COUNTING TOTALLY REAL UNITS AND EIGENVALUE PATTERNS IN $\mathrm{SL}_n(\mathbb{Z})$ AND $\mathrm{Sp}_{2n}(\mathbb{Z})$ IN THIN TUBES

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Abstract. For a vector $v = (v_1, \dots, v_n)$ with $v_1 > \cdots > v_n$ and $\sum v_i = 0$, we study the *directional entropy* of two arithmetic objects:
(1) the logarithmic embeddings of degree-n totally real units, and
(2) the logarithmic eigenvalue data of $\mathrm{SL}_n(\mathbb{Z})$.
In each case, the entropy in the direction of v is

$$\mathsf{E}_n(v) = \rho_{\mathrm{SL}_n}(v) = \sum_{i=1}^{n-1} (n-i)\, v_i,$$

the value of the half-sum of positive roots of $\mathrm{SL}_n(\mathbb{R})$ evaluated at v. More precisely, the number of objects lying in a thin tube around the ray $\mathbb{R}_+ v$ and of norm at most T grows on the order of $\exp\big(\rho_{\mathrm{SL}_n}(v)\, T\big)$ as $T \to \infty$.

Because each eigenvalue data determines an $\mathrm{SL}_n(\mathbb{R})$-conjugacy class, this implies a lower bound of order $\exp\big(\rho_{\mathrm{SL}_n}(v) T\big)$ for the number of $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes with a prescribed eigenvalue data; we also obtain an upper bound of order $\exp\big(2\rho_{\mathrm{SL}_n}(v) T\big)$.

A parallel argument for the symplectic lattice $\mathrm{Sp}_{2n}(\mathbb{Z})$, taken in the symmetric direction $v = (v_1, \dots, v_n, -v_n, \dots, -v_1)$, $v_1 > \cdots > v_n > 0$, shows that

$$\mathsf{E}^{\mathrm{Sp}}_{2n}(v) = \rho_{\mathrm{Sp}_{2n}}(v) = \sum_{i=1}^{n} (n+1-i) v_i,$$

the half-sum of positive roots of $\mathrm{Sp}_{2n}(\mathbb{R})$.

1. Introduction

Classical arithmetic asks how many objects of a given kind—ideals, points, matrices, geodesics—fit inside a region that grows without bound. In higher rank, the natural "size" of an object is rarely a single number; instead it is a vector that records growth rates in several directions at once. When we restrict our attention to a thin tube around a fixed ray, the leading exponent of an exponential growth can be viewed as a *directional entropy*: it measures how densely the arithmetic set populates that ray.

This paper pinpoints an explicit linear functional that governs the directional entropy of the following two collections:

- the logarithmic embeddings of *all* totally real units of fixed degree n;
- the logarithmic eigenvalue data (=Jordan projections) of elements in $\mathrm{SL}_n(\mathbb{Z})$.

We prove that both collections exhibit the same entropy along every ray in the positive Weyl chamber. Going further, we count $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes that share a prescribed eigenvalue pattern. The Jordan-projection entropy yields an immediate lower bound for this count, and we provide an upper bound, which is conjecturally a true order of magnitude. We also address the analogous problem for the symplectic lattice $\mathrm{Sp}_{2n}(\mathbb{Z})$.

Totally real algebraic units. For an integer $n \geq 2$, let $\mathcal{K}_n^\star$ denote the set of totally real number fields K of degree n. For each $K \in \mathcal{K}_n^\star$, let Σ_K denote the set of all *ordered* embeddings of K into $\mathbb{R}$. Define

$$\mathcal{K}_n = \bigsqcup_{K\in\mathcal{K}_n^\star, \sigma\in\Sigma_K} (K, \sigma).$$

For $(K, \sigma) \in \mathcal{K}_n$ with $\sigma = (\sigma_1, \cdots, \sigma_n)$, define the logarithmic map

$$\Lambda_{K,\sigma} : K - \{0\} \to \mathbb{R}^n, \ \ \Lambda_{K,\sigma}(\mathsf{u}) = \big(\log|\sigma_1(\mathsf{u})|, \dots, \log|\sigma_n(\mathsf{u})|\big).$$

Denote by O_K the ring of integers of K and by $O_K^\times$ its unit group. Consider the hyperplane

$$\mathsf{H} = \{v = (v_1, \cdots, v_n) \in \mathbb{R}^n : \sum v_i = 0\}$$

and note that, by Dirichlet's unit theorem, $\Lambda_{K,\sigma}(O_K^\times)$ is a lattice in H (cf. [18]).

We extend $\Lambda_{K,\sigma}$ coordinate-wise to a map $\Lambda : \mathcal{K}_n - \{0\} \to \mathbb{R}^n$: $\Lambda(u) = \Lambda_{K,\sigma}(u)$ if $u \in (K, \sigma)$. Collect all totally real units of degree n in the disjoint union

$$O_n^\times = \bigsqcup_{(K,\sigma)\in\mathcal{K}_n} (O_K^\times, \sigma).$$

We propose the following notion of directional entropy for vectors in

$$\mathsf{H}_+ = \Big\{v = (v_1, \dots, v_n) \in \mathsf{H} : v_1 > \cdots > v_n\Big\}.$$

Definition 1.1 (Directional entropy)**.** Fix a norm $\|\cdot\|$ on $\mathbb{R}^n$. For $v \in \mathsf{H}_+$, define the *upper* and *lower* directional entropies of $O_n^\times$ in the direction v by

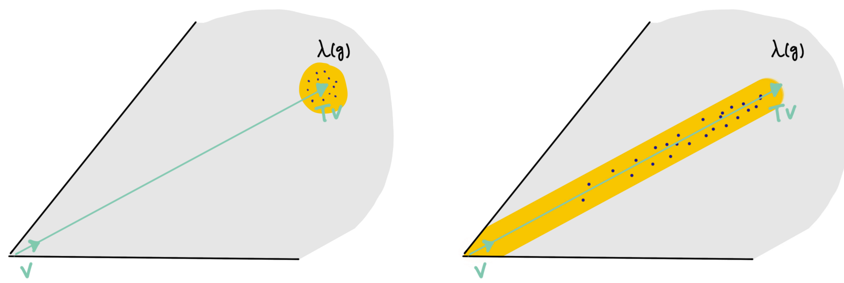
$$\overline{\mathsf{E}}_n(v) := \|v\| \cdot \lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log N_\varepsilon(T, v),$$

$$\underline{\mathsf{E}}_n(v) := \|v\| \cdot \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{T} \log N_\varepsilon(T, v)$$

where

$$N_\varepsilon(T, v) := \#\big\{u \in O_n^\times : \ \|\Lambda(u)\| \leq T, \ \|\Lambda(u) - \mathbb{R}_+ v\| < \varepsilon\big\}.$$

These quantities lie in $\{-\infty\} \cup [0, \infty)$, are independent of the choice of a norm, and are homogeneous of degree one in v. When they coincide, we write $\mathsf{E}_n(v)$ for their common value.



Remark 1.2. We call this quantity a *directional entropy* because it records the exponential growth rate of units whose logarithmic embeddings stay in a thin tube about the ray $\mathbb{R}_+ v$, mirroring standard entropy-type counts in dynamics.

We compute the directional entropy of $O_n^\times$ for every $v \in \mathsf{H}_+$:

Theorem 1.3. *For each* $v = (v_1, \cdots, v_n) \in \mathsf{H}_+$*, we have*

$$\mathsf{E}_n(v) = \sum_{i=1}^{n-1} (n-i) v_i.$$

We expect the same formula to hold for vectors lying on the walls of H_+.

The entropy $\mathsf{E}_n(v)$ may also be expressed via the discriminant of the model polynomial $q_{Tv}(x) = \prod_{i=1}^n (x - e^{Tv_i})$:

$$\lim_{T\to\infty} \frac{1}{T} \log \sqrt{\operatorname{Disc}(q_{Tv})} = \sum_{i=1}^{n-1} (n-i) v_i.$$

Remark 1.4. On the unit sphere for the max-norm $\{v \in \mathsf{H}_+ : \|v\|_{\max} = 1\}$, the entropy functional E_n reaches its supremum $\lfloor \frac{n^2}{4} \rfloor$ in the direction of $v = (1, \cdots 1, 0, -1, \cdots, -1)$ for n odd and $v = (1, \cdots, 1, -1, \cdots, -1)$ for n even, where the first $\lfloor n/2 \rfloor$-coordinates are 1. For instance,

$$\sup\{\mathsf{E}_4(v) : v \in \mathsf{H}_+, \|v\|_{\max} = 1\} = 4.$$

On the Euclidean unit sphere $\{\|v\|_{\mathrm{Euc}} = 1\}$, the maximum value of E_n is $\sqrt{\frac{n(n^2-1)}{12}}$, attained in the direction $(n-1, n-3, \cdots, -(n-3), -(n-1))$.

The following quantitative theorem yields Theorem 1.3.

Theorem 1.5. *Let* $v \in \mathsf{H}_+$*. For all sufficiently small* $\varepsilon > 0$*, we have*

$$\#\{\mathsf{u} \in O_n^\times : \|\Lambda(\mathsf{u}) - Tv\|_{\max} < \varepsilon\} \asymp_\varepsilon{}^{1} \exp\left(\sum_{i=1}^{n-1} (n-i) v_i T\right).$$

[1]We write $f(T) \asymp g(T)$ if there exist $C_1, C_2 > 0$ such that $C_1\, g(T) \le f(T) \le C_2\, g(T)$ for all $T \ge 1$. The notation $f(T) \asymp_\varepsilon g(T)$ has the same meaning, except that C_1 and C_2 may depend on ε.

More precisely,

$$2\left(\frac{4\varepsilon}{(n-1)3^n}\right)^{n-1} \le \liminf_{T\to\infty} \frac{\#\left\{\mathsf{u}\in O_n^\times : \|\Lambda(\mathsf{u}) - Tv\|_{\max} < \varepsilon\right\}}{\exp\left(\sum_{i=1}^{n-1}(n-i)v_i T\right)}$$
$$\le \limsup_{T\to\infty} \frac{\#\left\{\mathsf{u}\in O_n^\times : \|\Lambda(\mathsf{u}) - Tv\|_{\max} < \varepsilon\right\}}{\exp\left(\sum_{i=1}^{n-1}(n-i)v_i T\right)} \le 2(4\varepsilon)^{n-1}n!.$$

It is natural to ask whether the limit $\lim_{T\to\infty}\frac{\#\left\{\mathsf{u}\in O_n^\times : \|\Lambda(\mathsf{u}) - Tv\|_{\max} < \varepsilon\right\}}{\exp\left(\sum_{i=1}^{n-1}(n-i)v_i T\right)}$ exists, and, if so, what its value is. In Proposition 3.2, we give exponential error terms in the upper and lower bounds above; in particular, these error terms can be taken uniformly over all v in any fixed compact subset of H_+.

Eigenvalue patterns in $\mathrm{SL}_n(\mathbb{Z})$. An element $g\in\mathrm{SL}_n(\mathbb{R})$ is called *loxodromic* if its eigenvalues have pairwise distinct moduli; in particular, they are all real. For such $g\in\mathrm{SL}_n(\mathbb{R})$, write its eigenvalues as

$$\mathcal{E}(g) = \left(m_1(g)e^{\lambda_1(g)}, \dots, m_n(g)e^{\lambda_n(g)}\right) \tag{1.1}$$

with signs $m_i(g)\in\{\pm 1\}$ and ordering given by $\lambda_1(g)\ge\cdots\ge\lambda_n(g)$. Set

$$\lambda(g) \coloneqq \big(\lambda_1(g),\dots,\lambda_n(g)\big), \qquad m(g) \coloneqq \big(m_1(g),\dots,m_n(g)\big). \tag{1.2}$$

The vector $\lambda(g)$ is called the *Jordan projection* of g. Define the linear functional

$$\rho_{\mathrm{SL}_n}(v) = \tfrac{1}{2}\sum_{1\le i<j\le n}(v_i - v_j) = \sum_{i=1}^{n-1}(n-i)v_i$$

which is the half-sum of all positive roots of $\mathrm{SL}_n(\mathbb{R})$. For $v\in\mathsf{H}_+$ and $\varepsilon>0$, if T is sufficiently large, then any $\gamma\in\mathrm{SL}_n(\mathbb{R})$ with $\|\lambda(\gamma)-Tv\|<\varepsilon$ is loxodromic.

We prove the following counting results:

Theorem 1.6. *Let $v\in\mathsf{H}_+$ and let $m=(m_1,\cdots,m_n)\in\{\pm1\}$ be a sign pattern with $\prod_{i=1}^n m_i = 1$. Fix $\varepsilon>0$.*

(1) We have

$$\#\Big\{\lambda(\gamma) : \gamma\in\mathrm{SL}_n(\mathbb{Z}), \|\lambda(\gamma)-Tv\|\le\varepsilon, m(\gamma)=m\Big\} \asymp_\varepsilon e^{\rho_{\mathrm{SL}_n}(v)T},$$

where explicit upper and lower multiplicative constants are given in Theorem 4.1.

(2) There exist $C_1, C_2>0$ such that for all sufficiently large $T>1$,

$$C_1 e^{\rho_{\mathrm{SL}_n}(v)T} \le \#\Big\{[\gamma]\in[\mathrm{SL}_n(\mathbb{Z})] : \|\lambda(\gamma)-Tv\|\le\varepsilon, m(\gamma)=m\Big\} \le C_2 e^{2\rho_{\mathrm{SL}_n}(v)T}$$

where $[\mathrm{SL}_n(\mathbb{Z})]$ denotes the set of all $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes.

Observe that $2\rho_{\mathrm{SL}_n}(v)$ is precisely the volume growth exponent for the thin tubes around the ray $\mathbb{R}_+ v$: if $\mu(g) \in \mathsf{H}_+$ is the Cartan projection of g, i.e., the unique element such that $e^{\mu(g)} \in \mathrm{SO}(n) g \, \mathrm{SO}(n)$,

$$\mathrm{Vol}\Big\{g \in \mathrm{SL}_n(\mathbb{R}) : \|\mu(g) - Tv\| \le \varepsilon\Big\} \asymp_\varepsilon e^{2\rho_{\mathrm{SL}_n}(v)T},$$

with volume taken with respect to a Haar measure of $\mathrm{SL}_n(\mathbb{R})$ (cf. the proof of Theorem 6.4). Consequently, the Jordan–projection count in part (1) grows like the *square root* of this ambient volume growth.

Because each eigenvalue pattern determines an $\mathrm{SL}_n(\mathbb{R})$-conjugacy class, this yields a lower bound of order $e^{\rho_{\mathrm{SL}_n}(v)T}$, as stated in part (2), for the number of $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes with prescribed eigenvalue pattern; we also obtain an upper bound of order $e^{2\rho_{\mathrm{SL}_n}(v)T}$.

Similar to the directional entropy for $O_n^\times$, we also propose the following notion of the directional entropies for $\mathrm{SL}_n(\mathbb{Z})$:

Definition 1.7 (Directional entropy for $\mathrm{SL}_n(\mathbb{Z})$)**.** Let $v \in \mathsf{H}_+$ and let $m = (m_1, \cdots, m_n) \in \{\pm 1\}$ satisfy $\prod_{i=1}^n m_i = 1$. Define the *upper* and *lower* directional entropies by

$$\overline{\mathsf{E}}_{\mathrm{SL}_n(\mathbb{Z})}(v,m) := \|v\| \cdot \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log N_\varepsilon(T,v,m)}{T};$$

$$\underline{\mathsf{E}}_{\mathrm{SL}_n(\mathbb{Z})}(v,m) := \|v\| \cdot \lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{\log N_\varepsilon(T,v,m)}{T}$$

where

$$N_\varepsilon(T,v,m) = \#\big\{\lambda(\gamma) : \gamma \in \mathrm{SL}_n(\mathbb{Z}) : \|\lambda(\gamma) - \mathbb{R}_+ v\| \le \varepsilon, \|\lambda(\gamma)\| \le T, m(\gamma) = m\big\}.$$

Similarly, set

$$\overline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v,m) := \|v\| \cdot \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log M_\varepsilon(T,v,m)}{T};$$

$$\underline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v,m) := \|v\| \cdot \lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{\log M_\varepsilon(T,v,m)}{T}$$

where

$$M_\varepsilon(T,v,m) := \#\big\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})] : \|\lambda(\gamma) - \mathbb{R}_+ v\| \le \varepsilon, \|\lambda(\gamma)\| \le T, m(\gamma) = m\big\}.$$

As before, these quantities in $\{-\infty\} \cup [0,\infty)$ are norm-independent and homogeneous of degree one. When the lower and upper limits agree, we write $\mathsf{E}_{\mathrm{SL}_n(\mathbb{Z})}(v)$ and $\mathsf{E}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v)$, respectively.

As an immediate consequence of Theorem 1.6, we get

Theorem 1.8. *Let $v \in \mathsf{H}_+$ and $m \in \{\pm 1\}^n$ with $\prod_{i=1}^n m_i = 1$. Then*

$$\mathsf{E}_{\mathrm{SL}_n(\mathbb{Z})}(v,m) = \rho_{\mathrm{SL}_n}(v);$$

$$\rho_{\mathrm{SL}_n}(v) \;\le\; \underline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v,m) \;\le\; \overline{\mathsf{E}}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v,m) \;\le\; 2\rho_{\mathrm{SL}_n}(v).$$

We think that $\mathsf{E}^\star_{\mathrm{SL}_n(\mathbb{Z})}(v,m) = 2\rho_{\mathrm{SL}_n}(v)$ should be true, although we do not know how to prove this; see Conjecture 1.13 for a more general formulation.

Denote by $\mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}$ (resp. $[\mathrm{SL}_n(\mathbb{Z})]_{\mathrm{lox}}$) the set (resp. the set of $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes) of loxodromic elements of $\mathrm{SL}_n(\mathbb{Z})$. For $\gamma \in \mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}$, write

$$[\gamma]_{\mathbb{R}} \subset \mathrm{SL}_n(\mathbb{Z}) \qquad \text{and} \qquad [\gamma]_{\mathbb{Q}} \subset \mathrm{SL}_n(\mathbb{Z})$$

for its $\mathrm{SL}_n(\mathbb{R})$- and $\mathrm{SL}_n(\mathbb{Q})$-conjugacy classes, respectively. Because the centralizer of $\gamma \in \mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}$ is a maximal $\mathbb{Q}$-split torus and all such tori are conjugate under $\mathrm{SL}_n(\mathbb{Q})$, we have

$$[\gamma]_{\mathbb{R}} = [\gamma]_{\mathbb{Q}}.$$

Define the "class number"

$$h(\gamma) = \#\{\mathrm{SL}_n(\mathbb{Z})\text{-conjugacy classes inside } [\gamma]_{\mathbb{Q}}\}. \tag{1.3}$$

Since the eigenvalue pattern of a loxodromic element uniquely determines its $\mathrm{SL}_n(\mathbb{R})$-conjugacy class, the map $\gamma \mapsto \mathcal{E}(\gamma)$ gives a bijection

$$\{[\gamma]_{\mathbb{R}} : \gamma \in \mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}\} \Leftrightarrow \{\mathcal{E}(\gamma) : \gamma \in \mathrm{SL}_n(\mathbb{Z})_{\mathrm{lox}}\}.$$

Hence for any region $R \subset \mathbb{R}^n$,

$$\#\big\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})]_{\mathrm{lox}} : \mathcal{E}(\gamma) \in R\big\} \;=\; \sum_{\mathcal{E}(\gamma)\in R} h(\gamma).$$

In other words, the number of $\mathrm{SL}_n(\mathbb{Z})$-conjugacy classes whose eigenvalue pattern lies in R equals the count of those patterns, each weighted by its class number $h(\gamma)$.

Remark 1.9. In view of Theorem 1.6, one may expect that for any $\varepsilon > 0$,

$$h(\gamma) \ll_\varepsilon e^{\rho_{\mathrm{SL}_n}(\lambda(\gamma))(1+\varepsilon)}$$

for all loxodromic $\gamma \in \mathrm{SL}_n(\mathbb{Z})$.

For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $D = \mathrm{tr}(\gamma)^2 - 4$ square-free, the quantity $h(\gamma)$ coincides with the classical class number $h_K = \#\,\mathrm{Cl}(O_K)$ of the quadratic field $K = \mathbb{Q}(\sqrt{D})$ (see [19], [30], [18]). Moreover, the conjugacy classes $[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})]_{\mathrm{lox}}$ correspond bijectively to closed geodesics C_γ on the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^2$, with length given by $2\lambda_1(\gamma)$ [28]. Hence the prime geodesic theorem on modular surface ([29], [15]) implies

$$\#\Big\{[\gamma] \in [\mathrm{SL}_2(\mathbb{Z})]_{\mathrm{lox}} : T - \varepsilon \le \|\lambda(\gamma)\| < T + \varepsilon\Big\} \asymp_\varepsilon \frac{e^{2T}}{2T}.$$

In this case, Theorem 1.6 gives

$$\#\Big\{\mathcal{E}(\gamma) : \gamma \in \mathrm{SL}_2(\mathbb{Z})_{\mathrm{lox}} : T - \varepsilon \le \|\lambda(\gamma)\| \le T + \varepsilon\Big\} \asymp_\varepsilon e^{T},$$

which also follows from the elementary fact that $e^{\|\lambda(\gamma)\|}$ is essentially the size of the (integral) trace of γ.

Remark 1.10. Eskin-Mozes-Shah studied a *transversal* counting problem in [11]. Fix a loxodromic element $\gamma_0 \in \mathrm{SL}_n(\mathbb{Z})$ and let $p \in \mathbb{Z}[x]$ be its characteristic polynomial. Write $K = \mathbb{Q}(\alpha)$ for a root α of p. Assume that p is irreducible over $\mathbb{Q}$ and $\mathbb{Z}[\alpha] = \mathcal{O}_K$. By [11, Theorem 1.1], as $T \to \infty$,

$$\#\Big\{\gamma \in [\gamma_0]_{\mathbb{R}} : \|\gamma\| < e^T\Big\} \sim^{2} c_n \frac{h(\gamma_0)\, R_K}{\sqrt{\mathrm{Disc}(p)}} \exp\left(\tfrac{1}{2}(n^2 - n)T\right),$$

where

- $c_n > 0$ depends only on n;
- $h(\gamma_0)$ is the class number defined in (1.3);
- R_K is the regulator of K, i.e. the volume of $\mathsf{H}/\left(\Lambda_{K,\sigma}(O_K^\times)\right)$;

Thus [11] counts *integral matrices lying inside a* fixed $\mathrm{SL}_n(\mathbb{R})$*–conjugacy class*, whereas our results count *the number of* distinct $\mathrm{SL}_n(\mathbb{Z})$*–conjugacy classes* whose Jordan projections fall into a given tube.

Eigenvalue patterns in $\mathrm{Sp}_{2n}(\mathbb{Z})$**.** We also carry out a parallel analysis for the symplectic lattice $\mathrm{Sp}_{2n}(\mathbb{Z})$, obtaining analogous counting results and entropy estimates. Fix the symplectic form in (5.3) so that

$$\mathfrak{a}^+ = \Big\{v = \mathrm{diag}(v_1, \ldots, v_n, -v_n, \ldots, -v_1) : \; v_1 \geq \cdots \geq v_n \geq 0\Big\}$$

is a positive Weyl chamber of $\mathrm{Sp}_{2n}(\mathbb{R})$. An element $g \in \mathrm{Sp}_{2n}(\mathbb{R})$ is *loxodromic* precisely when its Jordan projection

$$\lambda(g) = \big(\lambda_1(g), \ldots, \lambda_n(g), -\lambda_n(g), \ldots, -\lambda_1(g)\big) \in \mathrm{int}\,\mathfrak{a}^+.$$

For such g, set

$$m(g) = \big(m_1(g), \ldots, m_n(g)\big) \in \{\pm 1\}^n,$$

so that, for each i, the two real eigenvalues of g are $m_i(g)\, e^{\pm\lambda_i(g)}$.

Let

$$\rho_{\mathrm{Sp}_{2n}}(v) = \sum_{i=1}^{n} (n + 1 - i)\, v_i$$

be the half-sum of all positive roots of $(\mathfrak{sp}_{2n}(\mathbb{R}), \mathfrak{a})$.

Theorem 1.11. *Let* $v \in \mathrm{int}\,\mathfrak{a}^+$ *and* $m \in \{\pm 1\}^n$*. Fix small* $0 < \varepsilon < 1$*.*

(1) We have

$$\#\Big\{(\lambda(\gamma), m(\gamma)) : \gamma \in \mathrm{Sp}_{2n}(\mathbb{Z}),\ \|\lambda(\gamma) - Tv\| \leq \varepsilon,\ m(\gamma) = m\Big\} \asymp_\varepsilon e^{\rho_{\mathrm{Sp}_{2n}}(v)T},$$

where explicit upper and lower multiplicative constants are given in Theorem 5.7.

(2) There exist $C_1, C_2 > 0$ *such that for all sufficiently large* $T > 1$*,*

$$C_1 e^{\rho_{\mathrm{Sp}_{2n}}(v)T} \leq \#\Big\{[\gamma] \in [\mathrm{Sp}_{2n}(\mathbb{Z})] : \|\lambda(\gamma) - Tv\| \leq \varepsilon, m(\gamma) = m\Big\} \leq C_2 e^{2\rho_{\mathrm{Sp}_{2n}}(v)T}.$$

[2] We write $f(T) \sim g(T)$ if $\lim_{T\to\infty} f(T)/g(T) = 1$.

Define the directional entropies $\mathsf{E}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m)$ and $\mathsf{E}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m)$ exactly as in Definition 1.7, with $\mathrm{SL}_n(\mathbb{Z})$ replaced everywhere by $\mathrm{Sp}_{2n}(\mathbb{Z})$.

Corollary 1.12. *For all $v \in \operatorname{int} \mathfrak{a}^+$ and $m \in \{\pm 1\}^n$, we have*

$$\mathsf{E}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m) = \rho_{\mathrm{Sp}_{2n}}(v);$$

$$\rho_{\mathrm{Sp}_{2n}}(v) \;\le\; \underline{\mathsf{E}}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m) \;\le\; \overline{\mathsf{E}}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v,m) \;\le\; 2\rho_{\mathrm{Sp}_{2n}}(v).$$

On the proof: We outline the proof of Theorem 1.6. The proof of Theorem 1.5 is entirely analogous; one simply uses the bijection between *primitive* units and their minimal polynomials. Let $v \in \mathsf{H}_+$ and m a sign pattern. We translate the geometric condition "$\lambda(\gamma)$ lies in $B_\varepsilon(Tv)$ with sign pattern m" into a purely arithmetic statement about integral polynomials, and then we count those polynomials. For a loxodromic element $\gamma \in \mathrm{SL}_n(\mathbb{Z})$, its eigenvalue pattern $\mathcal{E}(\gamma)$ is equivalent to its characteristic polynomial $p_\gamma(x)$. Requiring $\lambda(\gamma) \in B_\varepsilon(Tv)$ and $m(\gamma) = m$ forces the roots of p_γ to satisfy $m_i e^{Tv_i} + O(\varepsilon e^{Tv_i})$, $1 \le i \le n$. Let $\mathcal{Q}_T(v,m;\varepsilon)$ denote the collection of all monic integral polynomials with this property. Using Rouché's theorem, we observe that $p \in \mathcal{Q}_T(v,m;\varepsilon)$ iff each coefficient lies in an interval of length $(1+O(\varepsilon))e^{T(v_1+\cdots+v_i)}$. Hence $\mathcal{Q}_T(v,m;\varepsilon)$ coincides with an expanding box $\mathcal{P}_T(v,m;\varepsilon)$ inside $\mathbb{Z}^{n-1}$ whose side-lengths grow at precisely those exponential rates. Counting integral points in this expanding box is governed by the square-root of the discriminant of the model polynomial $q_{Tv,m}(x) = \prod_{i=1}^n (x - m_i e^{Tv_i})$ with $\mathrm{Disc}(q_{Tv,m}) \asymp e^{2\rho_{\mathrm{SL}_n}(v)T}$. Exactly the same reasoning works for the symplectic lattice $\mathrm{Sp}_{2n}(\mathbb{Z})$. Here one exploits the fact that the characteristic polynomials of $\mathrm{Sp}_{2n}(\mathbb{Z})$ matrices are *precisely* the integral monic reciprocal (palindromic) polynomials of degree $2n$ ([33], [20]). Because the reciprocal property simply folds the coefficient box in half, the counting again reduces to a volume estimate and the resulting exponent is $\rho_{\mathrm{Sp}_{2n}}(v) = \sum_{i=1}^n (n+1-i)v_i$. For other arithmetic groups, no tidy description is available for the integral polynomials that arise as characteristic polynomials. Even in the case of integral orthogonal groups, a clean criterion necessary for this approach to work does not seem to be known.

On the other hand, the upper bound for the conjugacy-class count in Theorem 1.6 is a special case of Theorem 6.2, which applies to any lattice in a semisimple real algebraic group. The proof proceeds by relating the Jordan projection to the Cartan projection and by applying the standard orbital-counting technique of Eskin-McMullen [10], which exploits the mixing of the G-action on $\Gamma\backslash G$ and the strong wavefront lemma ([14, Theorem 3.7]).

We conclude the introduction by formulating the following conjecture:

Conjecture 1.13. *Let Γ be an arithmetic lattice of a connected simple real algebraic group G. Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and let ρ_G be the half-sum of all positive roots of $(\mathfrak{g}, \mathfrak{a}^+)$, where $\mathfrak{g} = \operatorname{Lie} G$. For $v \in \operatorname{int} \mathfrak{a}^+$,* *define directional entropies* $\mathsf{E}_\Gamma(v)$ *and* $\mathsf{E}^\star_\Gamma(v)$ *as in Definition 6.1. Then*

$$\mathsf{E}_\Gamma(v) = \rho_G(v) \quad \text{and} \quad \mathsf{E}^\star_\Gamma(v) = 2\,\rho_G(v).$$

If G has rank-one, the prime geodesic theorem for rank-one locally symmetric manifolds (see, for instance, [29], [15], [21], [12], [27], [23], etc.) implies that $\mathsf{E}^\star_\Gamma(v) = 2\rho_G(v)$. As we shall see in Theorem 6.4, the corresponding entropy $\mathsf{E}^\star_\Gamma(v)$ defined via the Cartan projection is always $2\rho_G(v)$; it seems plausible that the Jordan and Cartan counts differ only by a polynomial factor, in which case the second equality in the above would indeed hold.

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2. Root separations and Proof of Theorem 1.5

Let $n \geq 2$. As $T \to \infty$, the number of monic integral polynomials of degree n whose roots are bounded by e^T grows in the order of $e^{n(n+1)T/2}$. If we additionally require the constant term to be ± 1, the growth rate drops to the order $e^{n(n-1)T/2}$. These orders remain unchanged when we restrict to totally real polynomials [3].

In this section, we fix a vector $v \in \mathsf{H}_+$ and a sign pattern

$$m = (m_1, \cdots, m_n) \in \{\pm 1\}^n,$$

and count those polynomials whose roots lie near the prescribed points

$$m_i e^{Tv_i} \quad 1 \leq i \leq n,$$

up to an additive error order $O(\varepsilon e^{Tv_i})$ for a fixed $\varepsilon > 0$. The proof relies on translating the information about the roots into precise size constraints on the polynomial's coefficients.

Definition 2.1. For $\varepsilon > 0$ and $T > 1$, denote by

$$\mathfrak{Q}_T(v, m; \varepsilon) \quad (\text{resp. } \mathfrak{Q}^{\text{irr}}_T(v, m; \varepsilon))$$

the set of all monic integral (resp. irreducible[3]) polynomials with roots $x_1, \cdots, x_n$ such that

$$|x_i - m_i e^{Tv_i}| \leq \varepsilon e^{Tv_i} \qquad \text{for all } i = 1, \cdots, n. \tag{2.1}$$

Set

$$\delta_v := \min_{1 \leq i \leq n-1} (v_1 + \cdots + v_i) \tag{2.2}$$

[3]Throughout the paper, irreducible means irreducible over $\mathbb{Z}$

Theorem 2.2. *Let $\varepsilon > 0$. As $T \to \infty$,*

$$\#\mathcal{Q}_T(v, m; \varepsilon) \asymp_\varepsilon \exp \tfrac{1}{2} \sum_{i<j} (v_i - v_j)T.$$

More precisely, there exist absolute constants $c_1, c_2 > 0$ such that for all T large enough depending on n and ε,

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1} e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T} \left(1 - c_1 \tfrac{n}{\varepsilon} e^{-\delta_v T}\right) \le \#\mathcal{Q}_T(v, m; \varepsilon)$$
$$\le (2\varepsilon)^{n-1}\, n!\, e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T} \left(1 + c_2 \tfrac{n}{\varepsilon} e^{-\delta_v T}\right)$$

and

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1} e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T} \left(1 - c_1 \tfrac{2^n}{\varepsilon} e^{-\min(\delta_v, \eta_v)T}\right) \le \#\mathcal{Q}_T^{\mathrm{irr}}(v, m; \varepsilon)$$
$$\le (2\varepsilon)^{n-1}\, n!\, e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T} \left(1 + c_2 \tfrac{2^n}{\varepsilon} e^{-\min(\delta_v, \eta_v)T}\right)$$

where $\eta_v > 0$ is defined in (2.15).

This theorem follows from three lemmas 2.4, 2.5 and 2.8 below.

To motivate Definition 2.3, let us first examine the size of each coefficient of the following reference polynomial

$$q_T(x) = q_{Tv,m}(x) := \prod_{i=1}^{n} (x - m_i e^{Tv_i}). \tag{2.3}$$

Writing $q_T(x) = \sum_{k=0}^{n} (-1)^{n-k} b_{n-k} x^k = x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1)^n b_n$, Vieta's formulas give:

$$b_i = \sum_{\substack{S \subset \{1,\dots,n\} \\ |S| = i}} \Big(\prod_{j \in S} m_j\Big) e^{T \sum_{j \in S} v_j} \quad 1 \le i \le n.$$

Therefore, for any $0 < \varepsilon < 1$, there exists $T_1 = T_1(v, \varepsilon) > 0$ such that for all $T \ge T_1$ and all $1 \le i \le n$,

$$(1 - \varepsilon) e^{T(v_1 + \cdots + v_i)} \ \le\ b_i M_i \ \le\ (1 + \varepsilon) e^{T(v_1 + \cdots + v_i)}$$

where $M_i = \prod_{j=1}^{i} m_j$.

Definition 2.3. For $0 < \varepsilon < 1$, let $\mathscr{P}_T(v, m; \varepsilon)$ be the set of all monic integral polynomials

$$p(x) = \sum_{i=0}^{n} (-1)^{n-i} a_{n-i} x^i$$

such that for all $1 \le i \le n$,

$$(1 - \varepsilon) e^{T(v_1 + \cdots + v_i)} \le a_i M_i \le (1 + \varepsilon) e^{T(v_1 + \cdots + v_i)}. \tag{2.4}$$

We also define $\mathscr{P}'_T(v, m; \varepsilon)$ to be the set of all monic integral polynomials $p(x) = \sum_{i=0}^{n} (-1)^{n-i} a_{n-i} x^i$ such that for all $1 \le i \le n$,

$$(1 - (i+1)\varepsilon) e^{T(v_1 + \cdots + v_i)} \le a_i M_i \le (1 + (i+1)\varepsilon) e^{T(v_1 + \cdots + v_i)}. \tag{2.5}$$

Note that for all $0 < \varepsilon < 1/(n+1)$, any $p \in \mathscr{P}'_T(v, m; \varepsilon)$ satisfies $a_n = M_n$. Throughout the paper, we repeatedly use the following simple identity:

$$\sum_{i=1}^{n-1}(v_1 + \cdots + v_i) = \tfrac{1}{2} \sum_{1 \le i < j \le n} (v_i - v_j) = \sum_{i=1}^{n-1}(n-i)v_i \tag{2.6}$$

where $\sum_{i=1}^n v_i = 0$ is used.

We immediately get the following by counting integral vectors in the axis-parallel boxes given by (2.4) and (2.5) from the classical theorem of Davenport [8]. Recall the constant $\delta_v > 0$ from (2.2).

Lemma 2.4. *For any $0 < \varepsilon < \frac{1}{n+1}$ and $T > 1$ large enough, we have we have*

$$(2\varepsilon)^{n-1} e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T}\left(1 - \tfrac{n}{\varepsilon}e^{-\delta_v T}\right) \le \#\mathscr{P}_T(v, m; \varepsilon) \le (2\varepsilon)^{n-1} e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T}\left(1 + \tfrac{n}{\varepsilon}e^{-\delta_v T}\right)$$

and

$$(2\varepsilon)^{n-1} n! e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T}\left(1 - \tfrac{n}{\varepsilon}e^{-\delta_v T}\right) \le \#\mathscr{P}'_T(v, m; \varepsilon) \le (2\varepsilon)^{n-1} n! e^{\frac{1}{2}\sum_{i<j}(v_i - v_j)T}\left(1 + \tfrac{n}{\varepsilon}e^{-\delta_v T}\right)$$

Proof. Davenport's theorem [8] gives that for any axix-parallel box $B = \prod_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$,

$$|\#(B \cap \mathbb{Z}^d) - \mathrm{Vol}(B)| \le \sum_{i=1}^d \prod_{j \ne i}(b_j - a_j) = \sum_{i=1}^d \frac{\mathrm{vol}(B)}{b_i - a_i}. \tag{2.7}$$

Setting $E_i = e^{(v_1 + \cdots + v_i)T}$ and $B = \prod_{i=1}^{n-1}[(1-\varepsilon)E_i, (1+\varepsilon)E_i]$, the number $\#\mathscr{P}_T(v, m; \varepsilon)$ is same as $\#(\mathbb{Z}^{n-1} \cap B)$ and hence the first claim follows by (2.6) and (2.7). The second claim follows similarly. □

Lemma 2.5 (Root approximation)**.** *Let $c_n = (n-1)3^n$. For any $0 < \varepsilon < 1/(4c_n)$, there exists $T_0 = T_0(v, \varepsilon) \ge 1$ such that for all $T \ge T_0$, every polynomial $p \in \mathscr{P}_T(v, m; \varepsilon)$ has n-distinct real roots $x_1, \cdots, x_n$ with*

$$|x_i - m_i e^{T v_i}| \le c_n \varepsilon e^{T v_i} \qquad \text{for all } i = 1, \cdots, n. \tag{2.8}$$

Conversely, any monic polynomial $p \in \mathbb{Z}[x]$ with roots $x_1, \cdots, x_n$ satisfying (2.8) belongs to $\mathscr{P}'_T(v, m; \varepsilon)$.

In other words, for all T sufficiently large,

$$\mathscr{P}_T(v, m; \frac{\varepsilon}{c_n}) \subset \mathscr{Q}_T(v, m; \varepsilon) \subset \mathscr{P}'_T(v, m; \varepsilon).$$

Proof. The second statement is a simple consequence of Vieta's formulas. Let $w_i = \sum_{j=1}^i v_j$ for each $1 \le i \le n-1$. Let q_T be as in (2.3) so that

$q_T(x) = \sum_{i=0}^{n}(-1)^{n-i}b_{n-i}x^i$. So for all $T \geq T_1(v,\varepsilon)$ and for all $1 \leq i \leq n-1$,

$$e^{Tw_i-\varepsilon} \leq b_i M_i \leq e^{Tw_i+\varepsilon}.$$

By increasing T_1 if necessary, we may assume that $e^{Tv_i} \geq 3e^{Tv_{i+1}}$ and $e^{Tw_i} \geq 3e^{Tw_{i+1}}$ for all i. Consequently $|e^{Tv_i} - e^{Tv_{i+1}}| \geq \left(\frac{2}{3}\right)e^{Tv_i}$.

Fix $0 < \varepsilon < (4c_n)^{-1}$. Then we have

$$3(n-1)(1+c_n\varepsilon)^{n-1} < c_n(2/3 - c_n\varepsilon)^{n-1}. \tag{2.9}$$

To check this, we note that $f(\varepsilon) = \frac{3(n-1)(1+c_n\varepsilon)^{n-1}}{c_n(2/3-c_n\varepsilon)^{n-1}}$ is a strictly increasing function on the interval $(0,(4c_n)^{-1})$ and $f((4c_n)^{-1}) = 1$.

For each $1 \leq j \leq n$, consider the discs

$$D_j = \{x \in \mathbb{C} : |x - m_j e^{Tv_j}| \leq c_n\varepsilon e^{Tv_j}\}.$$

Since $c_n\varepsilon \leq 1/4$, we have $(1+c_n\varepsilon)e^{Tv_{j+1}} < (1-c_n\varepsilon)e^{Tv_j}$ for all j and hence these discs are pairwise disjoint.

Let $p_T(x) = \sum_{i=0}^{n}(-1)^{n-i}a_{n-i}x^i$ be a polynomial in $\mathscr{P}_T(v,m;\varepsilon)$. We claim that for all T sufficiently large, $p_T(x)$ has precisely one root inside each disc D_j. Write

$$\Delta_T(x) \coloneqq q_T(x) - p_T(x) = \sum_{i=1}^{n-1}(-1)^{n-i}(b_{n-i} - a_{n-i})x^i.$$

From (2.4), $|b_{n-i} - a_{n-i}| \leq 3\varepsilon\, e^{Tw_{n-i}}$. Hence for all $x \in \partial D_j$, we have

$$|\Delta_T(x)| \leq 3\varepsilon\big(1 + c_n\varepsilon\big)^{n-1}\sum_{i=1}^{n-1} e^{Tw_{n-i}+iTv_j}. \tag{2.10}$$

On the other hand,

$$|q_T(x)| = \prod_{i=0}^{n}|x - m_i e^{Tv_i}| = c_n\varepsilon e^{Tv_j}\prod_{i<j}|x - m_i e^{Tv_i}| \cdot \prod_{i>j}|x - m_i e^{Tv_i}|.$$

For $i < j$,

$$|x - m_i e^{Tv_i}| \geq |m_i e^{Tv_i} - m_j e^{Tv_j}| - |x - m_j e^{Tv_j}| \geq \frac{2}{3}e^{Tv_i} - c_n\varepsilon e^{Tv_j} \geq (\frac{2}{3} - c_n\varepsilon)e^{Tv_i}.$$

For $i > j$, we similarly have

$$|x - m_i e^{Tv_i}| \geq (\frac{2}{3} - c_n\varepsilon)e^{Tv_j}.$$

Hence

$$|q_T(x)| \geq c_n\varepsilon\left(\frac{2}{3} - c_n\varepsilon\right)^{n-1} e^{T(\sum_{i<j} v_i)+(n-j+1)Tv_j}. \tag{2.11}$$

Let

$$S(T,j) = \sum_{i=1}^{n-1} e^{Tw_{n-i}+iTv_j} \quad\text{and}\quad R(T,j) = e^{T(\sum_{k<j} v_k)+(n-j+1)Tv_j}.$$

Let

$$\Delta_{i,j} = \left(\sum_{k<j} v_k + (n-j+1)v_j \right) - (v_1 + \cdots + v_{n-i} + iv_j)$$

so that

$$\frac{S(T,j)}{R(T,j)} = \sum_{i=1}^{n-1} e^{-T\Delta_{i,j}}. \tag{2.12}$$

We check that $\Delta_{i,j} \geq 0$ by writing

$$\Delta_{i,j} = \begin{cases} (n-i-(j-1))v_j - (v_j + \cdots + v_{n-i}) \geq 0 & \text{if } n-i \geq j-1, \\ (v_{n-i+1} + \cdots + v_{j-1}) - ((j-1)-(n-i))v_j \geq 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Hence by (2.12), we have

$$\frac{S(T,j)}{R(T,j)} \leq n-1.$$

Therefore by (2.10) and (2.11), and since ε satisfies (2.9), we get that for all $x \in \partial D_j$,

$$\begin{aligned} |\Delta_T(x)| \leq 3\varepsilon(1+c_n\varepsilon)^{n-1}S(T,j) &\leq (n-1)3\varepsilon(1+c_n\varepsilon)^{n-1}R(T,j) \\ &< c_n\varepsilon(2/3 - c_n\varepsilon)^{n-1}R(T,j) \leq |q_T(x)|, \end{aligned}$$

and hence

$$|\Delta_T(x)| < |q_T(x)|.$$

Hence by Rouché's theorem (cf. [2]), two polynomials $q_T(x)$ and $p_T(x)$ have the same number of zeros (counted with multiplicity) inside each D_j. Since D_j are pairwise disjoint and q_T has exactly one root in each D_j, the same holds for p_T. Since p_T has real coefficients and each D_j is invariant under complex conjugation, $p_T(x)$ has one *real* root x_j such that

$$|x_j - m_j e^{Tv_j}| \leq c_n \varepsilon e^{Tv_j}.$$

Hence $p_T \in \mathcal{Q}_T(v, m; c_n\varepsilon)$. This finishes the proof. $\square$

Denote by $\mathrm{Disc}(p) = \prod_{i \neq j}(x_i - x_j) = \prod_{i<j}(x_i - x_j)^2$ the discriminant of a polynomial p with roots $x_1, \cdots, x_n$. For the polynomial $q_{Tv,m}(x) = \prod_{i=1}^{n}(x - m_i e^{Tv_i})$, its discriminant $\mathrm{Disc}(q_{Tv,m})$ satisfies

$$\mathrm{Disc}(q_{Tv,m}) = e^{\left(\sum_{1 \leq i<j \leq n} v_i\right)T} \left(1 + O(e^{-\eta T})\right) \qquad \text{for some } \eta > 0$$

and hence

$$\lim_{T\to\infty} \tfrac{1}{T} \log \mathrm{Disc}(q_{Tv,m}) = \sum_{i<j}(v_i - v_j).$$

The following is a simple consequence of Lemma 2.5:

Corollary 2.6. *For all small $\varepsilon > 0$, there exist $T_0 = T_0(v, \varepsilon) > 0$ such that for all $T \geq T_0$, every polynomial $p_T \in \mathscr{P}_T(v, m; \varepsilon)$ satisfies*

$$(1 - C_n\varepsilon)e^{\sum_{i<j}(v_i - v_j)T} \leq \operatorname{Disc}(p_T) \leq (1 + C_n\varepsilon)e^{\sum_{i<j}(v_i - v_j)T}$$

where $C_n = n(n-1)/2$. In particular,

$$\lim_{T\to\infty} \tfrac{1}{T}\log \operatorname{Disc}(p_T) = \sum_{i<j}(v_i - v_j).$$

We will need the following estimates in the next lemma 2.8:

Lemma 2.7. *Let $\{1, \ldots, n\} = S_1 \sqcup S_2$ be a partition into two non-empty subsets so that $\sum_{i\in S_j} v_i = 0$ for $j = 1, 2$. Writing $S_1 = \{i_1 < \cdots < i_{\ell_1}\}$ and $S_2 = \{j_1 < \cdots < j_{\ell_2}\}$ with $\ell_1 + \ell_2 = n$, we have*

$$\sum_{k=1}^{\ell_1 - 1}(v_{i_1} + \cdots + v_{i_k}) \;+\; \sum_{k=1}^{\ell_2 - 1}(v_{j_1} + \cdots + v_{j_k}) \;<\; \sum_{k=1}^{n-1}(v_1 + \cdots + v_k)\,.$$

Proof. We first rewrite the right hand side (RHS) as

$$\sum_{k=1}^{n-1}(v_1 + \cdots + v_k) = \sum_{k=1}^{n-1}\sum_{i=1}^{k} v_i = \sum_{i=1}^{n-1}(n-i)\,v_i = \sum_{1\leq i<j\leq n} v_i.$$

Similarly, the left hand side, for each $j = 1, 2$,

$$\sum_{k=1}^{\ell_j - 1}(v_{i_1} + \cdots + v_{i_k}) = \sum_{i<j, i,j\in S_j} v_i,$$

and hence

$$\text{LHS} = \sum_{i<j, i,j \text{ in the same } S_*} v_i.$$

Set

$$D_v(S_1, S_2) := \sum_{k=1}^{n-1}(v_1 + \cdots + v_k) - \left(\sum_{k=1}^{\ell_1 - 1}(v_{i_1} + \cdots + v_{i_k}) \;+\; \sum_{k=1}^{\ell_2 - 1}(v_{j_1} + \cdots + v_{j_k})\right). \tag{2.14}$$

Hence

$$D_v(S_1, S_2) = (\text{RHS}) - (\text{LHS}) = \sum_{i<j, S(i)\neq S(j)} v_i,$$

where $S(i)$ denotes the block containing i. Add the same pairs with the complementary index:

$$\sum_{i<j, S(i)\neq S(j)} (v_i + v_j) = |S_2|\sum_{i\in S_1} v_i \;+\; |S_1|\sum_{j\in S_2} v_j = 0,$$

because each block has total sum 0. Hence $D_v(S_1, S_2) = -\sum_{i<j, S(i)\neq S(j)} v_j$; so

$$2D_v(S_2, S_2) = \sum_{i<j, S(i)\neq S(j)} (v_i - v_j).$$

Since $v_1 > \cdots > v_n$, every difference $v_i - v_j$ $(i < j)$ is strictly positive. There is at least one cross pair (the blocks are non-empty), so $D_v(S_1, S_2) > 0$. □

Define

$$\eta_v := \min D_v(S_1, S_2) > 0 \tag{2.15}$$

where $D_v(S_1, S_2)$ is as in (2.14) and the minimum is taken over all non-trivial partitions of $\{1, \cdots, n\} = S_1 \sqcup S_2$. The function $v \mapsto \eta_v$ is clearly continuous on H_+ and hence $\min_{v\in Q} \eta_v > 0$ for any compact subset $Q \subset \mathsf{H}_+$.

Lemma 2.8. *Let $0 < \varepsilon < 1$. As $T \to \infty$, the proportion of irreducible polynomials in $\mathscr{P}_T(v, m; \varepsilon)$ tends to 1 exponentially fast: there exists $T_0 = T_0(n, \varepsilon)$ such that for all $T \geq T_0$,*

$$\#\{p \in \mathscr{P}_T(v, m; \varepsilon) \text{ irreducible}\} = \#\mathscr{P}_T(v, m; \varepsilon) \cdot \left(1 + O(2^n \varepsilon^{-1} e^{-\eta_v T})\right)$$

where the implied constant is an absolute constant. The same type of estimate holds for $\mathscr{P}'_T(v, m; \varepsilon)$.

Proof. Let $T \geq T_0(v, \varepsilon)$ be as in Lemma 2.5. For any $p \in \mathscr{P}_T(v, m; \varepsilon)$, by Lemma 2.5, p has distinct roots $x_1, \cdots, x_n$ such that $|x_i - m_i e^{Tv_i}| \leq c_n \varepsilon e^{Tv_i}$ for each $1 \leq i \leq n$. Suppose that $p \in \mathscr{P}_T(v, m; \varepsilon)$ is reducible over $\mathbb{Z}$. We then have a partition of $\{1, \cdots, n\}$ as the disjoint union $S_1 \sqcup S_2$ of non-empty subsets such that $p(x) = f_1(x) f_2(x)$ where $f_j(x) = \prod_{k\in S_j}(x - x_k) \in \mathbb{Z}[x]$ for $j = 1, 2$. List elements of S_j as $j_1 > j_2 > \cdots > j_{\ell_j}$. Let $u_j = (v_{j_1}, \cdots, v_{j_{\ell_j}})$ and $M'_j = (m_{j_1}, \cdots, m_{j_{\ell_j}})$. It follows that

$$f_j \in \mathscr{P}_T(u_j, M'_j; \varepsilon).$$

Hence for all sufficiently large T, by Lemma 2.4, for all sufficiently large $T \geq 1$, we have

$$\#\mathscr{P}_T(u_j, M'_j; \varepsilon) \leq 2(2\varepsilon)^{\ell_j - 1} e^{T\sum_{k=1}^{\ell_j - 1}(v_{j_1} + \cdots + v_{j_k})}.$$

Since the constant terms of f_1 and f_2 are ± 1, we have $\sum_{i\in S_j} v_i = 0$. By Lemma 2.7, we have

$$\sum_{k=1}^{\ell_1 - 1}(v_{1_1} + \cdots + v_{1_k}) + \sum_{k=1}^{\ell_2 - 1}(v_{2_1} + \cdots + v_{2_k}) < \sum_{k=1}^{n-1}(v_1 + \cdots + v_k),$$

that is, $D_v(S_1, S_2) > 0$. Therefore we have

$$\frac{\#\mathscr{P}_T(u_1, M'_1; \varepsilon) \cdot \#\mathscr{P}_T(u_2, M'_2; \varepsilon)}{\#\mathscr{P}_T(v, m; \varepsilon)} \leq 8\varepsilon^{-1} e^{-D_v(S_1, S_2)T}.$$

Since this holds for any non-trivial partition of $\{1, \cdots, n\}$ into two non-empty subsets and there are at most 2^{n-1} number of such partitions, this proves the first claim by the definition of η_v. The proof for $\mathscr{P}'_T(v,m;\varepsilon)$ is similar. $\square$

Proof of Theorem 2.2. The first claim follows from Lemma 2.4 and Lemma 2.5. By Lemma 2.5 and Lemma 2.8, for all T sufficiently large, the cardinality of the set of *reducible* polynomials in $\mathscr{Q}_T(v,m;\varepsilon)$ is at most $c2^n\varepsilon^{-1}e^{-\eta_v T}\cdot \#\mathscr{Q}_T(v,m;\varepsilon)$ for some absolute constant $c>0$. Hence the second claim follows from this and the first claim. $\square$

3. Directional entropy of totally real algebraic units

We now apply our polynomial analysis to compute the directional entropy for totally real units of degree n. Fix $v\in \mathsf{H}_+$ and a sign pattern $m=(m_1,\cdots,m_n)\in\{\pm1\}^n$. We use the notation $\mathcal{K}_n, O_n^\times, \Sigma_K$, etc. from the introduction. For simplicity, for $\mathsf{u}\in (O_K^\times,\sigma)$, we write $\|\sigma(\mathsf{u})-me^{Tv}\|\le\varepsilon e^{Tv}$ to mean that $|\sigma_i(\mathsf{u})-m_ie^{Tv_i}|<\varepsilon e^{Tv_i}$ $\forall i$. For each $T>1$, define

$$\mathscr{U}_T(v,m;\varepsilon)=\bigcup_{(K,\sigma)\in\mathcal{K}_n}\left\{\mathsf{u}\in(O_K^\times,\sigma):\|\sigma(\mathsf{u})-me^{Tv}\|\le\varepsilon e^{Tv}\right\}.$$

Let $\mathscr{U}_T^{\text{prim}}(v,m;\varepsilon)$ be the set of all $\mathsf{u}\in(O_K^\times,\sigma)$, $(K,\sigma)\in\mathcal{K}_n$, such that the field $\mathbb{Q}(\mathsf{u})$ has degree n, or equivalently, $p(x)=\prod_{i=1}^n(x-\sigma_i(\mathsf{u}))$ is irreducible over $\mathbb{Z}$ for $\sigma=(\sigma_1,\cdots,\sigma_n)$.

Lemma 3.1. *Let $0\le\varepsilon<1/2$. Then for all sufficiently large $T>1$, we have*

$$\mathscr{U}_T(v,m;\varepsilon)=\mathscr{U}_T^{\text{prim}}(v,m;\varepsilon).$$

Proof. If $\mathsf{u}\in\mathscr{U}_T(v,m;\varepsilon)\cap(O_K^\times,\sigma)$ is non-primitive, then there is a subfield K_0 of K of degree $1<m<n$ such that $\mathsf{u}\in O_{K_0}^\times$ and each of the m-embeddings $K_0\hookrightarrow\mathbb{R}$ extends to precisely n/m-embeddings to K into $\mathbb{R}$. Therefore for some $i<j$, $\sigma_i(\mathsf{u})=\sigma_j(\mathsf{u})$. This implies that $|m_ie^{Tv_i}-m_je^{Tv_j}|\le\varepsilon e^{Tv_i}$. Since $v_i>v_j$ and hence $|m_ie^{Tv_i}-m_je^{Tv_j}|=e^{Tv_i}(1\pm e^{T(v_j-v_i)})\ge e^{Tv_i}/2$ for all sufficiently large T, u cannot be non-primitive if T is large enough. $\square$

Proposition 3.2. *Let $v\in\mathsf{H}_+$. For all small $\varepsilon>0$, we have*

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1}\le\liminf_{T\to\infty}\frac{\#\mathscr{U}_T(v,m;\varepsilon)}{e^{\frac12\sum_{i<j}(v_i-v_j)T}}\le\limsup_{T\to\infty}\frac{\#\mathscr{U}_T(v,m;\varepsilon)}{e^{\frac12\sum_{i<j}(v_i-v_j)T}}\le(2\varepsilon)^{n-1}\,n!\,.$$

More precisely, there exist absolute constants $c_1,c_2>0$ such that for all large $T>1$ depending on n and ε, we have

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1}e^{\frac12\sum_{i<j}(v_i-v_j)T}\left(1-c_12^n\varepsilon^{-1}e^{-\min(\delta_v,\eta_v)T}\right)\le\#\mathscr{U}_T(v,m;\varepsilon)$$
$$\le(2\varepsilon)^{n-1}\,n!\,e^{\frac12\sum_{i<j}(v_i-v_j)T}\left(1+c_22^n\varepsilon^{-1}e^{-\min(\delta_v,\eta_v)T}\right)$$

where $v\mapsto\delta_v$ and $v\mapsto\eta_v$ are positive continuous functions given in (2.2) and (2.15) respectively.

Proof. Let $\varepsilon > 0$ and $T \geq T_0(v, \varepsilon)$ as in Lemma 2.5. Let $p \in \mathcal{Q}_T^{\mathrm{irr}}(v, m; \varepsilon)$, and let K be its splitting field, which must be a totally real number field of degree n. Let $x_1, \cdots, x_n$ be the roots of p ordered so that $|x_1| > \cdots > |x_n|$. Since $x_i \in O_K$ and $\prod_{i=1}^n x_i = \prod_{i=1}^n m_i = \pm 1$, there exists a unit $\mathsf{u} \in O_K^\times$ and $\sigma = (\sigma_1, \cdots, \sigma_n) \in \Sigma_K$ such that $K = \mathbb{Q}(u)$ and $x_i = \sigma_i(\mathsf{u})$. Hence

$$\#\mathcal{Q}_T^{\mathrm{irr}}(v, m; \varepsilon) \leq \#\mathcal{U}_T^{\mathrm{prim}}(v, m; \varepsilon).$$

Conversely, let $\mathsf{u} \in \mathcal{U}_T^{\mathrm{prim}}(v, m; \varepsilon) \cap (O_K^\times, \sigma)$. Setting $p(x) = \prod(x - \sigma_i(\mathsf{u}))$, we have $p \in \mathcal{Q}_T^{\mathrm{irr}}(v, m; O(\varepsilon))$. Moreover, this map is injective for T large enough. To see this, suppose that there exist $\mathsf{u} \in \mathcal{U}_T(v, m; \varepsilon)^{\mathrm{prim}} \cap (O_K^\times, \sigma)$ and $\mathsf{u}' \in \mathcal{U}_T(v, m; \varepsilon)^{\mathrm{prim}} \cap O_{K'}^\times, \sigma')$ such that $p(x) = q(x)$ where $p(x) = \prod(x - \sigma_i(\mathsf{u}))$ and $q(x) = \prod(x - \sigma_i'(\mathsf{u}'))$. Since K and K' must be the splitting fields of p and q respectively, $K = K'$ and $\{\sigma_i'(\mathsf{u}') : i = 1, \cdots, n\} = \{\sigma_i(\mathsf{u}) : i = 1, \cdots, n\}$. Since the intervals $(Tv_i - \varepsilon, Tv_i + \varepsilon)$ are pairwise disjoint once T is sufficiently big, and $\log \sigma_i(\mathsf{u}), \log \sigma_i'(\mathsf{u}') \in (Tv_i - \varepsilon, Tv_i + \varepsilon)$ for all T sufficiently large, we must have $\sigma_i(\mathsf{u}) = \sigma_i'(\mathsf{u}')$ for all $1 \leq i \leq n$. Hence for all sufficiently large $T \gg 1$,

$$\#\mathcal{U}_T^{\mathrm{prim}}(v, m; \varepsilon) \leq \#\mathcal{Q}_T^{\mathrm{irr}}(v, m; \varepsilon).$$

Therefore the claim follows from Lemma 3.1 and Theorem 2.2. □

Observe that once T is sufficiently large depending only on v and ε, the sets $\mathcal{U}_T(v, m; \varepsilon)$, $m \in \{\pm 1\}^n$, are pairwise disjoint. Since

$$\{u \in O_n^\times : \|\Lambda(u) - Tv\| \leq \varepsilon\} = \bigsqcup_{m \in \{\pm 1\}^n} \mathcal{U}_T(v, m; \varepsilon),$$

Theorem 1.5 follows from Proposition 3.2.

Define

$$\mathsf{E}_n(v, m) = \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \log \#\mathcal{U}_T(v, m; \varepsilon),$$

if the limit exists. As an immediate consequence of Proposition 3.2, we have:

Theorem 3.3. *We have*

$$\mathsf{E}_n(v, m) = \tfrac{1}{2} \sum_{i<j} (v_i - v_j).$$

4. Eigenvalue entropy of $\mathrm{SL}_n(\mathbb{Z})$

In this section, we count the eigenvalue patterns of $\mathrm{SL}_n(\mathbb{Z})$ that lie in a thin tube around a fixed ray, invoking Theorem 2.2.

Fix $v \in \mathsf{H}_+$ and a sign pattern $m = (m_1, \cdots, m_n) \in \{\pm 1\}^n$ with $\prod_{i=1}^n m_i = 1$. Let

$$\rho_{\mathrm{SL}_n}(v) = \frac{1}{2} \sum_{i<j} (v_i - v_j);$$

be the half-sum of all positive roots of $\mathrm{SL}_n(\mathbb{R})$. For each loxodromic element $g \in \mathrm{SL}_n(\mathbb{R})$, let $\mathcal{E}(g)$, $\lambda(g)$ and $m(g)$ be its eigenvalue pattern, Jordan projection and sign pattern as defined in (1.1) and (1.2). Set

$$J_T(v,m;\varepsilon) := \#\Big\{(\lambda(\gamma), m(\gamma)) : \gamma \in \mathrm{SL}_n(\mathbb{Z}), \|\lambda(\gamma) - Tv\|_{\max} \le \varepsilon, m(\gamma) = m\Big\}.$$

Theorem 4.1. *For all small $\varepsilon > 0$, we have*

$$\left(\tfrac{2\varepsilon}{(n-1)3^n}\right)^{n-1} \le \liminf_{T\to\infty} \frac{\#J_T(v,m;\varepsilon)}{e^{\rho_{\mathrm{SL}_n}(v)T}} \le \limsup_{T\to\infty} \frac{\#J_T(v,m;\varepsilon)}{e^{\rho_{\mathrm{SL}_n}(v)T}} \le (2\varepsilon)^{n-1} n!.$$

In particular,

$$\mathsf{E}_{\mathrm{SL}_n(\mathbb{Z})}(v,m) = \rho_{\mathrm{SL}_n}(v). \tag{4.1}$$

Proof. There exists $T_1 = T_1(v,\varepsilon) > 0$ such that for all $T \ge T_1$ and for each $(\lambda(\gamma), m(\gamma)) \in J_T(v,m;\varepsilon)$, the polynomial $p(x) = \prod(x - m_i(\gamma)e^{\lambda_i(\gamma)})$ belongs to $\mathcal{Q}_T(v,m;\varepsilon)$. Since this gives an injective map, we have $\#J_T(v,m;\varepsilon) \le \#\mathcal{Q}_T(v,m;\varepsilon)$.

Let $f \in \mathcal{Q}_T(v,m;\varepsilon) = \sum_{i=0}^n (-1)^{n-i} a_{n-i} x^i$. Consider the companion matrix of f:

$$C_f = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{n+1}a_n \\ 1 & 0 & \cdots & 0 & (-1)^n a_{n-1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -a_2 \\ 0 & \cdots & 0 & 1 & a_1 \end{pmatrix}$$

Since $\det C_f = a_n = \prod_{i=1}^n m_i = 1$, we have $C_f \in \mathrm{SL}_n(\mathbb{Z})$. If $x_1, \cdots, x_n$ are distinct real roots of f ordered so that $|x_1| > \cdots > |x_n|$, then

$$|x_i - m_i e^{Tv_i}| \le \varepsilon e^{Tv_i}, \qquad 1 \le i \le n.$$

Therefore $\|\lambda(C_f) - Tv\| \le \varepsilon$ and $m(C_f) = m$. Hence the assignment $f \mapsto (\lambda(C_f), m(C_f))$ gives a map from the set $\mathcal{Q}_T(v,m;\varepsilon)$ to $J_T(v,m;\varepsilon)$. Since $(\lambda(C_f), m(C_f))$ describes all roots of f, this map is also injective. Therefore $\#J_T(v,m;\varepsilon) \ge \#\mathcal{Q}_T(v,m;\varepsilon)$. Hence the claim follows from Theorem 2.2. □

The lower bound stated below follows directly from Theorem 4.1 and the corresponding upper bound will be proved in Theorem 6.2 in a more general setting.

Theorem 4.2. *For each $v \in \mathsf{H}_+$, each sign pattern $m \in \{\pm 1\}$ with $\prod_{i=1}^n m_i = 1$, and $\varepsilon > 0$, there exist $C_1, C_2 > 0$ such that*

$$C_1 e^{\rho_{\mathrm{SL}_n}(v)T} \le \#\Big\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})] : \|\lambda(\gamma) - Tv\| \le \varepsilon, m(\gamma) = m\Big\} \le C_2 e^{2\rho_{\mathrm{SL}_n}(v)T}.$$

In particular,

$$\rho_{\mathrm{SL}_n}(v) \le \underline{\mathsf{E}}^{\star}_{\mathrm{SL}_n(\mathbb{Z})}(v,m) \ \le\ \overline{\mathsf{E}}^{\star}_{\mathrm{SL}_n(\mathbb{Z})}(v,m) \le 2\rho_{\mathrm{SL}_n}(v). \tag{4.2}$$

Corollary 4.3. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Let $v = v_{\|\cdot\|} \in \mathsf{H}_+$ be a unit vector such that $\max_{\|v\|=1} \rho_{\mathrm{SL}_n}(v) = \rho_{\mathrm{SL}_n}(v_{\|\cdot\|})$. There exists $C > 0$ such that that for all $T > 1$,*

$$\#\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})] : \|\lambda(\gamma)\| < T, \gamma \in \mathrm{SL}_n(\mathbb{Z})\} \geq Ce^{\rho_{\mathrm{SL}_n}(v_{\|\cdot\|})T}.$$

(1) For the Euclidean norm $\|\cdot\|_{\mathrm{Euc}}$, $\rho_{\mathrm{SL}_n}(v_{\|\cdot\|_{\mathrm{Euc}}}) = \sqrt{\frac{n(n^2-1)}{12}}$.
(2) For the maximum norm $\|\cdot\|_{\max}$, $\rho_{\mathrm{SL}_n}(v_{\|\cdot\|_{\max}})$ is $\lfloor n^2/4 \rfloor$.

Proof. Let $N(T) := \#\{\lambda(\gamma) : \gamma \in \mathrm{SL}_n(\mathbb{Z}), \|\lambda(\gamma)\| < T\}$. Since

$$N(T) \geq \#J_{m,T-\varepsilon}(v_{\|\cdot\|}, \varepsilon)$$

for any sign pattern m with $\prod m_j = 1$, the desired lower bound for $N(T)$ follows from Theorem 4.1.

Since $2\rho_{\mathrm{SL}_n}(v) = \sum_{k=1}^n (n+1-2k)v_k$, its maximum on the unit sphere (for the Euclidean norm) is attained in the direction of $(n+1-2k)_{k=1}^n$. Hence if we write $v_{\|\cdot\|_{\mathrm{Euc}}} = (v_1^*, \cdots, v_n^*)$, then

$$v_k^* = \frac{n+1-2k}{\sqrt{n(n^2-1)/3}}, \quad \text{and} \quad \rho_{\mathrm{SL}_n}(v_{\|\cdot\|_{\mathrm{Euc}}}) = \sqrt{\frac{n(n^2-1)}{12}}. \tag{4.3}$$

For the maximum norm, $v_{\|\cdot\|_{\max}}$ is given by $v_k = 1$ whenever $n+1-2k > 0$ and $v_k = -1$ whenever $n+1-2k < 0$ and $v_k = 0$ if $2k = n+1$. Then for $n = 2m$, $2\rho_{\mathrm{SL}_n}(\|v\|_{\max}) = \sum_{k=1}^m (2m+1-2k) + \sum_{k=m+1}^{2m} (2m+1-2k)(-1) = 2m^2 = n^2/2$, and for $n = 2m+1$, $2\rho_{\mathrm{SL}_n}(\|v\|_{\max}) = m(m+1) = (n^2-1)/2$. □

5. Reciprocal polynomials and Counting for $\mathrm{Sp}_{2n}(\mathbb{Z})$

In this section, we investigate directional entropies for the symplectic lattice $\mathrm{Sp}_{2n}(\mathbb{Z})$. Our estimates rely on the anaylsis of reciprocal polynomials.

Reciprocal polynomials. A monic polynomial $p \in \mathbb{R}[x]$ of degree $2n$ is called *reciprocal* (also called *palindromic*) if

$$p(x) = x^{2n} p(x^{-1}).$$

Equivalently,

$$p(x) = \sum_{k=0}^{2n} (-1)^{2n-k} a_{2n-k}\, x^k, \qquad a_0 = a_{2n} = 1, \quad a_i = a_{2n-i} \ \ (1 \leq i \leq n);$$

or

$$p(x) = \prod_{i=1}^n \big(x - x_i\big)\big(x - x_i^{-1}\big), \qquad x_1, \ldots, x_n \in \mathbb{C} - \{0\}.$$

Let

$$\mathfrak{a}^+ = \{v = \mathrm{diag}(v_1, \cdots, v_n, -v_n, \cdots, -v_1) : v_1 \geq \cdots \geq v_n \geq 0\}.$$

Fix

$$v \in \mathrm{int}\, \mathfrak{a}^+ \quad \text{and} \quad m = (m_1, \cdots, m_n) \in \{\pm 1\}^n.$$

Definition 5.1. Let $\varepsilon > 0$ and $T > 1$. Let $\mathscr{Q}_T^*(v, m; \varepsilon)$ (resp. $\mathscr{Q}_T^{*,\mathrm{irr}}(v, m; \varepsilon)$) be the set of all monic integral (resp. irreducible) reciprocal polynomials with roots $x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}$ such that for all $i = 1, \cdots, n$,

$$|x_i - m_i e^{Tv_i}| \le \varepsilon e^{Tv_i}, \;\; |x_i^{-1} - m_i e^{-Tv_i}| \le \varepsilon e^{-Tv_i}.$$

Set

$$\rho^*(v) = \sum_{i=1}^n (v_1 + \cdots + v_i) = \sum_{i=1}^n (n + 1 - i) v_i.$$

The following theorem is a direct combination of Lemma 5.3, Corollary 5.4 and Lemma 5.5:

Theorem 5.2. *Let $\varepsilon > 0$. As $T \to \infty$,*

$$\#\mathscr{Q}_T^*(v, m; \varepsilon) \asymp_\varepsilon e^{\rho^*(v)T};$$

more precisely,

$$\left(\tfrac{2\varepsilon}{(2n-1)3^{2n}}\right)^n \le \liminf_{T\to\infty} \frac{\#\mathscr{Q}_T^*(v, m; \varepsilon)}{e^{\rho^*(v)T}} \le \limsup_{T\to\infty} \frac{\#\mathscr{Q}_T^*(v, m; \varepsilon)}{e^{\rho^*(v)T}} \le (2\varepsilon)^n (n+1)!.$$

Moreover,

$$\#\mathscr{Q}_T^{*,\mathrm{irr}}(v, m; \varepsilon) = \#\mathscr{Q}_T^*(v, m; \varepsilon)(1 + O(e^{-\eta T}))$$

for some $\eta > 0$ depending only on v.

For $T > 1$, define the model polynomial

$$q_{Tv}(x) = \prod_{i=1}^n (x - m_i e^{Tv_i})(x - m_i e^{-Tv_i}) = \sum_{k=0}^{2n} (-1)^{2n-k} b_{2n-k} x^k.$$

Then for any $\varepsilon > 0$, and sufficiently large $T \gg 1$, we have that $b_0 = 1 = b_{2n}$, $b_i = b_{2n-i}$ and

$$(1 - \varepsilon) e^{T(v_1 + \cdots + v_i)} \le b_i M_i \le (1 + \varepsilon) e^{T(v_1 + \cdots + v_i)} \qquad \text{for all } 1 \le i \le n$$

where $M_i = \prod_{j=1}^i m_i$.

Let $\mathscr{P}_T^*(v, m; \varepsilon)$ be the set of all monic reciprocal polynomials

$$p(x) = \sum_{k=0}^{2n} (-1)^{2n-k} a_{2n-k} x^k \in \mathbb{Z}[x]$$

such that

$$(1 - \varepsilon) e^{T(v_1 + \cdots + v_i)} \le a_i M_i \le (1 + \varepsilon) e^{T(v_1 + \cdots + v_i)} \qquad \text{for all } 1 \le i \le n.$$

Let $\mathscr{P}_T^{**}(v, m; \varepsilon)$ be defined by the condition that

$$(1-(i+1)\varepsilon) e^{T(v_1 + \cdots + v_i)} \le a_i M_i \le (1+(i+1)\varepsilon) e^{T(v_1 + \cdots + v_i)} \qquad \text{for all } 1 \le i \le n.$$

Clearly we have:

Lemma 5.3. *For all sufficiently small $\varepsilon > 0$, we have, as $T \to \infty$,*

$$\#\mathscr{P}_T^*(v, m; \varepsilon) \sim (2\varepsilon)^n e^{\rho^*(v)T} \quad \textit{and} \quad \#\mathscr{P}_T^*(v, m; \varepsilon) \sim (2\varepsilon)^n (n+1)! e^{\rho^*(v)T}.$$

The following follows from Lemma 2.5:

Corollary 5.4 (Root approximation)**.** *For all sufficiently small $\varepsilon > 0$, there exists $T_0 = T_0(v, \varepsilon) > 1$ such that for all $T \geq T_0$, we have*

$$\mathscr{P}_T^*(v, m; \frac{\varepsilon}{c_{2n}}) \subset \mathscr{Q}_T^*(v, m; \varepsilon) \subset \mathscr{P}_T^{**}(v, m; \varepsilon)$$

where $c_{2n} = (2n-1)3^{2n}$.

Lemma 5.5. *For all sufficiently small $\varepsilon > 0$, there is $\eta > 0$ depending only on v such that for all T large enough, we have*

$$\frac{\#\{p \in \mathscr{P}_T^*(v, m; \varepsilon) \text{ irreducible}\}}{\#\mathscr{P}_T^*(v, m; \varepsilon)} = 1 + O(e^{-\eta T}).$$

Proof. Suppose that $p \in \mathscr{P}_T^*(v, m; \varepsilon)$ is *reducible*. Because p is *reciprocal*, every irreducible factor f of p forces its reciprocal $f^*(x) = x^{\deg f} f(x^{-1})$ to be a factor as well.

Consider first those p that factor as $p = f \cdot f^*$ with f irreducible of degree n. Write the roots of p as $x_i = m_i e^{T v_i}\big(1 + O(\varepsilon)\big)$ and $x_i^{-1} = m_i e^{-T v_i}\big(1 + O(\varepsilon)\big)$, $1 \leq i \leq n$. Since the constant term of f must be ± 1, if we set $P = \{i : f(x_i) = 0\} = \{j_1 < \cdots < j_\ell\}$, then $1 \leq \ell < n$. By Vieta's formula, the k-th coefficient of f is bounded by $\exp\left((v_{j_1} + \cdots + v_{j_k})T\right)$, up to a multiplicative constant. Taking the product over $k = 1, \cdots, \ell$, the coefficient box for f has volume at most a constant multiple of $\exp \sum_{k=1}^{\ell}(v_{j_1} + \cdots + v_{j_k})$. Because $\ell < n$ and $v_1 > \cdots > v_n > 0$, we have $\sum_{k=1}^{\ell}(v_{j_1} + \cdots + v_{j_k}) \leq \sum_{k=1}^{n}(v_1 + \cdots + v_k) - v_n = \rho^*(v) - v_n$. Hence

$$\#\{p \text{ with the factorization } p = f\, f^*\} \ll e^{(\rho^*(v) - v_n)T}. \tag{5.1}$$

All remaining reducible polynomials split as

$$p(x) = f_1(x)\, f_2(x), \qquad \deg f_j = 2s_j,\ 1 \leq s_j \leq n-1,\ s_1 + s_2 = n,$$

with each f_j itself a monic *reciprocal* polynomial in $\mathbb{Z}[x]$.

Let $S \subset \{1, \ldots, n\}$ record which conjugate-pairs $\{x_i, x_i^{-1}\}$ of roots of f_1 in decreasing modulus. Writing $S = \{i_1 > \cdots > i_{s_1}\}$, we obtain

$$f_1 \in \mathscr{P}_T^*\big(u_1, M_1'; \varepsilon\big), \quad f_2 \in \mathscr{P}_T^*\big(u_2, M_2'; \varepsilon\big),$$

where u_j collects the v-coordinates indexed by S and its complement, and M_j' the corresponding sign patterns.

By Lemma 5.3, we get

$$\#\Big\{p \in \mathscr{P}_T^*(v, m; \varepsilon) \text{ encoded by } S\Big\} \ll e^{\big(\rho^*(u_1) + \rho^*(u_2)\big)T}. \tag{5.2}$$

We claim that

$$\Delta(S) := \rho^*(v) - \big(\rho^*(u_1) + \rho^*(u_2)\big) > 0.$$

Write $w = (n, n-1, \ldots, 1)$ so that $\rho^*(v) = w \cdot v$. Inside the factors f_j the largest coefficient weight drops from n to at most $n-1$, while no weight increases. Because $v_1 > \cdots > v_n$, we get $w{\cdot}v > w'{\cdot}v$, where w' is the modified weight-vector attached to (u_1, u_2), implying the claim.

It now follows that

$$\#\{p \in \mathscr{P}_T^*(v, m; \varepsilon) \text{ reducible}\} \ll e^{(\rho^*(v)-\eta)T}$$

where $\eta := \min_S \Delta(S) > 0$. Combined with Lemma 5.3, this completes the proof. $\square$

Jordan projections of $\mathrm{Sp}_{2n}(\mathbb{Z})$. Let

$$G = \mathrm{Sp}_{2n}(\mathbb{R}) = \{g \in \mathrm{SL}_{2n}(\mathbb{R}) : g^t J_n g = J_n\} \quad J_n = \begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix} \tag{5.3}$$

where $\bar{I}_n$ is the anti-diagonal identity matrix.

Then $\mathfrak{a}^+$ is a positive Weyl chamber. For a loxodromic element $g \in G$, its Jordan projection is given by

$$\lambda(g) = (\lambda_1(g), \cdots, \lambda_n(g), -\lambda_n(g), \cdots, -\lambda_1(g)) \in \operatorname{int} \mathfrak{a}^+$$

and its eigenvalue datum is

$$\mathcal{E}(g) = (m_1(g)e^{\lambda_1(g)}, \cdots, m_n(g)e^{\lambda_n(g)}, m_n(g)e^{-\lambda_n(g)}, \cdots, m_1(g)e^{-\lambda_1(g)})$$

where $m_i(g) \in \{\pm 1\}$, $1 \le i \le n$.

Theorem 5.6. *([33], [20], [1]) Every integral monic reciprocal polynomial is the characteristic polynomial of some element of $\mathrm{Sp}_{2n}(\mathbb{Z})$.*

We define $\mathsf{E}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v, m)$ and $\mathsf{E}^\star_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v, m)$ exactly as in Definition 1.7, replacing $\mathrm{SL}_n(\mathbb{Z})$ by $\mathrm{Sp}_{2n}(\mathbb{Z})$ throughout.

Observe that

$$\rho^*(v) = \rho_{\mathrm{Sp}_{2n}}(v) = \sum_{i=1}^n (n+1-i)v_i$$

where $\rho_{\mathrm{Sp}_{2n}}$ is the half-sum of all positive roots of $(\mathfrak{sp}_{2n}(\mathbb{R}), \mathfrak{a})$.

Theorem 5.7. *Let $v \in \operatorname{int} \mathfrak{a}^+$ and $m = (m_1, \ldots, m_n) \in \{\pm 1\}^n$. For $\varepsilon > 0$, set*

$$J_T^{\mathrm{Sp}}(v, m; \varepsilon) := \Big\{(\lambda(\gamma), m(\gamma)) : \gamma \in \mathrm{Sp}_{2n}(\mathbb{Z}),\ \|\lambda(\gamma) - Tv\|_{\max} \le \varepsilon,\ m(\gamma) = m\Big\}.$$

For all sufficiently small $\varepsilon > 0$, we have

$$\left(\frac{2\varepsilon}{(2n-1)3^{2n}}\right)^n \le \liminf_{T\to\infty} \frac{\# J_T^{\mathrm{Sp}}(v, m; \varepsilon)}{e^{\rho_{\mathrm{Sp}_{2n}}(v)T}} \le \limsup_{T\to\infty} \frac{\# J_T^{\mathrm{Sp}}(v, m; \varepsilon)}{e^{\rho_{\mathrm{Sp}_{2n}}(v)T}} \le (2\varepsilon)^n (n+1)!.$$

Consequently

$$\mathsf{E}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v, m) = \rho_{\mathrm{Sp}_{2n}}(v).$$

Proof. For $(\lambda(\gamma), m(\gamma)) \in J_T^{\mathrm{Sp}}(v, m; \varepsilon)$, set

$$p(x) = \prod_{i=1}^n \big(x - m_i e^{\lambda_i(\gamma)}\big)\big(x - m_i e^{-\lambda_i(\gamma)}\big) \in \mathscr{Q}_T^*(v, m; \varepsilon).$$

The assignment $(\lambda(\gamma), m(\gamma)) \mapsto p(x)$ is injective, so

$$\# J_T^{\mathrm{Sp}}(v, m; \varepsilon) \;\leq\; \# \mathcal{Q}_T^*(v, m; \varepsilon).$$

Conversely, if $p \in \mathcal{Q}_T^*(v, m; \varepsilon)$, then Theorem 5.6 and Corollary 5.4 produce a $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})$ with $(\lambda(\gamma), m(\gamma)) \in J_T^{\mathrm{Sp}}(v, m; \varepsilon)$. The map $p \mapsto (\lambda(\gamma), m(\gamma))$ is injective, hence

$$\# \mathcal{Q}_T^*(v, m; \varepsilon) \;\leq\; \# J_T^{\mathrm{Sp}}(v, m; \varepsilon).$$

Hence the claim follows from Theorem 5.2. □

The lower bound below follows directly from the above theorem and the upper bound will be proved in Theorem 6.2.

Theorem 5.8. *Let $v \in \operatorname{int} \mathfrak{a}^+$ and $m = (m_1, \ldots, m_n) \in \{\pm 1\}^n$. For every $0 < \varepsilon < 1$, there exist $C_1, C_2 > 0$ such that*

$$C_1 e^{\rho_{\mathrm{Sp}_{2n}}(v) T} \leq \# \Big\{ [\gamma] \in [\mathrm{Sp}_{2n}(\mathbb{Z})] : \|\lambda(\gamma) - Tv\| \leq \varepsilon, m(\gamma) = m \Big\} \leq C_2 e^{2\rho_{\mathrm{Sp}_{2n}}(v) T}.$$

In particular,

$$\rho_{\mathrm{Sp}_{2n}}(v) \leq \underline{\mathsf{E}}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v, m) \;\leq\; \overline{\mathsf{E}}^{\star}_{\mathrm{Sp}_{2n}(\mathbb{Z})}(v, m) \leq 2\rho_{\mathrm{Sp}_{2n}}(v). \tag{5.4}$$

In [32], Yang establishes a bijection between the set of $\mathrm{Sp}_{2n}(\mathbb{Z})$–conjugacy classes and a distinguished subset of units of degree $2n$. Through this correspondence, Theorem 5.8 can be viewed as a result about the growth of that collection of algebraic units.

6. Upper bound for $\mathsf{E}^{\star}_{\Gamma}$ for a general lattice

Let G be a connected semisimple real algebraic group. Fix a Cartan involution so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition into ± 1 eigenspaces. Let $K < G$ be the maximal compact subgroup with Lie algebra $\mathfrak{k}$, and let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra with closed positive chamber $\mathfrak{a}^+$. Let ρ_G denote the half-sum of all positive roots of $(\mathfrak{g}, \mathfrak{a})$.

Write $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}^+$, and let $M = Z_K(A)$. Every $g \in G$ decomposes as a commuting product $g = g_h g_e g_u$ of hyperbolic, elliptic and unipotent elements, and the hyperbolic part g_h is G-conjugate to a unique element $\exp \lambda(g) \in A^+$; we call $\lambda(g)$ the *Jordan projection*. If $\lambda(g) \in \operatorname{int} \mathfrak{a}^+$ we say g is *loxodromic*; then g_u is the identity and g_e is conjugate to an element $m(g) \in M$, unique up to M-conjugacy. We denote by $[\Gamma]_{\mathrm{lox}}$ the set of Γ-conjugacy classes of all loxodromic elements of Γ.

Definition 6.1 (Directional entropy for Γ)**.** Let $\Gamma < G$ be a lattice. Let $\|\cdot\|$ be any norm on $\mathfrak{a}$. For any vector $v \in \operatorname{int} \mathfrak{a}^+$, define the directional *Jordan-entropy* functions by

$$\overline{\mathsf{E}}_{\Gamma}(v) := \|v\| \cdot \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log N_\varepsilon(T, v)}{T}, \quad \underline{\mathsf{E}}_{\Gamma}(v) := \|v\| \cdot \lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{\log N_\varepsilon(T, v)}{T}$$

where $N_\varepsilon(T, v) = \# \{ \lambda(\gamma) : \gamma \in \Gamma : \|\lambda(\gamma) - \mathbb{R}_+ v\| \leq \varepsilon, \|\lambda(\gamma)\| \leq T \}$.

Similarly,

$$\overline{\mathsf{E}}^{\star}_{\Gamma}(v) := \|v\|\cdot\lim_{\varepsilon\to 0}\ \limsup_{T\to\infty}\frac{\log M_\varepsilon(T,v)}{T},\quad \underline{\mathsf{E}}^{\star}_{\Gamma}(v) := \|v\|\cdot\lim_{\varepsilon\to 0}\ \liminf_{T\to\infty}\frac{\log M_\varepsilon(T,v)}{T}$$

where $M_\varepsilon(T,v) := \#\{[\gamma]\in[\Gamma] : \|\lambda(\gamma)-\mathbb{R}_+v\|\le\varepsilon, \|\lambda(\gamma)\|\le T\}$. These definitions are independent of the choice of a norm. When the lower and upper values coincide, we write $\mathsf{E}_\Gamma(v)$ and $\mathsf{E}^\star_\Gamma(v)$, respectively.

Theorem 6.2. *For all $v\in\operatorname{int}\mathfrak{a}^+$ and $\varepsilon>0$, there exists $C>0$ such that for all $T\ge 1$,*

$$\#\Big\{\,[\gamma]\in[\Gamma] : \|\lambda(\gamma)-\mathbb{R}_+v\|\le\varepsilon, \|\lambda(\gamma)\|\le T\Big\}\le Ce^{2\rho_G(v)T}.$$

In particular,

$$\overline{\mathsf{E}}^{\star}_{\Gamma}(v)\le 2\rho_G(v).$$

Cartan counting and Upper bound. Let $\mu(g)\in\mathfrak{a}^+$ denote the Cartan projection of $g\in G$, i.e. the unique element with

$$g\in Ke^{\mu(g)}K.$$

If we use the norm on $\mathfrak{a}$ induced from the Killing form on $\mathfrak{g}$, then for all $g\in G$, we have $\|\mu(g)\| = d(go,o)$ where $o=[K]\in G/K$ and d is the Riemannian distance on the symmetric space G/K. Counting lattice points subject to constraints on the Cartan projection $\mu(g)$ is considerably better understood than the analogous problem for the Jordan projection; see, for example, ([9], [10], [14], [5], [13], etc). In particular, following the method of Eskin-McMullen[10], we can count lattice points whose Cartan projections lie in prescribed tubes or cones by combining the mixing of the A–action on $\Gamma\backslash G$ with the strong wavefront lemma stated below.

Lemma 6.3 (Strong wavefront lemma)**.** [14, Theorem 3.7] *Let $\mathcal{C}\subset\operatorname{int}\mathfrak{a}^+$ be closed and at positive distance from every wall of $\mathfrak{a}^+$. For any neighborhoods $\mathcal{O}_K\subset K$ and $\mathcal{O}_A\subset A$ of e, there exists a neighborhood $U\subset G$ of e such that for any $g=k_1ak_2\in K(\exp\mathcal{C})K$, we have*

$$UgU\subset k_1\mathcal{O}_K\,a\mathcal{O}_A\,k_2O_K.$$

Theorem 6.4. *Let $\Gamma<G$ be a lattice and $v\in\operatorname{int}\mathfrak{a}^+$. For any $\varepsilon>0$ we have, as $T\to\infty$,*

$$\#\{\gamma\in\Gamma : \|\mu(\gamma)-Tv\|\le\varepsilon\}\ \sim\ C\,e^{2\rho_G(v)T}$$

for some constant $C=C(\varepsilon)>0$.

Proof. Fix $\varepsilon>0$ and put

$$b_{T,\varepsilon}=\{u\in\mathfrak{a}^+ : \|u-Tv\|<\varepsilon\},\qquad Z_T = K\,\exp(b_{T,\varepsilon})\,K.$$

For $g=k_1(\exp v)k_2\in K(\exp\mathfrak{a}^+)K$, the Haar measure is

$$dg=\prod_\alpha \sinh\alpha(v)dk_1dvdk_2,$$

where the product runs over all positive roots, counted with multiplicity [17]. We obtain that

$$\operatorname{Vol} Z_T \sim C_\varepsilon\, e^{2\rho_G(v)T} \tag{6.1}$$

for some constant $C_\varepsilon > 0$. Since $v \in \operatorname{int}\mathfrak{a}^+$, the set $b_{T,\varepsilon}$ has a positive distance from all walls of $\mathfrak{a}^+$. Lemma 6.3 and (6.1) imply that the family $\{Z_T\}_{T\gg 1}$ is *well-rounded*: for any $\eta > 0$, there exists an open neighborhood U_η of e in G such that

$$Z_{T-\eta} \subset \bigcap_{u_1,u_2\in U_\eta} u_1 Z_T u_2 \subset \bigcup_{u_1,u_2\in U_\eta} u_1 Z_T u_2 \subset Z_{T+\eta}$$

and

$$\limsup_{\eta\to 0} \frac{\operatorname{Vol}(Z_{T+\eta})}{\operatorname{Vol}(Z_{T-\eta})} = 1.$$

Define the counting function $F_T = F_{Z_T}$ on $(\Gamma\times\Gamma)\backslash(G\times G)$ by

$$F_T([g_1],[g_2]) = \sum_{\gamma\in\Gamma} \chi_{Z_T}(g_1^{-1}\gamma g_2)$$

so that $F_T([e],[e]) = \#\Gamma\cap Z_T$. If ϕ_η is the approximation of the identity function on G supported on the η-neighborhood of e in G and $\Phi_\eta([g]) = \sum_{\gamma\in\Gamma}\phi_\eta(\gamma g)$, then the standard unfolding argument gives that

$$\langle F_T, \Phi_\eta\otimes\Phi_\eta\rangle := \int F_T(x_1,x_2)\Phi_\eta(x_1)\Phi_\eta(x_2)dx_1dx_2 = \int_{g\in Z_T}\langle \Phi_\eta, g.\Phi_\eta\rangle_{L^2(\Gamma\backslash G)}dg.$$

Using strong mixing of the G-action on $L^2(\Gamma\backslash G)$ [16], we get

$$\langle F_T, \Phi_\eta\otimes\Phi_\eta\rangle \sim \frac{1}{\operatorname{Vol}(\Gamma\backslash G)}\operatorname{Vol} Z_T.$$

Noting that

$$\langle F_{T-\eta}, \Phi_\eta\otimes\Phi_\eta\rangle \le F_T([e],[e]) \le \langle F_{T+\eta}, \Phi_\eta\otimes\Phi_\eta\rangle,$$

the well-roundness property of the family $\{Z_T\}$ implies that

$$F_T([e],[e]) \sim \frac{1}{\operatorname{Vol}(\Gamma\backslash G)}\operatorname{Vol} Z_T.$$

□

The following can be deduced from [31, Theorem 1.2] for arithmetic lattices (see the proof of [24, Theorem 3.1]). For rank one groups, this is a standard fact which follows from the thick-thin decomposition of rank one locally symmetric manifolds of finite volume. Hence by Margulis arithmeticity theorem [22], we get:

Theorem 6.5. *Let $\Gamma < G$ be a lattice. There exists a compact subset $Q \subset G$ such that any compact AM-orbit in $\Gamma\backslash G$ is of the form $\Gamma\backslash\Gamma gAM$ for some $g \in Q$.*

Corollary 6.6. *For any lattice $\Gamma < G$, there is $C > 1$ such that for any conjugacy class $[\gamma] \in [\Gamma]_{\rm lox}$, there exists $\gamma' \in [\gamma]$ such that*

$$\|\lambda(\gamma) - \mu(\gamma')\| \le C.$$

Proof. Let Q be a compact subset in Theorem 6.5. We claim that there exists a representative $\gamma' \in [\gamma]$ such that

$$\gamma' \;=\; g\, e^{\lambda(\gamma)} m_\gamma\, g^{-1}, \qquad m_\gamma \in M,\; g \in Q.$$

To see this, since γ is loxodromic, its centralizer in G is of the form $hAMh^{-1}$ with $\Gamma\backslash\Gamma hAM$ compact [26]. Since $h = \gamma_0 g a_0 m_0 \in \Gamma QAM$ with $g_0 \in Q$ by Theorem 6.5 and $\gamma = he^{\lambda(\gamma)}mh^{-1}$ for some $m \in M$, it suffices to set

$$\gamma' = ge^{\lambda(\gamma)}(m_0 m m_0^{-1})g^{-1}.$$

Therefore there is $C > 1$ depending only on Q such that $\|\lambda(\gamma) - \mu(\gamma')\| \le C$ by [4, Lemma 4.6]. □

Since $\gamma' \in [\gamma]$, the map $[\gamma] \to \gamma'$ is an injective map to Γ. Hence we get:

Corollary 6.7. *Let $\Gamma < G$ be a lattice. For any bounded subset $B \subset \mathfrak{a}^+$,*

$$\#\{[\gamma] \in [\Gamma] : \lambda(\gamma) \in B\} < \infty.$$

Proof of Theorem 6.2. Let $B_\varepsilon(0) \subset \mathfrak{a}$ be the ball of radius ε about the origin, and fix $v \in \operatorname{int}\mathfrak{a}^+$. Suppose that $\gamma \in \Gamma$ satisfies $\lambda(\gamma) \in Tv + B_\varepsilon(0)$ for all sufficiently large T. Then since $v \in \operatorname{int}\mathfrak{a}^+$ and T is large, γ is loxodromic. Hence, by Corollary 6.6, there is $\gamma' \in [\gamma]$ such that $\|\lambda(\gamma) - \mu(\gamma')\| \le C$.

Thus, by the injectivity of the map $[\gamma] \to \gamma'$,

$$\#\big\{[\gamma] : \lambda(\gamma) \in Tv + B_\varepsilon(0)\big\} \;\le\; \#\big\{\gamma' \in \Gamma : \mu(\gamma') \in Tv + B_C(0)\big\}.$$

Applying Theorem 6.4 proves the claim.

Remark 6.8. In [25], Quint introduced the growth indicator

$$\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$$

of a Zariski dense discrete subgroup $\Gamma < G$. Let $\mathcal{L}_\Gamma$ be the limit cone of Γ, that is, the asymptotic cone of the Cartan projection $\mu(\Gamma)$. For $v \in \operatorname{int}\mathcal{L}_\Gamma$, it is equal to

$$\psi_\Gamma(v) = \|v\| \inf_{\mathcal{C}} \limsup_{T\to\infty} \frac{\log \#\{\gamma \in \Gamma : \|\mu(\gamma)\| \le T, \mu(\gamma) \in \mathcal{C}\}}{T}$$

where the infimum is taken over all open cones $\mathcal{C} \subset \mathfrak{a}^+$ containing v. If $\Gamma < G$ is a lattice, then $\mathcal{L}_\Gamma = \mathfrak{a}^+$ and $\psi_\Gamma = 2\rho_G$.

While $\psi_\Gamma(v) < +\infty$ for all $v \in \mathfrak{a}^+$ and for any discrete subgroup Γ, the directional entropy

$$\overline{E}^\star_\Gamma(v) = \|v\| \cdot \lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{\log \#\{[\gamma] \in \Gamma : \|\lambda(\gamma)\| \le T, \|\lambda(\gamma) - \mathbb{R}_+ v\| \le \varepsilon\}}{T}$$

may take the value $+\infty$; this already occurs for a normal subgroup of a cocompact lattice of $\mathrm{SL}_2(\mathbb{R})$ of infinite index. Theorem 6.2 shows that for Γ

lattice, $\overline{E}^\star_\Gamma(v) \le \psi_\Gamma(v) = 2\rho_G(v)$ for all $v \in \operatorname{int}\mathfrak{a}^+$. It is shown in [6] that if Γ is a Zariski dense *Borel Anosov* subgroup of G, then $\overline{E}^\star_\Gamma(v) = \psi_\Gamma(v)$ for all $v \in \operatorname{int}\mathcal{L}_\Gamma$.

Upper bound without directional restriction. We will use the following for the upper bound:

Theorem 6.9. *Let $\Gamma < G$ be a lattice in G. If $\mathcal{C}$ is a convex cone in $\mathfrak{a}^+$ with non-empty interior and $\mathcal{C}_T = \{v \in \mathcal{C} : \|v\| < T\}$, then*

$$\#\Gamma \cap K\exp(\mathcal{C}_T)K \sim C \cdot e^{2\rho_G(u_\mathcal{C})T}T^{(\operatorname{rank} G-1)/2}$$

where $\|\cdot\|$ is the norm on $\mathfrak{a}$ induced from the Killing form on $\mathfrak{g}$ and $u_\mathcal{C}$ is the unique unit vector such that $2\rho_G(u_\mathcal{C}) = \max_{\|u\|=1, u\in\mathcal{C}} 2\rho_G(u)$.

Proof. In [14, Lemma 5.4], it is shown that for $\mathcal{C} = \mathfrak{a}^+$,

$$\operatorname{Vol}(K\exp(\mathcal{C}_T)K) \sim C \cdot e^{2\rho_G(u_\mathcal{C})T}T^{(\operatorname{rank} G-1)/2}.$$

The same proof works for any convex cone $\mathcal{C}$ with non-empty interior.

By Theorem 6.3, the family $Z_T = K\exp(\mathcal{C}_T)K$, $T \ge 1$ is well-rounded, as in the proof of Theorem 6.4. Consequently, by the same argument used there, we get

$$\#\Gamma \cap K\exp(\mathcal{C}_T)K \sim \operatorname{Vol}(Z_T).$$

□

Corollary 6.10. *Let $\Gamma < G$ be a lattice. There exist $C > 0$ such that for all $T > 1$,*

$$\#\{[\gamma] \in [\Gamma]_{\text{lox}} : \|\lambda(\gamma)\| < T\} \le Ce^{2\|\rho_G\|T}T^{(\operatorname{rank} G-1)/2}$$

where $\|\rho_G\| = \max_{u\in\mathfrak{a}^+, \|u\|=1} \rho_G(u)$.

Proof. Let $[\gamma] \to \gamma'$ be the injective map from the conjugacy classes of loxodromic elements to Γ given in Corollary 6.6. Therefore

$$\#\{[\gamma] \in [\Gamma]_{\text{lox}} : \|\lambda(\gamma)\| < T\} \le \#\{\gamma' \in \Gamma, \|\mu(\gamma')\| < T + C\}$$

where $C > 1$ is as in Corollary 6.6. Therefore the upper bound follows from Theorem 6.9. □

We remark that in [7], some upper bound for cocompact lattices of G was obtained. We record the following for $\mathrm{SL}_n(\mathbb{Z})$:

Corollary 6.11. *There exist $C_1, C_2 > 0$ such that for all $T > 1$,*

$$C_1 e^{d_n T/2} \le \#\{[\gamma] \in [\mathrm{SL}_n(\mathbb{Z})]_{\text{lox}} : \|\lambda(\gamma)\|_{\text{Euc}} < T\} \le C_2 T^{(n-2)/2} e^{d_n T}$$

where $d_n = \sqrt{\frac{n(n^2-1)}{3}}$.

Proof. The lower bound follows from Corollary 4.3. Since the norm on $\mathfrak{a}$ induced by the Killing form on $\mathfrak{sl}_n(\mathbb{R})$ is a constant multiple of the Euclidean norm on $\mathfrak{a}$, the upper bound follows from Corollary 6.10 and (4.3). □

References

[1] R. Ackermann. Achievable spectral radii of symplectic Perron-Frobenius matrices. *New York J. Math.* 17 (2011) 683-697.

[2] L. Ahlfors. Complex Analysis. *McGraw-Hill, New York 1979.*

[3] S. Akiyama and A. Petho. On the distribution of polynomials with bounded roots II. Polynomials with integer coefficients. *Uniform Distribution Theory.*, 9 (2024), 5-19.

[4] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.*, 7(1):1–47, 1997.

[5] Y. Benoist, and H. Oh. Effective equidistribution of S-integral points on symmetric varieties. *Annales de L'Institut Fourier*, Vol 62 (2012), 1889-1942

[6] M. Chow and H. Oh. Jordan and Cartan spectra in higher rank with applications to correlations. *arXiv arXiv:2308.16329*

[7] N. T. Dang and J. Li. Equidistribution and counting of periodic tori in the space of Weyl chambers. To appear in Commentarii Mathematici Helvetici.

[8] H. Davenport. On a principle of Lipschtiz. J. London Math. Soc. 26 (1951), 179-183.

[9] W. Duke, Z. Rudnick, P. Sarnak. Density of integer points on affine homogeneous varieties. *Duke Math. J.*,Vol 71, 1993, 181-209.

[10] A. Eskin and C. McMullen. Mixing, counting, and equidistribution. *Duke Math. J.*, Vol 71, 1993, 143-180.

[11] A. Eskin, S. Mozes and N. Shah. Unipotent flows and counting lattice points on homogeneous varieties. *Annals of Math.*, 143 (1996), 253-299.

[12] R. Gangolii and G. Warner. Zeta functions of Selberg's type for some noncompact quotients of symmetric spaces of rank one. *Nagoya Math. J.*, 78, (1980), 1-44.

[13] A. Gorodnik and A. Nevo. Counting lattice points. *Journal fur die reine und angewandte Mathematik (Crelles Journal)*, 2012(663):127-176.

[14] A. Gorodnik and H. Oh. Orbits of discrete subgroups on a symmetric space and the Furstenberg boundary. *Duke Math.* Vol 139 (2007), 483-525.

[15] D. A. Hejhal. The Selberg trace formula for $\mathrm{PSL}(2,\mathbb{R})$ *Vol. I. Lecture Notes in Mathematics.* Vol. 548. Springer-Verlag, Berlin-New York, 1976.

[16] R. Howe and C. Moore. Asymptotic properties of unitary representations. *J. Functional Analysis.* 32 (1979), no. 1, 72-96.

[17] A. Knapp. Representation theory of semisimple Lie groups, An overview based on examples. *Princeton university press.* 1986.

[18] H. Koch. Number theory: Algebraic numbers and functions. *GTM Math. AMS* Vol 24, 2000

[19] C. G. Latimer and C. C. MacDuffee. A correspondence between classes of ideals and classes of matrices. *Ann. Math.* 34 (1933) 313-316.

[20] D. Margalit and S. Spallone. A homological recipe for Pseudo-Anosovs. *Math. Res. Lett* 14 (2007), no.5 853-863.

[21] G. Margulis. *On some aspects of the theory of Anosov systems (translation of 1970 thesis).* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004.

[22] G. A. Margulis. Discrete subgroups of semisimple Lie groups. *Berlin Heidelberg New York.* Springer-Verlag, 1991.

[23] G. Margulis, A. Mohammadi, and H. Oh. Closed geodesics and holonomies for Kleinian manifolds. *Geom. Funct. Anal.*, 24(5):1608–1636, 2014.

[24] H. Oh. Finiteness of compact maximal flats of bounded volume. *Ergod. Th and Dynam. Sys.* Vol 24 (2004), 217–225.

[25] J.-F. Quint. Divergence exponentielle des sous-groupes discrets en rang supérieur. *Comment. Math. Helv.*, 77(3):563–608, 2002.

[26] G. Prasad and M. S. Raghunathan Cartan subgroups and lattices in semisimple groups. *Annals of Math.* Vol 96 (1972), 296-317.

[27] T. Roblin. Ergodicité et équidistribution en courbure négative. *Mém. Soc. Math. Fr. (N.S.)*, (95):vi+96, 2003.

[28] P. Sarnak Class numbers of indefinite binary quadratic forms. *Journal of Number theory*, Vol 2, Issue 2 (1982), 229-247.

[29] A. Selberg. On discontinuous groups in higher-dimensional symmetric spaces. *In Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960)*, 147–164. Tata Institute of Fundamental Research.

[30] O. Taussky. On a theorem of Latimer and Macduffee. *Canadian Journal of Mathematics* Volume 1 (1949) 300 - 302.

[31] G. Tomanov and B. Weiss. Closed orbits for actions of maximal tori on homogeneous spaces. *Duke Math. J.*, 119(2): 367-392.

[32] Q. Yang. Conjugacy classes in integral symplectic groups. *Linear Algebra Appl.* 418 (2006), 614–624

[33] Q. Yang. Decomposability of symplectic matrices over principal ideal domain. *J. Number theory*, Vol 149 (2015), 139–152

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