $\psi \in C_c^\infty(\Gamma\backslash G)^M$, as $y \to 0$,*

$$\int_{(N\cap\Gamma)\backslash N} \psi(n_x a_y)\, dx$$
$$= \sum_{\ell \in \hat{K}} c_n(\ell) \cdot \langle \psi, \phi_\ell \rangle \cdot y^{n-1-\delta} + O(\mathcal{S}_{2n-1}(\psi) \cdot y^{(n-1-\delta)+\frac{2\mathbf{s}_\Gamma}{2n+1}})$$

where $\mathcal{S}_{2n-1}(\psi)$ denotes the L^2-Sobolev norm of ψ of order $2n - 1$.
Moreover

$$c_n(\ell) = O((|\ell| + 1)^{(n-2)/2}) \quad and \quad \sum_{\ell \in \hat{K}} |c_n(\ell)\langle \psi, \phi_\ell \rangle| = O(\mathcal{S}_2(\psi)).$$

Remark 1.4. (1) When Γ is a lattice in G, i.e., when $\delta = n - 1$,
an effective equidistribution for expanding closed horospheres is
well known, via the mixing of the geodesic flow and the thick-
ening argument. This argument goes back to the 1970 thesis of
Margulis [23] and was generalized by Eskin and McMullen [13].
For $n = 2$, Sarnak [34] obtained a sharper result, based on the
study of Eisenstein series.

(2) In principle, our methods should extend to prove an analogous
result for $G = \mathrm{Isom}^+(\mathbb{H}^n)$ for any $n \geq 2$; however computations
needed to understand ϕ_ℓ's seem very intricate as the dimension
gets higher.

(3) When ψ is K-invariant, Theorem 1.3 was obtained in [20]. See
also [18] for its extensions to other rank one Lie groups. See [2]
and [35] for the study of non-closed horocycles for geometrically
finite surfaces.

As mentioned before, our approach in proving Theorem 1.3 is based
on the existence of L^2-eigenfunctions on $\Gamma \backslash \mathbb{H}^n$ for $\delta > (n - 1)/2$ and
hence cannot be applied to Γ with $\delta \leq (n - 1)/2$. However a non-
effective version of Theorem 1.3 is available for any $\delta > 0$; this was
obtained in [30] when $(N \cap \Gamma) \backslash N$ is compact and in [26] in general.
In these papers, the coefficient of the main term was given in terms
of the Burger-Roblin measure associated to the stable horospherical
foliation. In applications to counting problems, it is much handier to
have this coefficient in terms of a measure instead of an infinite sum.
For this reason, we present an alternative formulation of Theorem 1.3
in Theorem 1.6.

1.4. **Equidistribution in ergodic terms.** Let ν_j denote the Patterson-
Sullivan measure on the limit set $\Lambda(\Gamma)$ associated to the basepoint
$j = (0_{n-1}, 1) \in \mathbb{H}^n$, which is unique up to a constant multiple.

Sullivan gave an explicit construction of the base eigenfunction $\phi_0 \in$
$L^2(\Gamma \backslash G)^K$ using ν_j:

$$\phi_0(n_x a_y) = \int_{u \in \mathbb{R}^{n-1}} \left(\frac{(|u|^2 + 1)y}{|x - u|^2 + y^2} \right)^\delta \, d\nu_j(u). \qquad (1.5)$$

Here and also later, we identify $\mathbb{C} = \mathbb{R}^2$ for $n = 3$, so that $|x - u|^2 = (x_1 - u_1)^2 + (x_2 - u_2)^2$ for $x = x_1 + ix_2$ and $u = (u_1, u_2)$. We normalize ν_j so that $\|\phi_0\|_2 = 1$ [38].

Define the measure $\tilde{m}_N^{\mathrm{BR}}$ on G in the Iwasawa coordinates $G = KAN$: for $\psi \in C_c(G)$,

$$\tilde{m}_N^{\mathrm{BR}}(\psi) = \int_{KAN} \psi(ka_y n_x) y^{\delta - 1} dx dy d\nu_j(k(0)).$$

This measure is left Γ-invariant and right N-invariant, and the Burger-Roblin measure m_N^{BR} (associated to the stable horospherical subgroup N) is the measure on $\Gamma \backslash G$ induced from $\tilde{m}_N^{\mathrm{BR}}$. The BR measure m_N^{BR} is an infinite measure whenever $0 < \delta < n - 1$ [26].

Theorem 1.6. *Let $n = 2$ or 3 and $(n - 1)/2 < \delta \leq n - 1$. For any $\psi \in C_c^\infty (\Gamma \backslash G)^M$, as $y \to 0$,*

$$\int_{(N \cap \Gamma) \backslash N} \psi(n_x a_y) \, dx = \kappa_\Gamma \cdot m_N^{\mathrm{BR}}(\psi) \cdot y^{n - 1 - \delta}$$

$$+ O(\mathcal{S}_{2n-1}(\psi) \cdot y^{(n-1-\delta) + \frac{2s_\Gamma}{2n+1}})$$

where $\kappa_\Gamma = \int_{x \in \mathbb{R}^{n-1}} (1 + |x|^2)^{-\delta} dx \cdot \int_{n_x \in (N \cap \Gamma) \backslash N} (1 + |x|^2)^\delta d\nu_j(x)$.

1.5. **Effective orbital counting and Affine sieves in sectors.** Let Q be a quadratic form over $\mathbb{Q}$ of signature $(n, 1)$ and $v_0 \in \mathbb{Z}^{n+1}$ a non-zero vector such that $Q(v_0) = 0$. Let G_0 denote the identity component of $\mathrm{SO}_Q(\mathbb{R})$. As well known, G_0 is isomorphic to $G = \mathrm{PSL}_2(\mathbb{R})$ (for $n = 2$) and $G = \mathrm{PSL}_2(\mathbb{C})$ (for $n = 3$) as real Lie groups. Let $\Gamma < G_0(\mathbb{Z})$ be a geometrically finite subgroup with $\delta > (n - 1)/2$ such that $v_0 \Gamma$ is discrete. For each square-free integer d, let Γ_d be a subgroup of Γ containing $\{\gamma \in \Gamma : \gamma \equiv e \pmod{\mathrm{d}}\}$ and satisfying $\mathrm{Stab}_\Gamma v_0 = \mathrm{Stab}_{\Gamma_d} v_0$.

By a theorem of Bourgain, Gamburd and Sarnak [5], there exists a uniform spectral gap, say $s_0 > 0$, for all Γ_d, d square-free with no factors in a fixed set of finitely many primes.

Consider the representation $G \to G_0$ such that N is contained in $\mathrm{Stab}_G(v_0)$, and fix a norm $\|\cdot\|$ on $\mathbb{R}^{n+1}$. For any subset $\Omega \subset K$ and $T > 0$, define the sector

$$S_T(\Omega) := \{v \in v_0 A\Omega : \|v\| < T\}.$$

Define q_Ω to be the maximum of $0 \leq q \leq 1$ such that

$$\nu_j(\epsilon\text{-neighborhood of } \partial(\Omega^{-1}(0))) \ll \epsilon^q \text{ for all small } \epsilon > 0. \quad (1.7)$$

Note that if $\partial(\Omega^{-1}(0)) \cap \Lambda(\Gamma) = \emptyset$, then $q_\Omega = 1$. We will say Ω admissible if $q_\Omega > 0$.

Theorem 1.8. *Let Ω be an admissible left M-invariant Borel subset of K. Then for any $\gamma \in \Gamma$, as $T \to \infty$,*

$$\#\{v \in v_0 \Gamma_d \gamma \cap S_T(\Omega)\} = \frac{\Xi_{v_0}(\Gamma, \Omega)}{[\Gamma : \Gamma_d]} \cdot T^\delta + O(T^{\delta - \frac{8\mathbf{s}_0}{n(n+9)(2n+1)q_\Omega}}).$$

Identifying Γ with its pull back in G, $\Xi_{v_0}(\Gamma, \Omega)$ is given by

$$\Xi_{v_0}(\Gamma, \Omega) = \kappa_\Gamma \int_{k \in \Omega^{-1}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta}. \qquad (1.9)$$

As ν_j is supported on the limit set $\Lambda(\Gamma)$, $\Xi_{v_0}(\Gamma, \Omega) > 0$ if and only if the interior of $\Omega^{-1}(0)$ intersects $\Lambda(\Gamma)$.

Given an integer-valued polynomial F on the orbit $v_0 \Gamma$, Theorem 1.8 has an application in studying integral points $\mathbf{x}$ lying in a fixed sector with $F(\mathbf{x})$ having at most R prime factors (including multiplicities). For instance, the following theorem can be deduced from Theorem 1.8 using the same analysis as in [20, section 8].

Theorem 1.10. *Suppose that $\Omega \subset K$ is an admissible subset such that the interior of $\Omega^{-1}(0)$ intersects $\Lambda(\Gamma)$. Then there exist $R \geq 1$ (depending on $\mathbf{s}_0$) and a finite set $\mathcal{B}$ of primes such that for each $1 \leq i \leq n + 1$,*

$$\#\{\mathbf{x} \in v_0 \Gamma \cap S_T(\Omega) : x_i \text{ has at most } R \text{ prime factors outside } \mathcal{B}\} \asymp \frac{T^\delta}{\log T}$$

where $\mathbf{x} = (x_1, \cdots, x_{n+1})$ and $f(T) \asymp g(T)$ means that their ratio is between two positive constants uniformly for all $T \gg 1$.

Theorem 1.8 is proved in [26] without an error term. When the norm is K-*invariant* and $\Omega = K$, it was also proved in [20]. Theorem 1.10 for $\Omega = K$ has been obtained in [20] (also see [21]).

1.6. **Organization:** Sections 2-4 are devoted to understanding the base eigenfunctions ϕ_ℓ's and their integrals over closed N-orbits. In section 2, we find a *computable* recursive formula (Theorem 2.11) for a raising operator among M-invariant vectors in a general complementary series representation of G. Using this, in section 3, we obtain an explicit description of ϕ_ℓ's which turn out to be related to the Legendre polynomials for $n = 3$. Understanding each ϕ_ℓ as a function of $\Gamma \backslash G$, rather than as a vector in the Hilbert space $L^2(\Gamma \backslash G)$, is crucial in our approach, as we need to deal with several convergence issues of the integrals of ϕ_ℓ's as well as to thicken the N-integrals of ϕ_ℓ's uniformly over all ℓ's. In section 4, we compute the N-integrals of ϕ_ℓ's and compute $c_n(\ell)$'s explicitly (modulo c_0). In section 5, we carry out the thickening of the N-integrals of ϕ_ℓ's uniformly. Since ϕ_ℓ's are not supported on

compact subsets of $\Gamma \backslash G$, this step is delicate, as we need to ensure that there is at most a polynomial error term in ℓ in this procedure. The equidistribution theorems 1.3 and 1.6 are proved in section 6 and 7 respectively. In section 8, we deduce Theorem 1.8 and Theorem 1.1 from Theorem 1.6.

Added in print: Soon after we submitted the first version of our paper to the arXive, we received a preprint by Vinogradov [39], which also proves Theorem 1.1 (with a weaker error term) using different methods.

2. Ladder operators

2.1. **Notations and Preliminaries.** Let G be $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$. Hence as a real Lie group, G is isomorphic to the identity component of $\mathrm{SO}(n,1)$ for $n=2$ and 3 respectively. In this subsection, we introduce notations which will be used throughout the paper and review some basic facts about representations of G. Let K be a maximal compact subgroup of G. Denoting by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. Let $A = \exp(\mathfrak{a})$ where $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ and let M be the centralizer of A in K.

Define the symmetric bi-linear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by

$$\langle X, Y \rangle := \frac{1}{2(n-1)} B(X,Y)$$

where $B(X,Y) = \mathrm{Tr}(\mathrm{ad}X \mathrm{ad}Y)$ denotes the Killing form for $\mathfrak{g}$. The reason for this normalization is so that the Riemmanian metric on G/K induced by $\langle \cdot, \cdot \rangle$ has constant curvature -1.

Let $\{X_i\}$ be a basis for $\mathfrak{g}_\mathbb{C}$ over $\mathbb{C}$; put $g_{ij} = \langle X_i, X_j \rangle$ and let g^{ij} be the (i,j) entry of the inverse matrix of (g_{ij}). The element

$$\mathcal{C} = \sum g^{ij} X_i X_j$$

is called the Casimir element of $\mathfrak{g}_\mathbb{C}$ (with respect to $\langle \cdot, \cdot \rangle$). It is well-known that this definition is independent of the choice of a basis and that $\mathcal{C}$ lies in the center of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ of $\mathfrak{g}_\mathbb{C}$.

For $G = \mathrm{PSL}_2(\mathbb{R})$, set

$$K = \mathrm{PSO}(2) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, \pi) \right\};$$

and for $G = \mathrm{PSL}_2(\mathbb{C})$, we set $K = \mathrm{PSU}(2)$. In both cases, we set N to be the strict upper triangular subgroup of G and A the diagonal subgroup consisting of positive diagonals. We have the Iwasawa decomposition $G = NAK$: any element g of G is written uniquely as $g = n_x a_y k$ where $n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N$, $a_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}$ and $k \in K$.

Note that $M = \{e\}$ and $M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, \pi) \right\}$ respectively.

Set $\mathbb{H}^n = \{(x, y) : x \in \mathbb{R}^{n-1}, y > 0\}$ and $j = (0_{n-1}, 1)$. The group G acts on $\mathbb{H}^n$ via the extension of the Möbius transformation action on the boundary $\mathbb{R}^{n-1} \cup \{\infty\}$. Under this action, we have $K = \mathrm{Stab}_G(j)$ and $\mathbb{H}^n = G/K \simeq \exp(\mathfrak{p})$. Now the Laplacian operator Δ on $\mathbb{H}^n$ is respectively given by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad \text{and} \quad \Delta = -y^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial}{\partial y}$$

according as $n = 2$ or 3. By Kuga's lemma, for all $\psi \in C^\infty(G)^K = C^\infty(\mathbb{H}^n)$, we have

$$\mathcal{C}(\psi) = -\Delta(\psi).$$

Consider the following elements of $\mathfrak{g}$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For $G = \mathrm{PSL}_2(\mathbb{R})$, since $\{H, E, F\}$ is a basis for $\mathfrak{g}_{\mathbb{C}}$, we can compute $\mathcal{C}$ by a direct method and obtain:

Lemma 2.1. *For $n = 2$, we have*

$$\mathcal{C} = \tfrac{1}{4} H^2 + \tfrac{1}{2}(EF + FE).$$

We set $\mathcal{C}_K = E - F = \frac{\partial}{\partial \theta}$.

To compute the Casimir element for $n = 3$, note that the Lie algebra of $K = \mathrm{PSU}(2)$ is

$$\mathfrak{k} = \left\{ X \in \mathrm{M}_2(\mathbb{C}) \ : \ \bar{X}^t = -X, \ \mathrm{tr}(X) = 0 \right\}.$$

The elements $\frac{i}{2} H, X_1 := \frac{1}{2}(E - F), X_2 := \frac{i}{2}(E + F)$ generate $\mathfrak{k}$ as a real vector space. Set

$$D := \tfrac{i}{2} H \in \mathfrak{k}_{\mathbb{C}}, \quad E^+ := X_1 - iX_2 \in \mathfrak{k}_{\mathbb{C}} \quad \text{and} \quad E^- := -X_1 - iX_2 \in \mathfrak{k}_{\mathbb{C}}. \tag{2.2}$$

The elements $\frac{1}{2} H, Y_1 := \frac{1}{2}(E + F)$ and $Y_2 := \frac{i}{2}(E - F)$ form a basis of $\mathfrak{p}$ over $\mathbb{R}$. Set

$$\tilde{H} := \tfrac{1}{2} H \in \mathfrak{p}_{\mathbb{C}}, \quad R := Y_1 - iY_2 \in \mathfrak{p}_{\mathbb{C}} \quad \text{and} \quad L := -Y_1 - iY_2 \in \mathfrak{p}_{\mathbb{C}}. \tag{2.3}$$

Lemma 2.4. *For $n = 3$, we have*

$$\mathcal{C} = \mathcal{C}_K + \tilde{H}^2 - \tfrac{1}{2}(RL + LR)$$

where $\mathcal{C}_K := -D^2 + \tfrac{1}{2}(E^+E^- + E^-E^+)$ is the Casimir element of $\mathfrak{k}_{\mathbb{C}}$ (up to a constant multiple).

Proof. Note that $\{D, E^+, E^-, \tilde{H}, R, L\}$ forms a basis of $\mathfrak{g}_{\mathbb{C}}$. We check $[D, E^\pm] = \pm iE^\pm$, $[D, R] - iR$, $[D, L] = -iL$, $[E^+, E^-] = -2iD$, $[E^+, R] = [E^-, L] = [D, \tilde{H}] = 0$, $[E^-, \tilde{H}] = -L, [E^+, \tilde{H}] = -R$, $[E^-, R] = -2\tilde{H}$, $[E^+, L] = -2\tilde{H}$, $[R, \tilde{H}] = -E^+$, $[L, \tilde{H}] = E^-$, and $[R, L] = 2iD$.

Using these relations, we can compute the matrix g_{ij} used in the definition of $\mathcal{C}$ and obtain the above formula for $\mathcal{C}$. Since $\{D, E^+, E^-\}$ forms a basis of $\mathfrak{k}_{\mathbb{C}}$, we compute that $-D^2 + \tfrac{1}{2}(E^+E^- + E^-E^+)$ is the Casimir element of $\mathfrak{k}_{\mathbb{C}}$ (up to a scalar multiple) using the above relations. $\square$

2.2. **Complementary series Representation of G.** Let V be an infinite dimensional irreducible unitary representation of G. Denote by V^K and V^M the subspaces of K-invariant and M-invariant vectors respectively. We assume that V^K is non-trivial and fix a unit vector in $v_0 \in V^K$, which is unique up to a scalar multiple.

Let V^∞ denote the set of smooth vectors of V, i.e., $v \in V^\infty$ if the map $g \mapsto gv$ is a smooth function $G \to V$. Every element of $\mathfrak{g}$ acts as a differential operator on V^∞: for $X \in \mathfrak{g}$ and $v \in V^\infty$,

$$\pi(X)(v) := \frac{d}{dt}(\exp(tX).v)\Big|_{t=0}$$

where $\exp(X) = \sum_{j=0}^\infty \frac{X^j}{j!}$ denotes the usual exponential map $\mathfrak{g} \to G$. This action extends to the action of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ on V^∞.

Denote by $\hat{G}$ the unitary dual of G. A representation $\pi \in \hat{G}$ is called *tempered* if for any K-finite vectors v, w of π, the matrix coefficient function $g \mapsto \langle \pi(g)v, w \rangle$ belongs to $L^{2+\epsilon}(G)$ for any $\epsilon > 0$. It follows from the classification of $\hat{G}$ (cf. [19, Thm. 16.2-3]) that the non-tempered spectrum of $\hat{G}$ consists of the trivial representation and the complementary series representation $(\pi_s, \mathcal{H}_s)$ parametrized by $\frac{n-1}{2} < s < (n-1)$, where $\mathcal{C}$ acts on $\mathcal{H}_s^\infty$ by the scalar $s(s - n + 1)$.

We fix $V = \mathcal{H}_s$ for $\frac{n-1}{2} < s < (n-1)$. If $n = 2$, the unitary dual $\hat{K}$ can be parametrized by $\mathbb{Z}$ so that $\ell \in \hat{K}$ corresponds to the one dimensional representation V_ℓ on which $\mathcal{C}_K$ acts by the scalar $-4\ell^2$.

For $n = 3$, $\hat{K}$ can be parametrized by $\mathbb{Z}_{\geq 0}$ so that $\ell \in \hat{K}$ corresponds to the irreducible $2\ell + 1$ dimensional representation V_ℓ.

Lemma 2.5. *For each $\ell \geq 0$, $\mathcal{C}_K$ acts on V_ℓ as the scalar $\ell(\ell + 1)$.*

Proof. If w_ℓ is the highest weight vector of V_ℓ, then $D(w_\ell) = i\ell w_\ell$. Using $[E^+, E^-] = -2iD$, we can write $\mathcal{C}_K = -D^2 - iD + E^- E^+$. Hence $\mathcal{C}_K(w_\ell) = \ell^2 w_\ell + \ell w_\ell$ since $E^+(w_\ell) = 0$. Since $\mathcal{C}_K$ acts by the scalar on V_ℓ as V_ℓ is irreducible, the claim follows.   □

As a K-representation, we write

$$V = \oplus_{\ell \in \hat{K}} m_\ell V_\ell$$

where the multiplicity m_ℓ of V_ℓ is at most one for each ℓ (see the remark following Theorem 4.5 of [40]); in fact we show in the next subsection that $m_\ell = 1$ for all $\ell \in \hat{K}$. We also have that the space V_ℓ^M is at most one dimensional [11].

2.3. **Ladder operators.** In this subsection, we will compute the Ladder operators which maps V_ℓ^M to $V_{\ell+1}^M$ and use them to obtain an explicit recursive formula for a unit vector of V_ℓ^M starting from v_0. These are well-known for $n = 2$.

2.3.1. *The case $G = \mathrm{PSL}_2(\mathbb{R})$.* Set

$$\mathcal{R} = \frac{1}{2}(\pi(H) + i\pi(E + F)) \quad \text{and} \quad \overline{\mathcal{R}} = \frac{1}{2}(\pi(H) - i\pi(E + F)).$$

These are called raising and lowering operators respectively and it is well-known that $\mathcal{R}(V_\ell) = V_{\ell+1}$ and $\overline{\mathcal{R}}(V_\ell) = V_{\ell-1}$ for any $\ell \in \mathbb{Z}$ (cf. [1, Prop. 2.5.2]. For a fixed unit vector $v_0 \in V^K$, put

$$v_\ell = \begin{cases} \mathcal{R}^\ell(v_0) & \text{if } \ell \geq 0 \\ \overline{\mathcal{R}}^{|\ell|}(v_0) & \text{if } \ell < 0 \ . \end{cases}$$

Then $V = \oplus_{\ell \in \hat{K}} \mathbb{C} v_\ell$ and for $\ell \geq 0$, we have (see [7]):

$$\|v_{\pm\ell}\|_2 = \frac{\sqrt{\Gamma(s + \ell)\Gamma(1 - s + \ell)}}{\sqrt{\Gamma(s)\Gamma(1 - s)}}. \tag{2.6}$$

where $\Gamma(x)$ denotes the Gamma function and $\|v\|$ denotes the norm of $v \in V$: $\|v\| = \sqrt{\langle v, v \rangle}$.

2.3.2. *The case $G = \mathrm{PSL}_2(\mathbb{C})$.* Recall the elements $D, E^+, E^-, \tilde{H}, R, L$ from (2.2) and (2.3). For each $\ell \geq 0$, the K-space V_ℓ is the irreducible representation of K of dimension $2\ell + 1$. The operators $E^{\pm}$ move between different M-types inside each fixed V_ℓ and R (resp. L) maps the highest (resp. lowest) weight vector space of each V_ℓ into the highest (resp. lowest) weight vector space of $V_{\ell+1}$.

We will show that the following differential operator $\mathcal{Z}_\ell$ maps V_ℓ^M to $V_{\ell+1}^M$: for each $\ell \geq 0$, set

$$\mathcal{Z}_\ell := \tfrac{1}{2}\left(RE^- + LE^+ - 2(\ell+1)\tilde{H} \right).$$

Lemma 2.7. *For each $\ell \in \mathbb{Z}_{\geq 0}$,*

(1) $\mathcal{C}_K \mathcal{Z}_\ell = \mathcal{Z}_\ell \mathcal{C}_K - i(RE^- - LE^+)D - 2\tilde{H}(\mathcal{C}_K + D^2) + 2(\ell+1)(\mathcal{Z}_\ell + \ell\tilde{H})$;

(2) $D\mathcal{Z}_\ell = \mathcal{Z}_\ell D$;

(3) $\mathcal{Z}_\ell \tilde{H} = \tilde{H}\mathcal{Z}_\ell + \mathcal{C} - 2\mathcal{C}_K - D^2 - \tilde{H}^2$.

Proof. Since $\mathcal{C}_K = -D^2 + \tfrac{1}{2}(E^+E^- + E^-E^+)$, we compute

(1) $\mathcal{C}_K \tilde{H} = \tilde{H}\mathcal{C}_K - 2\mathcal{Z}_\ell - 2\ell\tilde{H}$,

(2) $\mathcal{C}_K R = R\mathcal{C}_K - 2iDR - 2\tilde{H}E^+$, and

(3) $\mathcal{C}_K L = L\mathcal{C}_K + 2iDL - 2\tilde{H}E^-$.

These relations imply

$$\mathcal{C}_K \mathcal{Z}_\ell = \tfrac{1}{2}\left(RE^- + LE^+ - 2(\ell+1)\tilde{H} \right)\mathcal{C}_K$$
$$- \left\{ iDRE^- - iDLE^+ + \tilde{H}(E^+E^- + E^-E^+) - 2(\ell+1)(\mathcal{Z}_\ell + \ell\tilde{H}) \right\}.$$

Using $E^+E^- = \mathcal{C}_K + D^2 - iD$, $E^-E^+ = \mathcal{C}_K + D^2 + iD$, $DRE^- = RE^-D$ and $DLE^+ = LE^+D$, we compute that $\mathcal{C}_K \mathcal{Z}_\ell$ is equal to

$$\mathcal{Z}_\ell \mathcal{C}_K - i(RE^-D - LE^+D) - \tilde{H}(2\mathcal{C}_K + 2D^2) + 2(\ell+1)\mathcal{Z}_\ell + 2(\ell+1)\ell\tilde{H}.$$

For (2), we note that $[D, R] = iR$, $[D, L] = -iL$, $[\tilde{H}, D] = 0$, $[D, E^{\pm}] = \pm iE^{\pm}$. Hence

$$D\mathcal{Z}_\ell = \tfrac{1}{2}\left(DRE^- + DLE^+ - 2(\ell+1)D\tilde{H} \right)$$
$$= \tfrac{1}{2}\left\{ (iR + RD)E^- + (-iL + LD)E^+ - 2(\ell+1)\tilde{H}D \right\}$$
$$= \tfrac{1}{2}\left\{ RE^- + LE^+ - 2(\ell+1)\tilde{H} \right\}D = \mathcal{Z}_\ell D.$$

(3) can be proved similarly using $\mathcal{C} = \mathcal{C}_K + \tfrac{1}{2}(2\tilde{H}^2 - RL - LR)$. $\square$

Proposition 2.8. *For each $\ell \geq 0$,*

$$\mathcal{Z}_\ell(V_\ell^M) \subset V_{\ell+1}^M.$$

Proof. Note that $v \in V_\ell$ if and only if $\mathcal{C}_K v = \ell(\ell + 1)v$, and $v \in V^M$ if and only if $Dv = 0$.

Let $v \in V_\ell^M$. Using $Dv = 0$ and Lemma 2.7 (1), we deduce that

$$\mathcal{C}_K(\mathcal{Z}_\ell v) = \ell(\ell + 1)\mathcal{Z}_\ell v - 2\ell(\ell + 1)\tilde{H}v + 2(\ell + 1)\mathcal{Z}_\ell v + 2\ell(\ell + 1)\tilde{H}v$$
$$= (\ell + 1)(\ell + 2)\mathcal{Z}_\ell v.$$

Hence $\mathcal{Z}_\ell v \in V_{\ell+1}$. By Lemma 2.7 (2), we have

$$D\mathcal{Z}_\ell v = \mathcal{Z}_\ell(Dv) = 0,$$

and hence $\mathcal{Z}_\ell v \in V_{\ell+1}^M$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Fixing a unit vector $v_0 \in V^K$, define v_ℓ, $\ell \geq 1$, recursively:

$$v_\ell := \mathcal{Z}_\ell(v_{\ell-1}). \qquad\qquad (2.9)$$

Put

$$a_\ell := -2\ell + 1 \quad \text{and} \quad b_\ell := (\ell - 1)^2(\ell(\ell - 2) - s(s - 2)). \qquad (2.10)$$

Theorem 2.11 (Recursive formula for v_ℓ). *For $\ell \geq 1$,*

$$v_\ell = a_\ell \tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}$$

where v_{-1} is understood as the zero vector.

Proof. For $\ell \geq 0$ and $m \geq 0$, we have

$$\mathcal{Z}_\ell = \tfrac{1}{2}(RE^- + LE^+ - 2(\ell + 1)\tilde{H})$$
$$= \tfrac{1}{2}(RE^- + LE^+ - 2(m + 1)\tilde{H} + (-2(\ell + 1) + 2(m + 1))\tilde{H})$$
$$= \mathcal{Z}_m + (m - \ell)\tilde{H}$$

and hence

$$\mathcal{Z}_\ell v_m = v_{m+1} + (m - \ell)\tilde{H}v_m. \qquad\qquad (2.12)$$

We need to show that for $\ell \geq 1$,

$$v_\ell = a_\ell \tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}.$$

Since $E^+ v_0 = E^- v_0 = 0$, we have $v_1 = \mathcal{Z}_0 v_0 = -\tilde{H}v_0$. For the induction process, assume $v_\ell = a_\ell \tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}$. We deduce that, using Lemma 2.7 (3) and (2.12),

$$v_{\ell+1} = \mathcal{Z}_\ell v_\ell$$
$$= a_\ell(\mathcal{Z}_\ell \tilde{H}v_{\ell-1}) + b_\ell(\mathcal{Z}_\ell v_{\ell-2})$$
$$= a_\ell(\tilde{H}\mathcal{Z}_\ell + \mathcal{C} - 2\mathcal{C}_K - D^2 - \tilde{H}^2)v_{\ell-1} + b_\ell(v_{\ell-1} - 2\tilde{H}v_{\ell-2}).$$

Observe that $\mathcal{C}v_{\ell-1} = \lambda v_{\ell-1}$ with $\lambda = -s(2-s)$, $\mathcal{C}_K v_{\ell-1} = \ell(\ell-1)v_{\ell-1}$ and $Dv_{\ell-1} = 0$.

Using the induction hypothesis, we deduce

$$
\begin{aligned}
v_{\ell+1} &= a_\ell(\tilde{H}(v_\ell - \tilde{H}v_{\ell-1}) + (\lambda - 2\ell(\ell-1))v_{\ell-1} - \tilde{H}^2 v_{\ell-1}) \\
&\quad + b_\ell(v_{\ell-1} - 2\tilde{H}v_{\ell-2}) \\
&= a_\ell\tilde{H}v_\ell - 2\tilde{H}(a_\ell\tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}) + (a_\ell(\lambda - 2\ell(\ell-1)) + b_\ell)v_{\ell-1} \\
&= (a_\ell - 2)\tilde{H}v_\ell + (a_\ell(\lambda - 2\ell(\ell-1)) + b_\ell)v_{\ell-1} \\
&= a_{\ell+1}\tilde{H}v_\ell + b_{\ell+1}v_{\ell-1}.
\end{aligned}
$$

This finishes the proof. $\qquad\square$

For $x > 0$, the Gamma function $\Gamma(x)$ is defined to be the integral $\int_0^\infty e^{-t}t^{-x-1}dt$: it satisfies $\Gamma(x+1) = x\Gamma(x)$ and for each positive integer ℓ, $\Gamma(\ell) = (\ell-1)!$.

Lemma 2.13. *For each $\ell \geq 0$,*

$$
\|v_\ell\| = \frac{\ell!}{\sqrt{2\ell+1}} \frac{\sqrt{\Gamma(s+\ell)\Gamma(2-s+\ell)}}{\sqrt{\Gamma(s)\Gamma(2-s)}}.
$$

In particular, $v_\ell \neq 0$ for each $\ell \geq 0$.

Proof. Note that v_ℓ's are mutually orthogonal to each other. By Lemma 2.11, we have $v_\ell = a_\ell\tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}$. Therefore

$$
\begin{aligned}
\|v_\ell\|^2 &= \left\langle a_\ell\tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}, v_\ell \right\rangle = \left\langle a_\ell\tilde{H}v_{\ell-1}, v_\ell \right\rangle \\
&= -\left\langle a_\ell v_{\ell-1}, \tilde{H}v_\ell \right\rangle = -\left\langle a_\ell v_{\ell-1}, \frac{1}{a_{\ell+1}}v_{\ell+1} - \frac{b_{\ell+1}}{a_{\ell+1}}v_{\ell-1} \right\rangle \\
&= \frac{a_\ell b_{\ell+1}}{a_{\ell+1}}\|v_{\ell-1}\|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|v_\ell\|^2 &= \frac{a_\ell b_{\ell+1}}{a_{\ell+1}} \cdot \frac{a_{\ell-1}b_\ell}{a_\ell} \cdots \cdots \frac{a_{\ell+1-j}b_{\ell+2-j}}{a_{\ell+2-j}} \cdots \frac{a_1 b_2}{a_2}\|v_0\|^2 \\
&= \frac{a_1}{a_{\ell+1}} \cdot \prod_{j=1}^{\ell} b_{j+1} \\
&= \frac{1}{2\ell+1} \prod_{j=1}^{\ell} j^2(j^2 - 1 - s(s-2)) \\
&= \frac{(\ell!)^2}{2\ell+1} \prod_{j=1}^{\ell}(j+s-1)(j-s+1).
\end{aligned}
$$

Therefore the claim follows by the well-known properties of the Gamma function. □

Lemma 2.13 shows that $\mathcal{Z}_\ell$ maps a non-zero vector to a non-zero vector. Therefore Theorem 2.11 together with Lemma 2.13 implies the following:

Corollary 2.14. *For each $\ell \in \mathbb{Z}_{\geq 0}$,*

$$\mathcal{Z}_\ell(V_\ell^M) = V_{\ell+1}^M \quad and \quad V^M = \oplus_{\ell \in \hat{K}} \mathbb{C} v_\ell.$$

3. Explicit formulas for base eigenfunctions ϕ_ℓ

Let $G = \mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$, so that $G = \mathrm{Isom}^+(\mathbb{H}^n)$ for $n = 2, 3$ respectively. We keep the notations for $N, A, M, \mathcal{C}$ and Δ, etc. from the section 2. In particular, $\mathcal{C}$ satisfies $\mathcal{C}(\psi) = -\Delta(\psi)$ for all $\psi \in C^\infty(\Gamma\backslash G)^K = C^\infty(\mathbb{H}^n)$.

Let $\Gamma < G$ be a geometrically finite discrete subgroup with critical exponent $\frac{n-1}{2} < \delta < n - 1$. Let $\nu_j = \nu_j(\Gamma)$ denote the Patterson-Sullivan measure on the limit set $\Lambda(\Gamma)$ with respect to the basepoint $j = (0_{n-1}, 1) \in \mathbb{H}^n$. Up to a scaling, ν_j is the weak-limit as $t \to \delta^+$ of the family of measures

$$\nu_j(t) := \frac{1}{\sum_{\gamma \in \Gamma} e^{-td(j,\gamma j)}} \sum_{\gamma \in \Gamma} e^{-td(j,\gamma j)} \delta_{\gamma(j)}$$

where $\delta_{\gamma(j)}$ is the dirac measure at $\gamma(j)$.

We consider the Hilbert space $L^2(\Gamma\backslash G)$ where the inner product $\langle \psi_1, \psi_2 \rangle$ is given by

$$\langle \psi_1, \psi_2 \rangle = \int_{\Gamma\backslash G} \psi_1(g)\overline{\psi_2(g)} \, dg$$

where dg denotes a G-invariant measure on $\Gamma\backslash G$. As dg is an invariant measure, the action of G on $L^2(\Gamma\backslash G)$ by right translations gives rise to a unitary representation.

Since the complementary series representations exhaust the non-tempered spectrum of $\hat{G}$ (cf. [19, Thm. 16. 2-3]), we deduce the following from [37] and [22]: recall the notation $\mathcal{H}_s$ of the complementary series representation of G on which $\mathcal{C}$ acts by $s(s - n + 1)$.

Theorem 3.1. *There exist*

$$\alpha_0 = \delta(\delta - n + 1) > \alpha_1 \geq \cdots \geq \alpha_k > -(n-1)^2/4$$

such that

$$L^2(\Gamma\backslash G) = \mathcal{H}_{s_0=\delta} \oplus \cdots \oplus \mathcal{H}_{s_k} \oplus \mathcal{W}$$

where $(n-1)/2 < s_i < (n-1)$ is given by the equation $\alpha_i = s_i(s_i - n + 1)$ and $\mathcal{W}$ lies in the tempered spectrum of $\hat{G}$.

We set $V := \mathcal{H}_\delta$. The base eigenfunction $\phi_0 \in V^K$ for the Laplacian Δ can be explicitly written as the integral of the Poisson kernel against ν_j ([37]):

$$\phi_0(n_x a_y) = \int_{u \in \mathbb{R}^n} \hat{\phi}_u(x, y) \, d\nu_j(u)$$

where $\hat{\phi}_u(x, y) := \left(\frac{(|u|^2 + 1)y}{|x - u|^2 + y^2} \right)^\delta$. We normalize ν_j so that $\|\phi_0\|_2 = 1$.

For $n = 2$, we set

$$\psi_\ell = \begin{cases} \mathcal{R}^\ell(\phi_0) & \text{if } \ell \geq 0 \\ \overline{\mathcal{R}}^{|\ell|}(\phi_0) & \text{if } \ell < 0 \end{cases}$$

Sine $\overline{\mathcal{R}}$ is the complex conjugate of $\mathcal{R}$, $\psi_{-\ell} = \overline{\psi_\ell}$. Therefore for $n = 2$, it suffices to describe ψ_ℓ for $\ell \geq 0$.

For $n = 3$, we define ψ_ℓ recursively:

$$\psi_0 := \phi_0, \quad \text{and} \quad \psi_\ell = \mathcal{Z}_\ell(\psi_{\ell-1}) \quad \text{for each } \ell \geq 1.$$

Definition 3.2. Let $n = 2, 3$. For each $\ell \in \hat{K}$, define the unit vector in V_ℓ^M by:

$$\phi_\ell := \frac{\psi_\ell}{\|\psi_\ell\|_2} \in C^\infty(\Gamma \backslash G)^M \cap L^2(\Gamma \backslash G).$$

We have by Corollary 2.14,

$$V^M = \oplus_{\ell \in \hat{K}} \mathbb{C} \phi_\ell.$$

The rest of this section is devoted to obtaining pointwise bounds for these base eigenfunctions ϕ_ℓ's in terms of ϕ_0.

3.1. **Base eigenfunctions for $G = \mathrm{PSL}_2(\mathbb{R})$.**

Theorem 3.3. *Let $G = \mathrm{PSL}_2(\mathbb{R})$. For $\ell \in \mathbb{Z}_{\geq 0}$,*

$$\phi_\ell(n_x a_y) = \frac{\sqrt{\Gamma(1-\delta)\Gamma(\ell+\delta)}}{\sqrt{\Gamma(\delta)\Gamma(\ell+1-\delta)}} \cdot \int_{\mathbb{R}} \hat{\phi}_u(x, y) \left(\frac{(x-u)-iy}{(x-u)+iy} \right)^\ell d\nu_j(u).$$

In particular,

$$|\phi_{\pm\ell}(n_x a_y)| \ll \phi_0(n_x a_y)$$

with implied constant independent of ℓ.

Proof. We have

$$K = \{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, \pi) \}.$$

The raising operator $\mathcal{R}$ in the Iwasawa coordinates $n_x a_y k_\theta$ can be written as

$$\mathcal{R} = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial \theta}\right)$$

(see [1]). Since $\|\psi_\ell\|$ is given in (2.6), it suffices to show that

$$\psi_\ell(n_x a_y k_\theta) = \frac{e^{2\ell i\theta}\Gamma(\delta+\ell)}{\Gamma(\delta)}\int_\mathbb{R}\hat{\phi}_u(x,y)\left(\frac{(x-u)-iy}{(x-u)+iy}\right)^\ell\,d\nu_j(u). \qquad (3.4)$$

The case $\ell = 0$ is clear. To use an induction, we assume that (3.4) holds for ℓ. We compute

$$\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(\hat{\phi}_u(x,y)\cdot\left(\frac{(x-u)-iy}{(x-u)+iy}\right)^\ell\right)$$

$$= \delta\cdot\hat{\phi}_u(x,y)\cdot\left(\frac{(x-u)-iy}{(x-u)+iy}\right)^{\ell+1} + \ell\cdot\hat{\phi}_u(x,y)\cdot\left(\frac{(x-u)-iy}{(x-u)+iy}\right)^\ell\cdot\left(\frac{-2iy}{(x-u)+iy}\right)$$

and

$$\frac{1}{2i}\frac{\partial}{\partial \theta}\left(\mathcal{R}^\ell\phi_0\right)(n_x a_y k_\theta)$$

$$= \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)}\cdot\ell\cdot e^{2i\ell\theta}\int_\mathbb{R}\hat{\phi}_u(x,y)\left(\frac{(x-u)-iy}{(x-u)+iy}\right)^\ell\,d\nu_j(u).$$

Hence

$$\left(\mathcal{R}^{\ell+1}\phi_0\right)(n_x a_y k_\theta) = \frac{e^{2i(\ell+1)\theta}\Gamma(\delta+\ell+1)}{\Gamma(\delta)}\int_\mathbb{R}\hat{\phi}_u(x,y)\left(\frac{(x-u)-iy}{(x-u)+iy}\right)^{\ell+1}\,d\nu_j(u)\,.$$

Hence (3.4) holds for $\ell+1$, finishing the proof.

$\square$

3.2. Base eigenfunctions for $G = \mathrm{PSL}_2(\mathbb{C})$.

The case of $n = 3$ involves more complicated computations and it turns out that ϕ_ℓ's are not uniformly bounded by ϕ_0, but grow polynomially as $\ell \to \infty$ (see Theorem 3.8).

We parametrize elements of $K = \mathrm{PSU}(2)$ as

$$K = \left\{k_{\mu_1,\mu_2,\theta} : \begin{matrix}0 \le \theta < \frac{\pi}{2},\ 0 \le \mu_1 < \pi,\ 0 \le \mu_2 < 2\pi,\\ \mu_1 = 0 \text{ if } \theta = \frac{\pi}{2}\ \text{ and }\ \mu_2 = 0 \text{ if } \theta = 0\end{matrix}\right\}$$

where

$$k_{\mu_1,\mu_2,\theta} := \begin{pmatrix} e^{i\mu_1}\cos\theta & e^{i\mu_2}\sin\theta \\ -e^{-i\mu_2}\sin\theta & e^{-i\mu_1}\cos\theta \end{pmatrix}.$$

Using the coordinates $(x_1, x_2, y, \mu_1, \mu_2, \theta)$, the element $\tilde{H}$ defined in (2.3) is given by

$$\tilde{H} = -\cos(\mu_1 + \mu_2)\sin(2\theta)\, y\frac{\partial}{\partial x_1} - \sin(\mu_1 + \mu_2)\sin(2\theta)\, y\frac{\partial}{\partial x_2}$$
$$+ \cos(2\theta)\, y\frac{\partial}{\partial y} + \frac{\sin(2\theta)}{2}\frac{\partial}{\partial \theta}. \quad (3.5)$$

Recall

$$\hat{\phi}_u(x, y) = \left(\frac{(|u|^2 + 1)y}{|x - u|^2 + y^2}\right)^\delta.$$

Setting $\hat{\phi}_u(n_x a_y k) = \hat{\phi}_u(x, y, k) := \hat{\phi}_u(x, y)$, we may regard $\hat{\phi}_u$ as a function on G and define functions $\hat{\phi}_u^{(\ell)}$ on G by the recursive formula: for each $\ell \geq 1$,

$$\hat{\phi}_u^{(\ell)} := (-2\ell + 1)\tilde{H}(\hat{\phi}_u^{(\ell-1)}) + (\ell - 1)^2(\delta(2 - \delta) + \ell(\ell - 2))\hat{\phi}_u^{(\ell-2)}$$

(as before, the terms which are not defined are understood as 0).

Lemma 3.6. *For each $\ell \geq 0$, and $n_x a_y \in NA$, we have*

$$\psi_\ell(n_x a_y) = \int_{u \in \mathbb{R}^2} \hat{\phi}_u^{(\ell)}(x, y, e)d\nu_j(u).$$

Proof. By the recursive formula for $\mathcal{Z}_\ell$ given in Theorem 2.11, for $\ell \geq 1$, we have

$$\psi_{\ell+1} = \mathcal{Z}_\ell(\psi_{\ell-1}) = a_\ell \tilde{H}(\psi_\ell) + b_\ell \psi_{\ell-1}$$

where $a_\ell = -2\ell + 1$ and $b_\ell = (\ell - 1)^2(\delta(2 - \delta) + \ell(\ell - 2))$.

We prove the claim by induction. The case of $\ell = 0$ is by definition of ψ_0. Suppose the claim holds for all $j \leq \ell - 1$. Observe that all the terms in (3.5) for $\tilde{H}$ except for the term $y\frac{\partial}{\partial y}$ have $\sin(2\theta)$ and hence vanish when $\theta = 0$. Therefore

$$\psi_\ell(n_x a_y) = a_\ell y \frac{\partial \psi_{\ell-1}}{\partial y}(n_x a_y) + b_\ell \psi_{\ell-2}(n_x a_y)$$

$$= a_\ell y \frac{\partial}{\partial y}\int_{u \in \mathbb{R}^2} \hat{\phi}_u^{(\ell-1)}(x, y)d\nu_j(u) + b_\ell \int_{u \in \mathbb{R}^2} \hat{\phi}_u^{(\ell-2)}(x, y)d\nu_j(u)$$

$$= \int_{u \in \mathbb{R}^2} \left(a_\ell(\tilde{H}\hat{\phi}_u^{(\ell-1)})(x, y, e) + b_\ell \hat{\phi}_u^{(\ell-2)}(x, y, e)\right)d\nu_j(u)$$

$$= \int_{u \in \mathbb{R}^2} \hat{\phi}_u^{(\ell)}(x, y, e)d\nu_j(u).$$

$\square$

It turns out that the Legendre polynomials appear in the formula for ϕ_ℓ: let us denote by $\mathbf{P}_\ell(t)$ the Legendre polynomial of degree ℓ. It is defined by the recursive relation: $\mathbf{P}_0(t) = 1$, $\mathbf{P}_1(t) = t$ and

$$\ell \cdot \mathbf{P}_\ell(t) = (2\ell - 1)t\mathbf{P}_{\ell-1}(t) - (\ell - 1)\mathbf{P}_{\ell-2}(t). \quad (3.7)$$

Theorem 3.8. *Let $\ell \geq 0$. For any $n_x a_y \in NA$, we have*

$$\phi_\ell(n_x a_y) = \sqrt{2\ell + 1} \cdot \frac{\sqrt{\Gamma(2-\delta)\Gamma(\ell+\delta)}}{\sqrt{\Gamma(\delta)\Gamma(\ell+2-\delta)}} \cdot \int_{u \in \mathbb{R}^2} \hat{\phi}_u(x, y) \cdot \mathbf{P}_\ell \left(\frac{y^2 - |x-u|^2}{y^2 + |x-u|^2} \right) \, d\nu_j(u).$$

In particular,

$$|\phi_\ell(n_x a_y)| \ll \sqrt{\ell + 1} \cdot \phi_0(n_x a_y)$$

with the implied constant independent of ℓ.

Proof. Since $\|\psi_\ell\|$'s have been computed in Lemma 2.13, it suffices to show that

$$\psi_\ell(n_x a_y) = \ell! \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)} \int_{u \in \mathbb{R}^2} \hat{\phi}_u(x, y) \cdot \mathbf{P}_\ell \left(\frac{y^2 - |x-u|^2}{y^2 + |x-u|^2} \right) d\nu_j(u). \qquad (3.9)$$

Since $\psi_\ell(n_x a_y) = \int_{u \in \mathbb{R}^2} \hat{\phi}_u^{(\ell)}(x, y, e) d\nu_j(u)$ by Lemma 3.6, (3.9) follows if we show:

$$\hat{\phi}_u^{(\ell)}(x, y, e) = \ell! \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)} \cdot \hat{\phi}_u(x, y) \cdot \mathbf{P}_\ell \left(\frac{y^2 - |x-u|^2}{y^2 + |x-u|^2} \right). \qquad (3.10)$$

Put $B(x, y, u) = \frac{y^2 - |x-u|^2}{y^2 + |x-u|^2}$. Since

$$y \frac{\partial}{\partial y} \hat{\phi}_u(x, y) = -\delta \cdot \hat{\phi}_u(x, y) \cdot B(x, y, u) \text{ and } y \frac{\partial}{\partial y} B = -B^2 + 1,$$

we have

$$\hat{\phi}_u^{(1)}(x, y, e) = a_1 \left(y \frac{\partial}{\partial y} \hat{\phi}_u \right)(x, y, e)$$
$$= -\delta \hat{\phi}_u(x, y) \cdot a_1 B = \delta \hat{\phi}_u(x, y) \mathbf{P}_1(B)$$

proving the claim for $\ell = 1$. For the induction, we assume that for all $j \leq \ell-1$, $\hat{\phi}_u^{(j)}(x, y, e) = d_j \hat{\phi}_u(x, y) j! \cdot \mathbf{P}_j(B)$ where $d_\ell := \prod_{j=1}^{\ell}(\delta+j-1)$. Then

$$\hat{\phi}_u^{(\ell)}(x, y, e) = a_\ell \cdot \left(y \frac{\partial}{\partial y} \hat{\phi}_u^{(\ell-1)} \right)(x, y, e) + b_\ell \cdot \hat{\phi}_u^{(\ell-2)}(x, y, e) =$$

$$a_\ell d_{\ell-1} \cdot (\ell-1)! \cdot y \frac{\partial}{\partial y} \left(\hat{\phi}_u(x, y) \mathbf{P}_{\ell-1}(B) \right) + b_\ell d_{\ell-2} \cdot \hat{\phi}_u(x, y) \cdot (\ell-2)! \cdot \mathbf{P}_{\ell-2}(B).$$

Since $b_\ell = (\ell-1)^2(-\delta+\ell)(\delta+\ell-2)$, we have

$$d_{\ell-2} b_\ell = (\ell-1)^2 d_{\ell-1}(-\delta+\ell).$$

Therefore

$$\hat{\phi}_u^{(\ell)}(x, y, e)$$
$$= d_{\ell-1} \left\{ -a_\ell \delta B(\ell-1)! \mathbf{P}_{\ell-1}(B) - a_\ell(B^2-1)(\ell-1)! \mathbf{P}'_{\ell-1}(B) \right.$$
$$\left. +(\ell-1)^2(-\delta+\ell)(\ell-2)! \mathbf{P}_{\ell-2}(B) \right\} \hat{\phi}_u(x, y)$$
$$= d_{\ell-1} \left\{ -a_\ell \delta B(\ell-1)! \mathbf{P}_{\ell-1}(B) - \delta(\ell-1)^2(\ell-2)! \mathbf{P}_{\ell-2}(B) \right.$$
$$\left. -a_\ell(B^2-1)(\ell-1)! \mathbf{P}'_{\ell-1}(B) + \ell(\ell-1)^2(\ell-2)! \mathbf{P}_{\ell-2}(B) \right\} \hat{\phi}_u(x, y).$$

Since

$$(\ell-1)(\ell-1)! \mathbf{P}_\ell(B) = -a_\ell(B^2-1)(\ell-1)! \mathbf{P}'_{\ell-1}(B) + (\ell-1)^2 \ell(\ell-2)! \mathbf{P}_{\ell-2}(B),$$

we have

$$\hat{\phi}_u^{(\ell)}(x, y, e) = d_{\ell-1} \hat{\phi}_u(x, y) \cdot (\delta+\ell-1)\ell! \mathbf{P}_\ell(B)$$
$$= d_\ell \cdot \hat{\phi}_u(x, y) \cdot \ell! \mathbf{P}_\ell(B).$$

This proves (3.10), in view of the relation $\Gamma(t+1) = t\Gamma(t)$ and $d_\ell = \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)}$. Hence the first claim of the theorem is proved.

Since $|\mathbf{P}_\ell(t)| \leq 1$ for $|t| \leq 1$ (see p. 987 of [17]), the second claim follows from the first claim.   □

We do not know whether for all $n_x a_y k \in G$, $|\phi_\ell(n_x a_y k)|$ is uniformly bounded by $\ell^N \phi_0(n_x a_y)$ for some $N > 0$. However for our purpose it suffices to prove the following bound:

Theorem 3.11. *For each $\ell \geq 0$, there exists a constant $C_\ell > 0$ such that for any $n_x a_y k \in NAK$,*

$$|\phi_\ell(n_x a_y k)| \leq C_\ell \cdot \phi_0(n_x a_y).$$

Proof. We use notations from the proof of Theorem 3.8. We compute

$$y\frac{\partial}{\partial x_1}\hat{\phi}_u = \delta \cdot \hat{\phi}_u \cdot \frac{-2y(x_1-u_1)}{|x-u|^2+y^2}, \quad y\frac{\partial}{\partial x_2}\hat{\phi}_u = \delta \cdot \hat{\phi}_u \cdot \frac{-2y(x_2-u_2)}{|x-u|^2+y^2}$$

and

$$y\frac{\partial}{\partial y}\hat{\phi}_u = \delta \cdot \hat{\phi}_u \cdot \frac{|x-u|^2-y^2}{|x-u|^2+y^2}.$$

Let

$$A_1(x, y, u) = \frac{-2y(x_1-u_1)}{|x-u|^2+y^2}, \quad A_2(x, y, u) = \frac{-2y(x_2-u_2)}{|x-u|^2+y^2}$$

and

$$B(x, y, u) = \frac{|x-u|^2-y^2}{|x-u|^2+y^2}.$$

For $\mu = (\mu_1, \mu_2)$ and $0 \leq \theta < \pi$, define

$$\Phi_1(\mu, \theta) := -\sin(2\theta)\cos(\mu_1+\mu_2), \text{ and } \Phi_2(\mu, \theta) := -\sin(2\theta)\sin(\mu_1+\mu_2)$$

and $\Psi(\theta) := \cos(2\theta)$ so that by (3.5),

$$\tilde{H} = \Phi_1 y \frac{\partial}{\partial x_1} + \Phi_2 y \frac{\partial}{\partial x_2} + \Psi y \frac{\partial}{\partial y} + \frac{\sin(2\theta)}{2} \frac{\partial}{\partial \theta}.$$

Hence

$$\tilde{H}\hat{\phi}_u = \delta \cdot \hat{\phi}_u \cdot (\Phi_1 A_1 + \Phi_2 A_2 + \Psi B);$$

so

$$\hat{\phi}_u^{(1)} = \hat{\phi}_u \cdot a_1 \delta \cdot (\Phi_1 A_1 + \Phi_2 A_2 + \Psi B).$$

We compute

$$y\tfrac{\partial}{\partial x_1} A_1 = A_1^2 + B - 1, \quad y\tfrac{\partial}{\partial x_2} A_1 = A_1 A_2 + B - 1, \quad y\tfrac{\partial}{\partial y} A_1 = A_1 B;$$

$$y\tfrac{\partial}{\partial x_1} A_2 = A_1 A_2 + B - 1, \quad y\tfrac{\partial}{\partial x_2} A_2 = A_2^2 + B - 1, \quad y\tfrac{\partial}{\partial y} A_2 = A_2 B;$$

and

$$y\tfrac{\partial}{\partial x_1} B = A_1(B - 1), \quad y\tfrac{\partial}{\partial x_2} B = A_2(B - 1), \quad y\tfrac{\partial}{\partial y} B = (B - 1)(B + 1).$$

We also compute:

$$\tilde{H}(\Phi_1) = \Phi_1 \Psi, \quad \tilde{H}(\Phi_2) = \Phi_2 \Psi$$

and

$$\tilde{H}(\Psi) = -\Phi_1^2 - \Phi_2^2 = -1 + \Psi^2.$$

It follows that $\hat{\phi}_u^{(\ell)} = \hat{\phi}_u \cdot p_\ell(\Phi_1, \Phi_2, A_1, A_2, \Psi, B)$ where p_ℓ is a polynomial in Φ_1, Φ_2, A_1, A_2, Ψ and B, whose coefficients are given by monomials in $\{\pm 1, a_1, \ldots, a_\ell, b_1, \ldots, b_\ell, \delta\}$. Since the absolute values of $\Phi_1, \Phi_2, \Psi, A_1, A_2$ and B are all bounded above by 1, we deduce

$$\left| \hat{\phi}_u^{(\ell)}(x, y, k) \right| \leq C_\ell \cdot \hat{\phi}_u(x, y).$$

for some constant $C_\ell > 0$ independent of x, y, k. Since

$$\psi_\ell(n_x a_y k) = \int_{u \in \Lambda(\Gamma)} \hat{\phi}_u^{(\ell)}(x, y, k) d\nu_j(u),$$

the claim follows. $\square$

4. HOROSPHERICAL AVERAGE OF ϕ_ℓ

We let the notations $G = NAK, \Gamma, \delta, \phi_0, \phi_\ell$, etc., be as in Section 3. We assume that Γ is geometrically finite with $(n-1)/2 < \delta < (n-1)$ and that $\Gamma\backslash\Gamma N$ is closed in $\Gamma\backslash G$ in the whole section. The main goal of this section is to compute the horospherical average of ϕ_ℓ over $(\Gamma \cap N)\backslash N$ (Theorem 4.12). Consider $\mathbb{H}^n$ for $n = 2$ or 3 according as $G = \mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$. We let $X_0 \in \mathrm{T}^1(\mathbb{H}^n)$ be the upward normal vector based at $j = (0_{n-1}, 1) \in \mathbb{H}^n$. The map $g \mapsto g(X_0)$ induces the identification of G/M with $\mathrm{T}^1(\mathbb{H}^n)$. The horosphere in $\mathrm{T}^1(\mathbb{H}^n)$

corresponding to NM/M consists of the upward normal vectors on the horizontal plane containing j, and hence based at $\infty \in \partial(\mathbb{H}^n)$.

The assumption that $\Gamma \backslash \Gamma N$ is closed is equivalent to saying that either $\infty \notin \Lambda(\Gamma)$ or ∞ is a parabolic fixed point of Γ [10]. Recall $\xi \in \Lambda(\Gamma)$ is a parabolic fixed point if it is a unique fixed point in $\partial(\mathbb{H}^n)$ of an element of Γ. One of the important features of a geometrically finite group Γ is that any parabolic fixed point ξ of Γ is bounded, meaning that the stabilizer $\mathrm{Stab}_\Gamma(\xi)$ acts cocompactly on $\Lambda(\Gamma) - \{\xi\}$ [8].

Therefore our assumption on the closedness of $\Gamma \backslash \Gamma N$ implies the following:

Lemma 4.1. $N \cap \Gamma$ *acts cocompactly on* $\Lambda(\Gamma) - \{\infty\}$.

As mentioned in the introduction, the rank of ∞ is the rank of $\Gamma \cap N$ as a free abelian group.

Definition 4.2. *Given* $\psi \in C(\Gamma \backslash G)^M$, *define* $\psi^N \in C(\Gamma \backslash G)^M$ *by*

$$\psi^N(g) := \int_{n_x \in (N \cap \Gamma) \backslash N} \psi(n_x g) \, dx$$

where dx *denotes the Lebesgue measure on* $\mathbb{R}^n$, *provided the integral converges.*

Proposition 4.3. *There exists* $c_n(0) > 0$ *such that for all* $y > 0$,

$$\phi_0^N(a_y) = c_n(0) \cdot y^{n-1-\delta}.$$

Proof. In [20], it was shown that $\phi_0^N(a_y)$ converges absolutely and that there are constants $c_n(0) > 0$ and $d_n(0) \in \mathbb{R}$ such that for all $y > 0$

$$\phi_0^N(a_y) = c_n(0) y^{n-1-\delta} + d_n(0) y^\delta. \tag{4.4}$$

Since $\phi_0 > 0$ and (4.4) holds for all $y > 0$, it follows that $d_n(0) \geq 0$. We claim that $d_n(0) = 0$.

When $\infty \notin \Lambda(\Gamma)$, $\Lambda(\Gamma)$ is a bounded subset of $\mathbb{R}^{n-1}$ and we can show by direct computations:

$$\phi_0^N(a_y) = \begin{cases} \frac{\sqrt{\pi}\Gamma(\delta-\frac{1}{2})}{\Gamma(\delta)} \cdot \int_{u \in \Lambda(\Gamma)} (|u|^2 + 1)^\delta \, d\nu_j(u) \cdot y^{1-\delta} & \text{if } n = 2 \\ \frac{\pi}{\delta-1} \cdot \int_{u \in \Lambda(\Gamma)} (|u|^2 + 1)^\delta \, d\nu_j(u) \cdot y^{2-\delta} & \text{if } n = 3 \end{cases} \tag{4.5}$$

(see [20]).

Now suppose $\infty \notin \Lambda(\Gamma)$. As Γ is geometrically finite, Γ admits a polyhedron fundamental domain $\mathcal{F}$ in $\mathbb{H}^n$ such that $F_0 \times [Y_0, \infty)$ injects to $\mathcal{F}$ for some $Y_0 \gg 1$ where F_0 is a fundamental domain in $\mathbb{R}^{n-1}$ for $N \cap \Gamma$. Let $B_t = \{x \in F_0 : |x| < t\}$ for $t > 1$.

When ∞ is a bounded parabolic fixed point of rank $n-1$, take t_0 so that $B_{t_0} = F_0$, which is possible since F_0 is bounded in this case. Then

$$\int_{x \in B_{t_0}} \phi_0(n_x a_y) dx \geq \tfrac{d_n(0)}{2} y^\delta. \tag{4.6}$$

Therefore using the Cauchy-Schwartz inequality, we have

$$\|\phi_0\|_2^2 \geq \int_{Y_0}^\infty \int_{B_{t_0}} \phi_0(n_x a_y)^2 y^{-n} dx dy$$

$$\geq \tfrac{1}{\mathrm{vol}(B_{t_0})} \int_{Y_0}^\infty \left(\int_{B_{t_0}} \phi_0(n_x a_y) dx \right)^2 y^{-n} dy$$

$$\geq \tfrac{d_n(0)^2}{4\mathrm{vol}(B_{t_0})} \int_{Y_0}^\infty y^{2\delta-n} dy.$$

Since $\delta > (n-1)/2$, $\|\phi_0\|_2 = \infty$ unless $d_n(0) \neq 0$. Therefore $d_n(0) = 0$.

The remaining case is when $n = 3$ and ∞ is a bounded parabolic fixed point of rank one. In this case, it was shown in the proof of [20, Prop. 4.6] that for all sufficiently large $t \gg 1$, there exists $b_t \to 0$ as $t \to \infty$ such that for all $y > 0$,

$$\int_{x \in F_0 - B_t} \phi_0(n_x a_y) dx \leq b_t y^\delta.$$

Therefore, if $d_3(0)$ were positive, then for some large $t_0 > 0$, we would have

$$\int_{x \in B_{t_0}} \phi_0(n_x a_y) dx \geq \tfrac{d_3(0)}{2} y^\delta. \tag{4.7}$$

By repeating the same argument as in the previous case, this leads to a contradiction. $\square$

Lemma 4.8. *Let $y > 0$ and $k \in K$. For each $1 \leq i \leq n-1$, we have*

$$\int_{(N \cap \Gamma) \backslash N} \tfrac{\partial}{\partial x_i} \phi_\ell(n_x a_y k) \, dx = 0.$$

Proof. By Theorems 3.3 and 3.11, we have, for some $C_\ell > 0$,

$$|\phi_\ell(n_x a_y k)| \leq C_\ell \cdot \phi_0(n_x a_y). \tag{4.9}$$

Suppose $n = 3$. If $\infty \notin \Lambda(\Gamma)$ and hence $N \cap \Gamma = \{e\}$, $|\phi_0(n_x a_y k)| \to 0$ as $|x| \to \infty$. Hence, by (4.9), as $|x| \to \infty$,

$$|\phi_\ell(n_x a_y k)| \to 0.$$

Therefore

$$\int_{x_1 \in \mathbb{R}} \tfrac{\partial}{\partial x_1} \phi_\ell(n_x a_y k) \ dx_1$$

$$= \lim_{t \to \infty} \int_{-t}^{t} \tfrac{\partial}{\partial x_1} \phi_\ell(n_x a_y k) \ dx_1$$

$$= \lim_{t \to \infty} (\phi_\ell(n_{t+\sqrt{-1}x_2} a_y k) - \phi_\ell(n_{-t+\sqrt{-1}x_2} a_y k)) = 0.$$

The other case of $i = 2$ is symmetric to this one.

When ∞ is a bounded parabolic fixed point of rank one, we may assume without loss of generality that $N \cap \Gamma$ is generated by $n_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so that there exists a fundamental domain F_0 in $\mathbb{R}^2$ inside $\{(x_1, x_2) : 0 \le x_1 \le 1\}$. In this case, $\phi_0(n_x a_y) \to 0$ as $x_2 \to \infty$ [38]. Hence $|\phi_\ell(n_x a_y k)| \to 0$, as $x_2 \to \infty$. By a similar argument as above, this implies that

$$\int_{x_2 \in \mathbb{R}} \tfrac{\partial}{\partial x_2} \phi_\ell(n_x a_y k) \ dx_2 = 0.$$

On the other hand, by the $n_1 \in N \cap \Gamma$-invariance of ϕ_ℓ,

$$\int_{x_1 \in [0,1]} \tfrac{\partial}{\partial x_1} \phi_\ell(n_x a_y k) \ dx_1 = \phi_\ell(n_1 n_{\sqrt{-1}x_2} a_y k) - \phi_\ell(n_{\sqrt{-1}x_2} a_y k) = 0.$$

If ∞ has rank 2, we may assume that $N \cap \Gamma$ is generated by n_1 and $n_{\sqrt{-1}}$. Then the the claim follows from Green's theorem and the invariance of ϕ_ℓ by $N \cap \Gamma$ as in the last argument. The case $n = 2$ can be shown similarly. $\square$

For $\ell \ge 0$, define

$$M_\ell(\theta) := \ell! \cdot \mathbf{P}_\ell(-\cos 2\theta). \tag{4.10}$$

Then $M_0(\theta) = 1$ and the recursive relation (3.7) for $\mathbf{P}_\ell$'s implies:

Lemma 4.11. (1) For $\ell \ge 1$,

$$M_\ell(\theta) = (-2\ell + 1) \cos(2\theta) M_{\ell-1}(\theta) - (\ell - 1)^2 M_{\ell-2}(\theta).$$

(2) For each $\ell \ge 1$,

$$2(\ell + 1) M_\ell(\theta) = 4(-2\ell + 1) \cos(2\theta) M_{\ell-1}(\theta)$$
$$+ (-2\ell + 1) \sin(2\theta) M'_{\ell-1}(\theta) + 2(\ell - 1)^2(\ell - 2) M_{\ell-2}(\theta).$$

The following theorem implies in particular that the integral $\phi_\ell^N(a_y)$ converges, which is a priori unclear as $(N \cap \Gamma) \backslash N$ is not compact in general.

Theorem 4.12. *For $\ell \geq 0$, we have*

$$\phi_\ell^N(a_y) = (-1)^{(n-2)\ell} \cdot c_n(0) \cdot \frac{\sqrt{\Gamma(n-1-\delta)\Gamma(\ell+\delta)}}{\sqrt{\Gamma(\delta)\Gamma(\ell+n-1-\delta)}} \cdot \sqrt{2(n-2)\ell+1} \cdot y^{n-1-\delta}.$$

In particular,

$$|\phi_\ell^N(a_y)| \ll \ell^{(n-2)/2} \cdot y^{n-1-\delta}$$

with the implied constant independent of $\ell \geq 1$.

Proof. By Theorems 3.3 and 3.8, we have

$$\left| \int_{(N\cap\Gamma)\backslash N} \phi_\ell(n_x a_y) \, dx \right| \leq \int_{(N\cap\Gamma)\backslash N} |\phi_\ell(n_x a_y)| \, dx$$

$$\ll_\ell \int_{(N\cap\Gamma)\backslash N} \phi_0(n_x a_y) \, dx.$$

Hence by Proposition 4.3, the integral $\phi_\ell^N(a_y)$ converges absolutely. Let $n = 2$. By (2.6), it suffices to show

$$\psi_\ell^N(a_y k_\theta) = e^{2\ell i\theta} c_2(0) \frac{\Gamma(\ell+1-\delta)}{\Gamma(1-\delta)} y^{1-\delta}. \tag{4.13}$$

The case $\ell = 0$ holds by Proposition 4.3. To use an induction, we assume (4.13) is true for ℓ. Then applying Lemma 4.8,

$$\psi_{\ell+1}^N(a_y k_\theta) = e^{2i\theta} \int_{(N\cap\Gamma)\backslash N} \left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\frac{1}{2i}\frac{\partial}{\partial \theta} \right) \psi_\ell(n_x a_y k_\theta) \, dx$$

$$= e^{2i\theta} \cdot \left(y\frac{\partial}{\partial y}\frac{1}{2i}\frac{\partial}{\partial \theta} \right) \psi_\ell^N(a_y k_\theta)$$

$$= e^{2i\theta} \cdot \left(y\frac{\partial}{\partial y}\frac{1}{2i}\frac{\partial}{\partial \theta} \right) \left(e^{2\ell i\theta} c_2(0) \frac{\Gamma(\ell+1-\delta)}{\Gamma(1-\delta)} y^{1-\delta} \right)$$

$$= e^{2(\ell+1)i\theta} \cdot \left(c_2(0) \cdot \frac{\Gamma(\ell+1-\delta)}{\Gamma(1-\delta)} \cdot ((1-\delta)+\ell) \, y^{1-\delta} \right)$$

$$= e^{2(\ell+1)i\theta} \cdot \left(c_2(0) \cdot \frac{\Gamma(\ell+1+(1-\delta))}{\Gamma(1-\delta)} y^{1-\delta} \right)$$

since $z\Gamma(z) = \Gamma(z+1)$. This proves (4.13) for $\ell+1$.

Let $n = 3$. Setting $q_\ell := c_3(0) \prod_{j=1}^\ell (j+1-\delta)$, we claim that

$$\psi_\ell^N(a_y k) = q_\ell \cdot y^{2-\delta} M_\ell(\theta) \tag{4.14}$$

where $M_\ell(\theta)$ is defined as in (4.10). Set $k = k_{\mu_1,\mu_2,\theta}$, $a_\ell = -2\ell+1$ and $b_\ell = (\ell-1)^2(\delta(2-\delta)+\ell(\ell-2))$ for simplicity.

Since ϕ_0 is fixed by K, $E^\pm(\phi_0) = 0$. Hence

$$\psi_1^N(a_y k) = a_1 \int_{(N\cap\Gamma)\backslash N} \tilde{H}(\phi_0)(n a_y k) \, dn.$$

By Proposition 4.3 and Lemma 4.8, using $\frac{\partial}{\partial\theta}\phi_0(n_x a_y k) = 0$ and (3.5), we have

$$\psi_1^N(a_y k) = a_1\left(\cos(2\theta)\, y\frac{\partial}{\partial y} + \frac{\sin(2\theta)}{2}\,\frac{\partial}{\partial\theta}\right)\psi_0^N(a_y k)$$

$$= a_1 c_3(0) \cdot \cos(2\theta)(2-\delta)y^{2-\delta}.$$

Hence (4.14) holds for $\ell = 1$.

Assuming that (4.14) holds for ℓ, we deduce using Lemma 4.11 that

$$\psi_{\ell+1}^N(a_y k) = a_{\ell+1}q_\ell\tilde{H}(y^{2-\delta}M_\ell(\theta)) + b_{\ell+1}q_{\ell-1}y^{2-\delta}M_{\ell-1}(\theta)$$

$$= (-2\ell-1)q_\ell(\cos(2\theta)(2-\delta)M_\ell(\theta) + \frac{\sin(2\theta)}{2}M_\ell'(\theta))y^{2-\delta}$$

$$+ \ell^2(\ell+1-\delta)q_{\ell-1}(\delta+\ell-1)y^{2-\delta}M_{\ell-1}(\theta).$$

By Lemma 4.11,

$$M_{\ell+1}(\theta) = (-2\ell-1)\cos(2\theta)M_\ell(\theta) - \ell^2 M_{\ell-1}(\theta)$$

and

$$(\ell+2)M_{\ell+1}(\theta) = 2(-2\ell-1)\cos(2\theta)M_\ell(\theta) + (-2\ell-1)\frac{\sin(2\theta)}{2}M_\ell'(\theta)$$
$$+ \ell^2(\ell-1)M_{\ell-1}(\theta).$$

Therefore we have

$$\psi_{\ell+1}^N(a_y k) = q_\ell(\ell+2-\delta)M_{\ell+1}(\theta)y^{2-\delta} = q_{\ell+1}M_{\ell+1}(\theta)y^{2-\delta},$$

proving (4.14) for $\ell+1$.

By the formula of $\|\psi_\ell\|_2$ given in Lemma 2.13, and using $|\mathbf{P}_\ell(t)| \leq 1$ for all $t \in [-1,1]$ and $\mathbf{P}_\ell(-1) = (-1)^\ell$, we finish the proof for $n = 3$. $\square$

5. Uniform thickening for ϕ_ℓ's

We continue the notations $G = NAK$, Γ, ϕ_ℓ, etc. from section 4. Assume that Γ is geometrically finite with $\frac{n-1}{2} < \delta < n-1$ and that $\Gamma\backslash\Gamma N$ is closed in $\Gamma\backslash G$. In this section, we approximate the integral $\phi_\ell^N(a_y)$ as the inner product $\langle a_y\phi_\ell, \rho_{\eta,\epsilon}\rangle = \int_{\Gamma\backslash G}\phi_\ell(ga_y)\rho_{\eta,\epsilon}(g)dg$ for a suitable test function $\rho_{\eta,\epsilon}$. This step enables us to relate the horospherical average $\phi_\ell^N(a_y)$ with the spectral decomposition of $L^2(\Gamma\backslash G)$ stated in Theorem 3.1.

Fix a fundamental domain F_0 for $N\cap\Gamma$ in $\mathbb{R}^{n-1}$ and choose a compact fundamental domain $F_\Lambda \subset F_0$ for $(N\cap\Gamma)\backslash\Lambda(\Gamma) - \{\infty\}$ given by Lemma 4.1.

Lemma 5.1. *Fix $\ell \geq 0$. Suppose $\Gamma\backslash\Gamma N$ is not compact. For any open subset $J \subset F_0$ containing $\mathcal{F}_\Lambda$, we have for all $0 < y < 1$,*

(1) $\int_{J^c}\phi_0(n_x a_y)\, dx \ll y^\delta$ *with the implied constant independent of ℓ;*

(2) $|\int_{F_0-J} \phi_\ell(n_x a_y)\, dx| \ll (\ell+1)^{(n-2)/2} y^\delta$ with the implied constant independent of ℓ.

Proof. Let $n = 2$. Then the non-compactness assumption on $\Gamma\backslash\Gamma N$ implies that $\infty \notin \Lambda(\Gamma)$ and hence $\epsilon_0 := \inf_{x \notin J, u \in \Lambda(\Gamma)} |x - u|$ is positive. Then by the change of variable $w = \frac{x-u}{y}$, we have

$$\int_{F_0-J} \phi_0(n_x a_y)\, dx \le 2y^{1-\delta} \int_{u \in \Lambda(\Gamma)} (u^2 + 1)^\delta d\nu_i(u) \cdot \int_{w=\epsilon_0/y}^\infty \left(\tfrac{1}{w^2+1}\right)^\delta dw.$$

The latter integral can be evaluated explicitly as an incomplete Beta function which has known asymptotics:

$$\int_{\epsilon_0/y}^\infty \left(\tfrac{1}{w^2+1}\right)^\delta dw = c\ \beta_{y^2/\epsilon_0^2}(\delta - 1/2, 1 - \delta),$$

where $\beta_z(\alpha, \beta) \ll z^\alpha$. Hence

$$\int_{F_0-J} \phi_0(n_x a_y)\, dx \ll y^{1-\delta} \cdot y^{2(\delta-1/2)} = y^\delta.$$

This proves (1) for $n = 2$, and we refer to [20, Proposition 3.7] for $n = 3$.

For the second claim, note that, by Theorem 3.8,

$$\left| \int_{x \in F_0-J} \phi_\ell(n_x a_y)\, dx \right| \le \int_{x \in F_0-J} |\phi_\ell(n_x a_y)|\ dx$$

$$\ll (\ell+1)^{(n-2)/2} \int_{F_0-J} \phi_0(n_x a_y)\, dx$$

since ϕ_0 is a positive function. Hence the claim (2) follows from (1). □

For $m \ge 0$, the associated Legendre function $\mathbf{P}_\ell^m(x)$ is defined by the following:

$$\mathbf{P}_\ell^m(x) = (-1)^m (1 - x^2)^{m/2} \tfrac{d^m}{dx^m}(\mathbf{P}_\ell(x)). \qquad (5.2)$$

We set $\mathbf{P}_\ell^{-m}(x) := (-1)^m \frac{(\ell-m)!}{(\ell+m)!} \mathbf{P}_\ell^m(x)$.

Let U_ϵ denote the ϵ-neighborhood of e in G for any $\epsilon > 0$. The following lemma controls the Lipschitz constants of ϕ_ℓ's for the action of K:

Lemma 5.3. *Let $\ell \ge 0$. For all sufficiently small $\epsilon > 0$,*

$$\phi_\ell(n_x a_y k) = (1 + O((\ell+1)^{n-1}\epsilon)) \cdot \phi_\ell(n_x a_y)$$

with the implied constant independent of ℓ, x, $y > 0$ and $k \in K_\epsilon :=$ $K \cap U_\epsilon$.

Proof. If $n = 2$, we have $\phi_\ell(n_x a_y k_\theta) = e^{2\ell i \theta} \phi_\ell(n_x a_y)$, and hence the claim follows easily as $e^{2\ell i \theta} = 1 + O((\ell + 1)\theta)$ for all θ small.

Let $n = 3$. We set $\mu = \mu_1 + \mu_2$. The operator $\mathcal{C}_K$ acts on M-invariant functions as follows:

$$-\mathcal{C}_K f(\mu, \theta) = \frac{1}{\sin^2 2\theta} \frac{\partial^2}{\partial \mu^2} f + \frac{1}{4 \sin 2\theta} \frac{\partial}{\partial \theta} \left(\sin 2\theta \frac{\partial}{\partial \theta} f \right).$$

Since $\mathcal{C}_K(\phi_\ell) = \ell(\ell + 1)\phi_\ell$, it follows from the theory of spherical harmonics (cf. [36]) that

$$\phi_\ell(n_x a_y k_{\mu_1, \mu_2, \theta}) = \sum_{m=-\ell}^{\ell} f_\ell^m(x, y) \cdot Y_\ell^m(\theta, \mu) \tag{5.4}$$

where $f_{\ell, m} \in C^\infty(\mathbb{C} \times \mathbb{R}_{>0})$ and

$$Y_\ell^m(\theta, \mu) = \frac{\sqrt{(2\ell+1)(\ell-m)!}}{\sqrt{4\pi(\ell+m)!}} \cdot \mathbf{P}_\ell^m(\cos(2\theta)) \cdot e^{im\mu}.$$

We have for $|\mu| < \epsilon$, $e^{im\mu} = 1 + O(\ell\epsilon)$ as $|m| \leq \ell$. Also, from the properties of the associated Legendre functions, we deduce that for $|\theta| < \epsilon$ and $|m| \leq \ell$,

$$\mathbf{P}_\ell^m(\cos 2\theta) = (1 + O(\ell\epsilon))\mathbf{P}_\ell^m(1).$$

Therefore for $|\theta| < \epsilon$ and $|\mu| < \epsilon$,

$$\mathbf{P}_\ell^m(\cos 2\theta) \cdot e^{im\mu} = (1 + O(\ell^2\epsilon))\mathbf{P}_\ell^m(1),$$

and hence

$$Y_\ell^m(\theta, \mu) = (1 + O(\ell^2\epsilon))Y_\ell^m(0, 0).$$

It follows that for all $\ell \geq 1$,

$$\phi_\ell(n_x a_y k_{\mu_1, \mu_2, \theta}) = (1 + O(\ell^2\epsilon)) \sum_{m=-\ell}^{\ell} f_\ell^m(x, y) \cdot Y_\ell^m(0, 0)$$

$$= (1 + O(\ell^2\epsilon))\phi_\ell(n_x a_y).$$

$\square$

Setting $N^- := \{n_x^- := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\}$ where x ranges over $\mathbb{R}$ (resp. $\mathbb{C}$) for $n = 2$ (resp. $n = 3$), the product map

$$N \times A \times M \times N^- \to G$$

is a diffeomorphism at a neighborhood of e. Let dk be the invariant probability measure on K. For $g = n_x a_y k$, $dg = y^{-n} dx\, dy\, dk$ defines a Haar measure on G. Let ν be a smooth measure on AMN^- such that $dn_x \otimes d\nu = dg$. Fix a bounded open domain $J \subset F_0$ which contains F_Λ and choose a compactly supported smooth function $0 \leq \eta \leq 1$ on N

with $\eta|_J = 1$. If ∞ is of rank $n - 1$, then set $J = F_0$; hence $\eta = 1$ on F_0.

Fix $\epsilon_0 > 0$ so that the multiplication map

$$\mathrm{supp}(\eta) \times (U_{\epsilon_0} \cap AMN^-) \to \mathrm{supp}(\eta)\left(U_{\epsilon_0} \cap AMN^-\right) \subset \Gamma\backslash G$$

is a bijection onto its image. For each $0 < \epsilon < \epsilon_0$, let $0 \le r_\epsilon \le 1$ be a non-negative smooth function in AMN^- whose support is contained in $W_\epsilon := (U_\epsilon \cap A)M(U_{\epsilon_0} \cap N^-)$ and $\int_{W_\epsilon} r_\epsilon \, d\nu = 1$.

We define the function $\rho_{\eta,\epsilon}$ on $\Gamma\backslash G$ as follows: for $g = n_x a_y m n_{x'}^-$

$$\rho_{\eta,\epsilon}(g) = \begin{cases} \eta(n_x) \cdot r_\epsilon(a_y m n_{x'}^-) & \text{for } g \in \mathrm{supp}(\eta)W_\epsilon \\ 0 & \text{for } g \notin \mathrm{supp}(\eta)W_\epsilon. \end{cases}$$

Proposition 5.5. *For any $\ell \in \mathbb{Z}_{\ge 0}$, we have for all $0 < \epsilon \ll 1$ and $y > 0$,*

$$\phi_\ell^N(a_y) =$$

$$\langle a_y \cdot \phi_\ell, \rho_{\eta,\epsilon} \rangle + O(\epsilon(\ell+1)^{\frac{n-2}{2}} y^{n-1-\delta} + (\ell+1)^{\frac{3n-4}{2}} y^{n-\delta}) + O_\eta((\ell+1)^{\frac{n-2}{2}} y^\delta)$$

with the implied constants independent of ℓ.

Proof. Let $h = a_{y_0} n_x^- m \in W_\epsilon$. Then for $n \in N$ and $y > 0$, we have

$$n h a_y = n a_{y y_0} n_{yx}^- m.$$

As the product map $A \times N \times K \to G$ is a diffeomorphism and hence a bi-Lipschitz map in a neighborhood of e, there exists $q \ge 1$ such that the ϵ-neighborhood of e in G is contained in the product $A_{q\epsilon} N_{q\epsilon} K_{q\epsilon}$ for all small $\epsilon > 0$. Therefore we may write

$$n_{yx}^- = a_{y_1} n_{x_1} k_1 \in A_{qy\epsilon_0} N_{qy\epsilon_0} K_{qy\epsilon_0}$$

and hence

$$n h a_y = n a_{y y_0 y_1} n_{x_1} k_1 m$$

$$= n(a_{y y_0 y_1} n_{x_1} a_{y y_0 y_1}^{-1}) a_{y y_0 y_1} k_1 m = n n_{x_1 y y_0 y_1} a_{y y_0 y_1} k_1 m.$$

By Lemma 5.3,

$$\phi_\ell(n_x h a_y) = \phi_\ell(n_x n_{x_1 y y_0 y_1} a_{y y_0 y_1})(1 + O((\ell+1)^{n-1} y)).$$

By Theorem 4.12, we may write $\phi_\ell^N(a_y) = c_n(\ell) y^{n-1-\delta}$ with $c_n(\ell) \ll (\ell+1)^{(n-2)/2}$. Hence, using Lemma 5.1,

$$\int_{(N\cap\Gamma)\backslash N} \phi_\ell(n_x h a_y) \cdot \eta(n_x) dx$$

$$= (1 + O((\ell+1)^{n-1} y)) \int_{N\cap\Gamma\backslash N} \phi_\ell(n_x n_{x_1 y y_0 y_1} a_{y y_0 y_1}) \cdot \eta(n_x) dx.$$

Since $x_1 y y_0 y_1 = 1 + O(y^2)$,

$$\int_{(N\cap\Gamma)\backslash N} \phi_\ell(n_x h a_y) \cdot \eta(n_x) dx$$

$$= (1 + O((\ell+1)^{n-1} y)) \int_{N\cap\Gamma\backslash N} \phi_\ell(n_x a_{yy_0 y_1}) \cdot (\eta(n_x) + O(y^2)) dx$$

$$= \int_{N\cap\Gamma\backslash N} \phi_\ell(n_x a_{yy_0 y_1}) \cdot \eta(n_x) dx + O((\ell+1)^{\frac{3n-4}{2}} y^{n-\delta})$$

$$+ O_\eta(y^2 + (\ell+1)^{n-1} y^3)(\ell+1)^{\frac{n-2}{2}} y^{n-1-\delta})$$

$$= c_n(\ell) y^{n-1-\delta}(1 + O(\epsilon)) + O((\ell+1)^{\frac{3n-4}{2}} y^{n-\delta}) + O_\eta((\ell+1)^{\frac{n-2}{2}}(y^{n+1-\delta} + y^\delta))$$

$$= c_n(\ell) y^{n-1-\delta} + O(\epsilon(\ell+1)^{\frac{n-2}{2}} y^{n-1-\delta} + (\ell+1)^{\frac{3n-4}{2}} y^{n-\delta}) + O_\eta((\ell+1)^{\frac{n-2}{2}} y^\delta)$$

as $|y_0 - 1| = O(\epsilon)$ and $|y_1 - 1| = O(y\epsilon)$.

As $\int r_\epsilon d\nu(h) = 1$, we deduce from Lemma 5.1 that

$$\langle a_y \phi_\ell, \rho_{\eta,\epsilon} \rangle = \int_{W_\epsilon} r_\epsilon(h) \left(\int_{N\cap\Gamma\backslash N} \phi_\ell(n_x h a_y) \eta(n_x) \, dx \right) d\nu(h)$$

$$= c_n(\ell) y^{n-1-\delta} + O(\epsilon(\ell+1)^{\frac{n-2}{2}} y^{n-1-\delta} + (\ell+1)^{\frac{3n-4}{2}} y^{n-\delta}) + O_\eta((\ell+1)^{\frac{n-2}{2}} y^\delta).$$

Since $\phi_\ell^N(a_y) = c_n(\ell) y^{n-1-\delta}$, this proves the claim. $\square$

The above proposition implies:

Corollary 5.6. *For any $\ell \geq 0$,*

$$|\langle a_y.\phi_\ell, \rho_{\eta,\epsilon} \rangle| \ll (\ell+1)^{\frac{3n-4}{2}} y^{n-1-\delta}$$

where the implied constant is independent of $\ell \geq 0$, $0 < \epsilon < 1$ and $0 < y < 1$.

6. Equidistribution of a closed horosphere

As before, let $n = 2$ or 3, and $G = \mathrm{PSL}_2(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{C})$ accordingly. Let $\Gamma < G$ be a torsion-free geometrically finite discrete subgroup with $(n-1)/2 < \delta < (n-1)$. We assume that $(N \cap \Gamma)\backslash N$ is closed. In this section, we prove the main effective equidistribution theorem for $(N \cap \Gamma)\backslash N a_y$ as $y \to 0$.

Let $\{X_i\}$ be a basis of the Lie algebra of G. For $\psi \in C^\infty(\Gamma\backslash G) \cap L^2(\Gamma\backslash G)^M$, we consider the following L^2-Sobolev norm $\mathcal{S}_m(\psi)$:

$$\mathcal{S}_m(\psi) = \sum \|X(\psi)\|_2$$

where the sum is taken over all monomials X in X_i's of order at most m. By Theorem 3.1, we can fix $(n-1)/2 < s_1 < \delta$ so that there is no eigenvalue of Δ between $s_1(n-1-s_1)$ and $\delta(n-1-\delta)$ in $L^2(\Gamma\backslash\mathbb{H}^n)$. That is, no complementary series representation of G with parameter $s_1 < s < \delta$ is contained in $L^2(\Gamma\backslash G)$.

Lemma 6.1. *For any $\psi_1, \psi_2 \in L^2(\Gamma\backslash G)^M \cap C^\infty(\Gamma\backslash G)$ and $0 < y < 1$, we have*
$$\langle a_y.\psi_1, \psi_2 \rangle = \sum_{\ell \in \hat{K}} \langle \psi_1, \phi_\ell \rangle \langle a_y.\phi_\ell, \psi_2 \rangle + O\left(y^{n-1-s_1} \cdot \mathcal{S}_{n-1}(\psi_1) \cdot \mathcal{S}_{n-1}(\psi_2)\right).$$

Proof. We write $L^2(\Gamma\backslash G) = V \oplus V^\perp$ where $V = \mathcal{H}_\delta$ is the complementary series representation of G of parameter δ. By Theorem 3.1 and the choice of s_1, the orthogonal complement $V^\perp$ does not contain any complementary series with parameter $s > s_1$. Since $V^M = \oplus_{\ell \in \hat{K}} \mathbb{C}\phi_\ell$, we can write
$$\psi_1 = \sum_{\ell \in \hat{K}} \langle \psi_1, \phi_\ell \rangle \phi_\ell + \psi_1^\perp$$
with $\psi_1^\perp \in V^\perp$. Hence
$$\langle a_y.\psi_1, \psi_2 \rangle = \sum_{\ell \in \hat{K}} \langle \psi_1, \phi_\ell \rangle \langle a_y.\phi_\ell, \psi_2 \rangle + \langle a_y.\psi_1^\perp, \psi_2 \rangle.$$

On the other hand, since $V^\perp$ does not contain any complementary series with parameter $s > s_1$, we have
$$|\langle a_y.\psi_1^\perp, \psi_2 \rangle| \ll y^{n-1-s_1} \cdot \mathcal{S}_{n-1}(\psi_1) \cdot \mathcal{S}_{n-1}(\psi_2)$$
(see [21, Prop. 3.3] and [20, Cor. 5.6]). $\square$

We refer to [20, Lem 6.5] for the next lemma:

Lemma 6.2. *For $\psi \in C_c^\infty(\Gamma\backslash G)$, there exists $\widehat{\psi} \in C_c^\infty(\Gamma\backslash G)$ such that*
  (1) *for all small $\epsilon > 0$, $h \in U_\epsilon$, and $g \in \Gamma\backslash G$,*
$$|\psi(g) - \psi(gh)| \le \epsilon \cdot \widehat{\psi}(g).$$
  (2) *for any $i \ge 1$, $\mathcal{S}_i(\widehat{\psi}) \ll \mathcal{S}_{2n-1}(\psi)$ with the implied constant depending only on $\mathrm{supp}(\psi)$.*

Lemma 6.3. *For any $\psi \in C_c^\infty(\Gamma\backslash G)$, $\ell \ge 0$, and $i \ge 1$, we have*
$$|\langle \psi, \phi_\ell \rangle| \ll (\ell+1)^{-2i} \|\mathcal{C}_K^i \psi\|_2.$$
In particular, for each $i \ge 1$,
$$\sum_{\ell \in \hat{K}} (|\ell|+1)^{2i} \cdot |\langle \psi, \phi_\ell \rangle| \ll \|\mathcal{C}_K^{i+1} \psi\|_2.$$

Proof. Recall that the operator $\mathcal{C}_K$ acts on each V_ℓ as a scalar α_ℓ, where $\alpha_\ell = -4\ell^2$ for $n = 2$ and and $\alpha_\ell = \ell(\ell + 1)$ for $n = 3$. Moreover, for any smooth vectors v, w, $|\langle \mathcal{C}_K v, w \rangle| = |\langle v, \mathcal{C}_K w \rangle|$, as $\mathcal{C}_K$ is skew-adjoint for $n = 2$ and $\mathcal{C}_K$ is adjoint for $n = 3$. Therefore

$$|\langle \mathcal{C}_K^i \psi, \phi_\ell \rangle| = |\langle \psi, \mathcal{C}_K^i \phi_\ell \rangle| = |\alpha_\ell^i| \cdot |\langle \psi, \phi_\ell \rangle|.$$

Hence for all $i \geq 1$,

$$|\langle \psi, \phi_\ell \rangle| \leq |\alpha_\ell|^{-i} \cdot \|\mathcal{C}_K^i \psi\|_2.$$

$\square$

By Theorem 4.12, we may write

$$\phi_\ell^N(a_y) = c_n(\ell) \cdot y^{n-1-\delta}$$

where

$$c_n(\ell) = (-1)^{(n-2)\ell} \cdot c_n(0) \cdot \frac{\sqrt{\Gamma(n-1-\delta)\Gamma(\ell+\delta)}}{\sqrt{\Gamma(\delta)\Gamma(\ell+n-1-\delta)}} \cdot \sqrt{2(n-2)\ell + 1}. \quad (6.4)$$

Theorem 6.5. *Fix* $0 < \mathbf{s}_\Gamma < \delta - s_1$. *For any* $\psi \in C_c^\infty(\Gamma\backslash G)^M$,

$$\psi^N(a_y) = \sum_{\ell \in \hat{K}} c_n(\ell) \langle \psi, \phi_\ell \rangle y^{n-1-\delta} + \mathcal{S}_{2n-1}(\psi) O(y^{n-1-\delta+\frac{2\mathbf{s}_\Gamma}{(2n+1)}}).$$

Proof. Fix $\psi \in C_c^\infty(\Gamma\backslash G)^M$. When ∞ has rank $n-1$, we set $J = F_0$. In other cases, it was shown in [20] that there exists a bounded open subset J of F_0 such that $\psi(n_x a_y) = 0$ for all $x \in F_0 - J$ and all $0 < y < 1$. We assume that J contains F_Λ, which is a bounded fundamental domain for the action of $(N \cap \Gamma)$ in $\Lambda(\Gamma) - \{\infty\}$.

Choose a non-negative function $\eta \in C_c^\infty(N \cap \Gamma \backslash N)$ such that $\eta|_J = 1$. Then

$$I_\eta(\psi)(a_y) := \int_{(N \cap \Gamma)\backslash N} \psi(n_x a_y) \eta(n_x) \, dx = \psi^N(a_y) .$$

Let $\epsilon_0, W_\epsilon, r_\epsilon, \rho_{\eta,\epsilon}$ be as defined in section 5 with respect to J and η. Since r_ϵ is the approximation of the identity in A direction, $\mathcal{S}_{n-1}(\rho_{\eta,\epsilon}) = O_\eta(\epsilon^{(-2n+1)/2})$. For any $0 < y < 1$, and any small $\epsilon > 0$, we have (see the proof of Prop. 6.6 in [20])

$$|I_\eta(\psi)(a_y) - \langle a_y.\psi, \rho_{\eta,\epsilon} \rangle| \ll (\epsilon + y) \cdot I_\eta(\widehat{\psi})(a_y). \quad (6.6)$$

Setting $\psi_0(g) := \psi(g)$, we define for $1 \leq i \leq k$, inductively

$$\psi_i(g) := \widehat{\psi}_{i-1}(g)$$

where $\widehat{\psi}_{i-1}$ is given by Lemma 6.2.

Let k be an integer bigger than $1 + \frac{(n-1-\delta)(2n+1)}{2(\delta-s_1)}$.

Applying Lemma 6.2 to each ψ_i, we obtain for $0 \leq i \leq k-1$,

$$I_\eta(\psi_i)(a_y) = \langle a_y.\psi_i, \rho_{\eta,\epsilon} \rangle + O\left((\epsilon + y) \cdot I_\eta(\widehat{\psi_i})(a_y)\right)$$
$$= \langle a_y.\psi_i, \rho_{\eta,\epsilon} \rangle + O\left((\epsilon + y) \cdot I_\eta(\psi_{i+1})(a_y)\right)$$

and

$$I_\eta(\psi_k)(a_y) = \langle a_y.\psi_k, \rho_{\eta,\epsilon} \rangle + O_\eta\left((\epsilon + y)\mathcal{S}_{n-1}(\psi_k)\right).$$

By Lemma 6.3,

$$|\langle \psi_i, \phi_\ell \rangle| = O(|\ell| + 1)^{-(2n-2)}\mathcal{S}_{2n-2}(\psi_i)$$

and by Lemma 6.2, $\mathcal{S}_j(\psi_i) \ll \mathcal{S}_{2n-1}(\psi)$ for all $j \geq 1$.

Since $|\langle a_y.\phi_\ell, \rho_{\eta,\epsilon} \rangle| \ll (|\ell| + 1)^{(3n-4)/2}y^{n-1-\delta}$ by Corollary 5.6, we have

$$\sum_{\ell \in \hat{K}} \langle \psi_i, \phi_\ell \rangle \langle a_y.\phi_\ell, \rho_{\eta,\epsilon} \rangle = O(\mathcal{S}_{2n-1}(\psi_i) \cdot y^{n-1-\delta}).$$

By Lemma 6.1, we deduce that for each $1 \leq i \leq k-1$,

$$\langle a_y.\psi_i, \rho_{\eta,\epsilon} \rangle = \sum_{\ell \in \hat{K}} \langle \psi_i, \phi_\ell \rangle \langle a_y.\phi_\ell, \rho_{\eta,\epsilon} \rangle + O\left(y^{n-1-s_1} \cdot \mathcal{S}_{n-1}(\psi_i) \cdot \mathcal{S}_{n-1}(\rho_{\eta,\epsilon})\right)$$
$$= O(\mathcal{S}_{2n-1}(\psi) \cdot y^{n-1-\delta}) + O(y^{n-1-s_1} \cdot \mathcal{S}_{n-1}(\psi_i)\mathcal{S}_{n-1}(\rho_{\eta,\epsilon}))$$
$$= \mathcal{S}_{2n-1}(\psi) \cdot O(y^{n-1-\delta} + \epsilon^{-(2n-1)/2}y^{n-1-s_1}).$$

Hence for any $0 < y < \epsilon$, using Proposition 5.5, we deduce

$$I_\eta(\psi)(a_y) = \langle a_y.\psi, \rho_{\eta,\epsilon} \rangle + \sum_{j=1}^{k-1} O\left(\langle a_y.\psi_j, \rho_{\eta,\epsilon} \rangle (\epsilon + y)^j\right) + O_\psi((\epsilon + y)^k)$$

$$= \langle a_y.\psi, \rho_{\eta,\epsilon} \rangle + \mathcal{S}_{2n-1}(\psi)O(\epsilon \cdot y^{n-1-\delta} + \epsilon^{-(2n-1)/2}y^{n-1-s_1} + \epsilon^k)$$
$$= \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle \langle a_y.\phi_\ell, \rho_{\eta,\epsilon} \rangle + \mathcal{S}_{2n-1}(\psi)O(\epsilon \cdot y^{n-1-\delta} + \epsilon^{-(2n-1)/2}y^{n-1-s_1} + \epsilon^k)$$
$$= \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle c_n(\ell)y^{n-1-\delta}+$$

$$\sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle \left(O(\epsilon(|\ell| + 1)^{\frac{n-2}{2}}y^{n-1-\delta} + (|\ell| + 1)^{\frac{3n-4}{2}}y^{n-\delta}) + O_\eta((|\ell| + 1)^{(n-2)/2}y^\delta)\right)$$

$$\quad + \mathcal{S}_{2n-1}(\psi)O(y^\delta + \epsilon y^{n-1-\delta} + \epsilon^{-(2n-1)/2}y^{n-1-s_1} + \epsilon^k).$$

Since $|\langle \psi, \phi_\ell \rangle| \ll (|\ell| + 1)^{-(2n-2)}O(\mathcal{S}_{2n-2}(\psi))$, we have

$$\sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle (|\ell| + 1)^{(3n-4)/2} = O(1), \quad \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle (|\ell| + 1)^{(n-2)/2} = O(1).$$

Hence we deduce

$$I_\eta(\psi)(a_y) = \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle \, c_n(\ell) y^{n-1-\delta}$$

$$+ \, \mathcal{S}_{2n-1}(\psi) O(\epsilon y^{n-1-\delta} + y^{n-\delta} + y^\delta + \epsilon^{-(2n-1)/2} y^{n-1-s_1} + \epsilon^k).$$

By equating $\epsilon \cdot y^{n-1-\delta}$ and $\epsilon^{-(2n-1)/2} y^{n-1-s_1}$, we put $\epsilon = y^{2(\delta-s_1)/(2n+1)}$ and obtain

$$I_\eta(\psi)(a_y) = \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle \, c_n(\ell) y^{n-1-\delta} + \mathcal{S}_{2n-1}(\psi) O(y^{n-1-\delta+\frac{2(\delta-s_1)}{2n+1}}).$$

$\square$

7. Comparing main terms from different approaches

The coefficient $\sum_{\ell \in \hat{K}} c_n(\ell) \langle \psi, \phi_\ell \rangle$ of the main term in Theorem 6.5 is related to the space average of ψ with respect to the Burger-Roblin measure (which we will call the BR measure for short) by [30] and [20] (also see [26]).

Recall that $\phi_0 = \phi_0^\Gamma$ is given by

$$\phi_0(x + jy) = \int_{\mathbb{R}^{n-1}} \left(\frac{(|u|^2+1)y}{|x-u|^2+y^2} \right)^\delta \, d\nu_j(u)$$

for the Patterson-Sullivan measure $\nu_j = \nu_j^\Gamma$ on the boundary $\partial(\mathbb{H}^n)$. Note that $\phi_0^\Gamma(j) = |\nu_j^\Gamma|$. As before, we normalize ν_j so that $\|\phi_0\|_2 = 1$.

For $\xi \in \partial(\mathbb{H}^n)$ and $z_1, z_2 \in \mathbb{H}^n$, recall the Busemann function:

$$\beta_\xi(z_1, z_2) = \lim_{s \to \infty} d(z_1, \xi_s) - d(z_2, \xi_s)$$

where ξ_s is a geodesic ray tending to ξ as $s \to \infty$. Using the identification of $\mathrm{T}^1(\mathbb{H}^n)$ and G/M, we give the definition of the Bowen-Margulis-Sullivan measure m^{BMS} on $\Gamma \backslash G/M$. For $u \in \mathrm{T}^1(\mathbb{H}^n)$, we denote by u^+ and u^- the forward and the backward endpoints of the geodesic determined by u, respectively. The correspondence $u \mapsto (u^+, u^-, \beta_{u^-}(j, \pi(u)))$ gives a homeomorphism between the space $\mathrm{T}^1(\mathbb{H}^n)$ with $(\partial(\mathbb{H}^n) \times \partial(\mathbb{H}^n) - \{(\xi, \xi) : \xi \in \partial(\mathbb{H}^n)\}) \times \mathbb{R}$ where $\pi : G \to G/K = \mathbb{H}^n$ is the canonical projection. Define the measure $\tilde{m}^{\mathrm{BMS}}$ on G/M:

$$d\tilde{m}^{\mathrm{BMS}}(u) = e^{\delta \beta_{u^+}(j, \pi(u))} \, e^{\delta \beta_{u^-}(j, \pi(u))} \, d\nu_j(u^+) d\nu_j(u^-) dt$$

where $t = \beta_{u^-}(j, \pi(u))$. This measure is left Γ-invariant and hence induces a measure m^{BMS} on $\Gamma \backslash G/M$.

Roblin obtained the following interesting identity in his thesis [31]:

Theorem 7.1 (Roblin)**.** *For $\delta > (n-1)/2$,*

$$\|\phi_0\|_2^2 = |m^{\mathrm{BMS}}| \cdot \int_{\mathbb{R}^{n-1}} \tfrac{dx}{(1+|x|^2)^\delta}.$$

As we have normalized ν_j so that $\|\phi_0\|_2 = 1$ and $\phi_0(j) = |\nu_j|$, we deduce

$$\frac{1}{|m^{\mathrm{BMS}}|} = \int_{\mathbb{R}^{n-1}} \tfrac{dx}{(1+|x|^2)^\delta}.$$

To describe the equidistribution result of $(N \cap \Gamma)\backslash N a_y$ from [30], we recall the measure $\tilde{m}_N^{\mathrm{BR}}$ defined in the introduction: for $\psi \in C_c(G/M)$,

$$\tilde{m}_N^{\mathrm{BR}}(\psi) = \int_{KAN} \psi(k a_y n_x) y^{\delta-1} dx \, dy \, d\nu_j(k(0)).$$

The BR measure m_N^{BR} (associated to the stable horospherical subgroup N) is the measure on $\Gamma\backslash G/M$ induced from $\tilde{m}_N^{\mathrm{BR}}$.

We define the measure μ_N^{PS} on N by

$$d\mu_N^{\mathrm{PS}}(n_x) = e^{-\delta \beta_x(j, x+j)} d\nu_j(x) = (1 + |x|^2)^\delta d\nu_j(x).$$

This induces a measure on $(N \cap \Gamma)\backslash N$ for which we use the same notation μ_N^{PS}. Since μ_N^{PS} is supported in $(N \cap \Gamma)\backslash (\Lambda(\Gamma) - \{\infty\})$, which is compact, we have $\mu_N^{\mathrm{PS}}((N \cap \Gamma)\backslash N) < \infty$.

The following is proved by Roblin [30] when $(N \cap \Gamma)\backslash N$ is compact and in [26] in general.

Theorem 7.2. *Let $\delta > 0$ and $(N \cap \Gamma)\backslash N$ closed. For any $\psi \in C_c(\Gamma\backslash G)^M$,*

$$\lim_{y \to 0} y^{\delta-n+1} \cdot \psi^N(a_y) = \frac{\mu_N^{\mathrm{PS}}(N \cap \Gamma\backslash N)}{|m^{\mathrm{BMS}}|} m_N^{\mathrm{BR}}(\psi).$$

Comparing the coefficients of the main terms of Theorem 7.2 and Theorem 6.5, and using Theorem 7.1, we deduce the following identity of the Burger-Roblin measure considered as a distribution on $\Gamma\backslash G$:

Theorem 7.3. *Let $\delta > (n-1)/2$. For any $\psi \in C_c^\infty(\Gamma\backslash G)$,*

$$\kappa_\Gamma \cdot m_N^{\mathrm{BR}}(\psi) = \sum_{\ell \in \hat{K}} c_n(\ell) \langle \psi, \phi_\ell \rangle$$

with $c_n(\ell)$ as in (6.4) and $\kappa_\Gamma = \int_{\mathbb{R}^{n-1}} \tfrac{dx}{(1+|x|^2)^\delta} \cdot \int_{n_x \in (N \cap \Gamma)\backslash N} (1 + |x|^2)^\delta d\nu_j(x)$.

Now Theorem 1.6 is a direct consequence of Theorem 7.3 and Theorem 1.3.

8. Application to counting in sectors

Let $n = 2$ or 3. Let Q be a real quadratic form of signature $(n, 1)$ and $v_0 \in \mathbb{R}^{n+1}$ be a non-zero vector such that $Q(v_0) = 0$. Let Γ_0 be a geometrically finite subgroup of the identity component of $\mathrm{SO}_Q(\mathbb{R})$. Suppose that $\delta > (n-1)/2$ and that $v_0\Gamma_0$ is discrete.

Let $\| \cdot \|$ be *any* norm in $\mathbb{R}^{n+1}$ and set $B_T := \{v \in \mathbb{R}^{n+1} : \|v\| < T\}$. Let $G = \mathrm{PSL}_2(\mathbb{R})$ if $n = 2$ and $\mathrm{PSL}_2(\mathbb{C})$ if $n = 3$. Let $\iota : G \to \mathrm{SO}_Q(\mathbb{R})$ be a representation so that the stabilizer of v_0 in G via ι is NM. Let $\Gamma := \iota^{-1}(\Gamma_0)$.

8.1. **Counting I.** For $g \in G$, we write $\kappa(g)$ for the K-coordinate of g in the Iwasawa decomposition $G = NAK$. As before, M denotes the centralizer of A in K. Fixing a function f on $M\backslash K$, define the counting function F_T on $\Gamma\backslash G$ by

$$F_T(g) = \sum_{\gamma \in N \cap \Gamma \backslash \Gamma} \chi_{B_T}(v_0\gamma g) f([\kappa(\gamma g)])$$

where χ_{B_T} denotes the characteristic function of B_T. Since $\kappa(g) = \kappa(ng)$ for any $n \in N$ and $g \in G$, F_T is well-defined.

For $k \in K$ and $\psi \in C_c(G)$, define $\psi^k \in C_c(G)^M$ by

$$\psi^k(g) = \int_{m \in M} \psi(gmk)dm$$

where dm denotes the probability Haar measure of M. Similarly, for $\Psi \in C_c(\Gamma\backslash G)$, we set $\Psi^k(g) = \int_{m \in M} \Psi(gmk)dm$.

Lemma 8.1. *For $\Psi \in C_c(\Gamma\backslash G)$ and for any Borel function f on K, we have*

$$\langle F_T, \Psi \rangle = \int_{k \in M\backslash K} f([k]) \int_{y > \|v_0 k\| T^{-1}} \left(\int_{(N \cap \Gamma)\backslash N} \Psi^k(n_x a_y)\, dx \right) y^{-n} dy dk$$

where dk denotes the probability Haar measure on K, also understood as the invariant measure on $M\backslash K$.

Proof. We compute

$$
\langle F_T, \Psi \rangle = \int_{\Gamma \backslash G} \sum_{\gamma \in N \cap \Gamma \backslash \Gamma} \chi_{B_T}(v_0 \gamma g) f(\kappa(\gamma g)) \Psi(g) \, dg
$$

$$
= \int_{(N \cap \Gamma) \backslash G} \chi_{B_T}(v_0 g) f(\kappa(g)) \Psi(g) \, dg
$$

$$
= \int_{a_y k \in AK} \chi_{B_T}(v_0 a_y k) f(k) \left(\int_{(N \cap \Gamma) \backslash N} \Psi(n_x a_y k) \, dx \right) y^{-n} dy dk
$$

$$
= \int_{k \in M \backslash K} \int_{y > \|v_0 k\| T^{-1}} f(k) \left(\int_{(N \cap \Gamma) \backslash N} (\int_{m \in M} \Psi(n_x a_y m k) dm) \, dx \right) y^{-n} dy dk.
$$

$\square$

By Theorem 1.6, for any $\Psi \in C^\infty(\Gamma \backslash G)^M$ and $k \in M \backslash K$, we have, as $y \to 0$,

$$
\int_{(N \cap \Gamma) \backslash N} \Psi^k(n_x a_y) dx = \kappa_\Gamma \cdot m_N^{\mathrm{BR}}(\Psi^k) \cdot y^{n-1-\delta} + \mathcal{S}_{2n-1}(\Psi) O(y^{(n-1-\delta) + \frac{2s_\Gamma}{2n+1}}).
$$

Therefore, we deduce from Lemma 8.1:

Theorem 8.2. *For any $\Psi \in C^\infty(\Gamma \backslash G)$ and a bounded Borel function f on $M \backslash K$, we have, as $T \to \infty$,*

$$
\langle F_T, \Psi \rangle = \tfrac{\kappa_\Gamma}{\delta} \cdot \left(\int_{k \in M \backslash K} \tfrac{m_N^{\mathrm{BR}}(\Psi^k) \cdot f(k)}{\|v_0 k\|^\delta} \, dk \right) \cdot T^\delta + O(\mathcal{S}_{2n-1}(\Psi) T^{\delta - \frac{2s_\Gamma}{2n+1}}).
$$

8.2. Counting II. For a left M-invariant Borel subset $\Omega \subset K$ and $T > 0$, define

$$
S_T(\Omega) := \{v \in v_0 A\Omega : \|v\| < T\}.
$$

For a subset I of $\mathbb{Z}_{\geq 0}$, let $\{\Gamma_d < \Gamma_0 : d \in I\}$ be a family of subgroups of finite index which satisfies $\mathrm{Stab}_{\Gamma_0} v_0 = \mathrm{Stab}_{\Gamma_d} v_0$ and which has a uniform spectral gap, say, s_0.

Set

$$
\Xi_{v_0}(\Gamma_0, \Omega) := \tfrac{\kappa_{\iota^{-1}(\Gamma_0)}}{\delta} \int_{k^{-1} \in \Omega} \tfrac{d\nu_j^\Gamma(k(0))}{\|v_0 k^{-1}\|^\delta}. \tag{8.3}
$$

We deduce the following from Theorem 8.2:

Theorem 8.4. *Let Ω be an admissible M-invariant Borel subset of K and let $q_\Omega > 0$ be as in (1.7). For any $\gamma' \in \Gamma_0$,*

$$
\#(v_0 \Gamma_d \gamma' \cap S_T(\Omega)) = \tfrac{\Xi_{v_0}(\Gamma_0, \Omega)}{[\Gamma_0 : \Gamma_d]} T^\delta + O(T^{\delta - \frac{8s_0}{n(n+9)(2n+1)q_\Omega}})
$$

with the implied constant independent of d and γ'.

Proof. Let $\Gamma := \iota^{-1}(\Gamma_0)$ and $\gamma_0 := \iota^{-1}(\gamma')$. Recall that U_ϵ denotes an ϵ-neighborhood of e in G. By abuse of notation, we use the notation Γ_d to denote $\iota^{-1}(\Gamma_d)$. Moreover, for all sufficiently small $\epsilon > 0$, and for $K_\epsilon := U_\epsilon \cap K$, we have, for $\Omega_{\epsilon+} = \Omega K_\epsilon$ and $\Omega_{\epsilon-} = \cap_{k \in K_\epsilon} \Omega k$,

$$\nu_j(\Omega_{\epsilon+}^{-1}(0) - \Omega_{\epsilon-}^{-1}(0)) = O(\epsilon^{q_\Omega}). \tag{8.5}$$

By the strong wave front lemma [14, Theorem 4.1], there exists $0 < \ell_0 < 1$ such that for $T \gg 1$,

$$S_T(\Omega)U_{\ell_0\epsilon} \subset S_{(1+\epsilon)T}(\Omega_{\epsilon+}) \quad \text{and} \quad S_{(1-\epsilon)T}(\Omega_{\epsilon-}) \subset \cap_{u \in U_{\ell_0\epsilon}} S_T(\Omega)u.$$

Let $\psi_\epsilon \in C_c^\infty(G)$ be a non-negative function supported in $U_{\ell_0\epsilon}$ with integral one, and define the following function of $\Gamma_d \backslash G$:

$$\Psi_{\Gamma_d,\epsilon}(g) = \sum_{\gamma \in \Gamma_d} \psi_\epsilon(\gamma g).$$

Define the counting function F_T^Ω on $\Gamma_d \backslash G$ by

$$F_T^\Omega(g) = \sum_{\gamma \in N \cap \Gamma \backslash \Gamma_d} \chi_{S_T(\Omega)}(v_0 \gamma g) = \sum_{\gamma \in N \cap \Gamma \backslash \Gamma_d} \chi_{B_T}(v_0 \gamma g) \chi_\Omega([\kappa(\gamma g)]);$$

this is well-defined as $\Gamma_d \cap N = \Gamma \cap N$ by the assumption on Γ_d. Note that $F_T^\Omega(\gamma_0) = \#v_0 \Gamma_d \gamma_0 \cap S_T(\Omega)$ and that

$$F_{(1-\epsilon)T}^{\Omega_{\epsilon-}}(\gamma_0 g) \le F_T^\Omega(\gamma_0) \le F_{(1+\epsilon)T}^{\Omega_{\epsilon+}}(\gamma_0 g)$$

for all $g \in U_{\ell_0\epsilon}$. On the other hand,

$$\int_{\Gamma_d \backslash G} F_{(1\pm\epsilon)T}^{\Omega_{\epsilon\pm}}(\gamma_0 g)\Psi_{\Gamma_d,\epsilon}(g)dg = \int_{\Gamma_d \backslash G} F_{(1\pm\epsilon)T}^{\Omega_{\epsilon\pm}}(g)\Psi_{\Gamma_d,\epsilon}(\gamma_0^{-1}g)dg.$$

Therefore, if we set $\Psi_{\Gamma_d,\epsilon}^{\gamma_0}(g) := \Psi_{\Gamma_d,\epsilon}(\gamma_0^{-1}g)$, we have

$$\langle F_{(1-\epsilon)T}^{\Omega_{\epsilon-}}, \Psi_{\Gamma_d,\epsilon}^{\gamma_0} \rangle \le F_T^\Omega(\gamma_0) \le \langle F_{(1+\epsilon)T}^{\Omega_{\epsilon+}}, \Psi_{\Gamma_d,\epsilon}^{\gamma_0} \rangle$$

where the inner product has taken place in $L^2(\Gamma_d \backslash G)$. Since $\phi_0^\Gamma(e) = |\nu_j^\Gamma|$ and $\|\phi_0^\Gamma\|_2 = 1$, we note that for all positive integer d,

$$\nu_j^{\Gamma_d} = \frac{1}{\sqrt{[\Gamma:\Gamma_d]}}\nu_j^\Gamma.$$

Therefore it follows that $\kappa_{\iota^{-1}(\Gamma_d)} = \frac{1}{\sqrt{[\Gamma:\Gamma_d]}}\kappa_{\iota^{-1}(\Gamma)}$, and that for any Γ-invariant continuous function f on G/M,

$$m_{\Gamma_d,N}^{\text{BR}}(f) = \frac{1}{\sqrt{[\Gamma:\Gamma_d]}}m_{\Gamma,N}^{\text{BR}}(f).$$

We use the following (see [26, Prop. 6.2], or [20, Sec. 7]):

$$\int_{k \in \Omega} \frac{m_{\Gamma,N}^{\text{BR}}(\Psi_{\Gamma,\epsilon}^k)}{\|v_0 k\|^\delta}dk = \int_{k \in \Omega^{-1}} \frac{d\nu_j^\Gamma(k(0))}{\|v_0 k^{-1}\|^\delta} \cdot (1 + O(\epsilon)). \tag{8.6}$$

Since $\dim(G) = n(n+1)/2$, we compute $\mathcal{S}_{2n-1}(\Psi_\epsilon) = O(\epsilon^{-(n^2+9n-4)/4})$. Hence putting these together and using Theorem 8.2, we have

$$\langle F^{\Omega_{\epsilon\pm}}_{(1\pm\epsilon)T}, \Psi^{\gamma_0}_{\Gamma_d,\epsilon}\rangle$$

$$= \frac{\kappa_\Gamma \cdot (1\pm\epsilon)^\delta}{\delta[\Gamma:\Gamma_d]} T^\delta \int_{k\in\Omega^{-1}_{\epsilon\pm}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta} + O(\epsilon T^\delta + \epsilon^{-(n^2+9n-4)/4} T^{\delta-\frac{2s_0}{2n+1}})$$

$$= \frac{\kappa_\Gamma \cdot T^\delta}{\delta[\Gamma:\Gamma_d]} \int_{k\in\Omega^{-1}_{\epsilon\pm}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta} + O(\epsilon T^\delta + \epsilon^{-(n^2+9n-4)/4} T^{\delta-\frac{2s_0}{2n+1}})$$

$$= \frac{\kappa_\Gamma \cdot T^\delta}{\delta[\Gamma:\Gamma_d]} \int_{k\in\Omega^{-1}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta} + O(\epsilon^{q_\Omega} T^\delta + \epsilon^{-(n^2+9n-4)/4} T^{\delta-\frac{2s_0}{2n+1}}).$$

Hence, by equating $\epsilon^{-(n^2+9n-4)/4} T^{-\frac{2s_0}{(2n+1)}} = \epsilon^{q_\Omega}$, we deduce

$$F^\Omega_T(e) = \frac{\kappa_\Gamma \cdot T^\delta}{\delta[\Gamma:\Gamma_d]} \int_{k\in\Omega^{-1}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta} + O(T^{\delta-\frac{8s_0}{n(n+9)(2n+1)q_\Omega}}).$$

This finishes the proof of Theorem 8.4.

$\square$

8.3. Let $\mathcal{P}$ be an Apollonian packing as in Theorem 1.1. Let

$$Q(x_1, x_2, x_3, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2$$

be the Descartes quadratic form, which has signature $(3,1)$. Let $\mathcal{A}$ denote the Apollonian group, i.e., the subgroup of $O_Q(\mathbb{Z})$ generated by

$$S_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The critical exponent of $\mathcal{A}$ is equal to α, which is the Hausdorff dimension of the residual set of $\mathcal{P}$ and is a geometrically finite group (cf. [20]).

In [20, Sec. 2], it was shown that there exists a vector v_0 with $Q(v_0) = 0$, whose coordinates are given by the curvatures of four mutually tangent circles of $\mathcal{P}$ and that

$$N_T(\mathcal{P}) = \{v \in v_0\mathcal{A} : \|v\|_{\max} < T\} + 3$$

for all $T \gg 1$.

Therefore Theorem 8.4 implies, as $q_K = 1$:

Corollary 8.7. *For some $c_{\mathcal{P}} > 0$,*

$$N_T(\mathcal{P}) = c_{\mathcal{P}} \cdot T^{\alpha} + O(T^{\alpha - 2\mathbf{s}_{\mathcal{A}}/63}).$$

Moreover, if we set $\mathcal{A}_0 < \mathrm{SO}(Q)^{\circ}$ to be a torsion free finite index subgroup of $\mathcal{A}$ and write $v_0\mathcal{A}$ as the disjoint union $\cup_{i=1}^{m} v_i \mathcal{A}_0$, then

$$c_{\mathcal{P}} = \sum_{i=1}^{m} \Xi_{v_i}(\mathcal{A}_0, K) \tag{8.8}$$

where Ξ_{v_i} is defined as in (8.3).

On the other hand, it can be deduced from the main results in [27] that

$$\lim_{T \to \infty} \frac{N_T(\mathcal{P})}{T^{\alpha}} = c_A \cdot \mathcal{H}_{\alpha}(\mathrm{Res}(\mathcal{P}))$$

where $c_A > 0$ is a constant independent of $\mathcal{P}$ (cf. [28] for details). Therefore Theorem 1.1 follows from Corollary 8.7.

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