EFFECTIVE EQUIDISTRIBUTION OF S-INTEGRAL POINTS ON SYMMETRIC VARIETIES

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ABSTRACT. Let K be a global field of characteristic not 2. Let $\mathbf{Z} = \mathbf{H}\backslash\mathbf{G}$ be a symmetric variety defined over K and S a finite set of places of K. We obtain counting and equidistribution results for the S-integral points of $\mathbf{Z}$.

Our results are effective when K is a number field.

1. INTRODUCTION

1.1. **General overview.** Consider a finite system of polynomial equations with integral coefficients. Its set of solutions defines an arithmetic variety $\mathbf{Z} \subset \mathbb{C}^d$ defined over $\mathbb{Z}$. For a set S of primes including the infinite prime ∞, let $\mathbb{Z}_S$ denote the ring of S-integers of $\mathbb{Q}$, that is, the set of rational numbers whose denominators are products of primes in S. If $S = \{\infty\}$, $\mathbb{Z}_S$ is simply the ring of integers $\mathbb{Z}$, and if S consists of all the primes, then $\mathbb{Z}_S$ is the field $\mathbb{Q}$ of rational numbers. For any subring R of $\mathbb{C}$, we denote by $\mathbf{Z}_R$ the set of points in $\mathbf{Z}$ with coordinates in R. One of the fundamental questions in number theory is to understand the properties of sets $\mathbf{Z}_{\mathbb{Z}_S}$. In this paper, we obtain effective counting and equidistribution results of the S-integral points, for S-finite, in the case when $\mathbf{Z}$ is a symmetric variety.

The counting question in this set-up has been intensively studied for the *integral* points via several different methods. The first method is due to Duke, Rudnick and Sarnak in 1993 [17], utilizing the theory of automorphic forms. Almost at the same time, Eskin and McMullen developed a different approach, making a more direct use of the mixing properties of semisimple real algebraic groups [21]. The third method , due to Eskin, Mozes and Shah [22], is based on the ergodic theory on homogeneous spaces, in particular, the work of Ratner [47] on ergodic invariant measures for unipotent flows, as well as the works of Dani and Margulis ([11], [12]) on the linearization methods and the non-divergence of unipotent flows.

The approach of [21] using mixing properties has several advantages over the others in our viewpoint. First it does not require the deep theory of automorphic forms, avoiding technical difficulties in dealing with the Eisenstein series as in [17]. Secondly, although this was never addressed in [21], in principle it also gives

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a rate of convergence which the ergodic method of [22] does not give. Thirdly the method can be extended to other global fields of positive characteristic, which is again hard to be achieved via the ergodic method.

For these reasons, we develop the approach of Eskin and McMullen [21] in this paper in order to obtain *effective* results for the general S-integral points on symmetric varieties.

Let $\mathbf{G}$ be a connected semisimple simply connected $\mathbb{Q}$-group and $\mathbf{H}$ a symmetric $\mathbb{Q}$-subgroup of $\mathbf{G}$. Set

$$X_S := \Gamma\backslash\mathbf{G}_S \quad \text{and} \quad Y_S := \Gamma\backslash\Gamma\mathbf{H}_S$$

where $\mathbf{G}_S := \prod_{p\in S}\mathbf{G}(\mathbb{Q}_p)$, $\mathbf{H}_S := \prod_{p\in S}\mathbf{H}(\mathbb{Q}_p)$ and Γ is an S-arithmetic subgroup of $\mathbf{G}_S$.

Two main ingredients of our proofs are

(1) the effective equidistribution of the translates $Y_S g$ in X_S for any $g \in \mathbf{G}_S$;
(2) the effective well-roundedness of the height balls $B_T := \{x \in Z_S : \mathrm{H}_S(x) < T\}$ where $Z_S \simeq \mathbf{H}_S\backslash\mathbf{G}_S$ is a Zariski closed subvariety of a K-vector space V, and H_S is the height function on V_S given by $\mathrm{H}_S((x_p)) := \prod_{p\in S}\|x_p\|_p$ for $(x_p)_{p\in S} \in V_S$.

For (1), we use the mixing properties of S-algebraic semisimple groups, with a rate of convergence. To relate the mixing property of $\mathbf{G}_S$ with the equidistribution problem (even in non-effective form) for the translates of $\mathbf{H}_S$, we need to verify certain geometric property, which was named the wavefront property by [21], for an S-algebraic symmetric variety. We prove this using the polar decompositions for non-archimedean symmetric spaces obtained in [4] specifically for this purpose. We emphasize that the wave front property is precisely the reason that our proofs work in the setting of an S-algebraic symmetric variety for S finite. This property does not hold for a general homogeneous variety even over the reals. In the case when Y_S is compact, and hence a uniform approximation of Y_S by neighborhoods in G_S is possible, the arguments in [21] can be more or less easily adapted to a proof of (1), based on the effective mixing and the wave-front property. Most of the technical difficulties in proving (1) arises in the case of a non-compact Y_S. Even in the case when $K = \mathbb{Q}$ and $S = \{\infty\}$, our paper seems to be the first place where a complete proof of (1) is given. Although (1) has been announced in [17] in this case, proofs have been given only for Y_S compact.

Now turning our attention to (2), recall first that the well-roundness of family $\{B_T\}$ of compact subsets in Z_S means that for all small $\varepsilon > 0$, the boundaries of B_T can be approximated by neighborhoods whose volume is of ε-order compared to the volume of B_T uniformly for all large T. For the norm-balls in a real symmetric variety (i.e., $S = \{\infty\}$), this property was verified in [17]. In effectivising this property, one needs to control how large T has to be taken for a given $\varepsilon > 0$, for instance, obtaining an estimate on the *derivative of the volume of B_T*, among

other things. In the case of a group variety, a similar estimate was made in [41] by the brute force computation using the decomposition of a Haar measure of the group, and was used to obtain an effective counting of lattice orbits in a group variety. We achieve this in a complete generality of a homogeneous variety of an algebraic group, using a theorem of Jeanquartier on fiber integrations [34], Hironaka's resolution of singularities [31], Denef's theorems on p-adic local zeta functions ([14], [15]). We remark that this is one of crucial ingredients of [27].

We state and prove all of our effective results for any number field K and the non-effective versions for any global field of characteristic not 2.

We remark that the approach for counting via mixing was first used by Margulis in his thesis [38] which was written in 1970 but was published only thirty years later. Recently similar mixing properties in an adelic setting have been used in the study of rational points of *group* varieties (see [9], [25], and [30]). We also mention that for the case of group varieties, the effective counting result was obtained for integral points in [41], under some additional assumptions. We refer to [35], [29], [23], [20], [28], [18], [40] etc., for other types of counting and equidistribution results.

1.2. **Main results.** We now give a precise description of the main results of this paper.

Let K be a global field of characteristic not 2, i.e. a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_q(t)$ where q is an odd prime. Let $\mathbf{Z}$ be a symmetric variety in a vector space $\mathbf{V}$ defined over K. That is, there exist a connected algebraic almost K-simple group $\mathbf{G}$ defined over K, a K-representation $\rho : \mathbf{G} \to \mathbf{GL}(\mathbf{V})$ with finite kernel and a non-zero point $z_0 \in \mathbf{V}_K$ whose stabilizer $\mathbf{H}$ in $\mathbf{G}$ is a symmetric K-subgroup of $\mathbf{G}$ such that $\mathbf{Z} = z_0\mathbf{G}$. By a symmetric K-subgroup of $\mathbf{G}$, we mean a K-subgroup whose identity component coincides with the identity component of the group of fixed points $\mathbf{G}^\sigma$ for an involution σ of $\mathbf{G}$ defined over K. We assume that the identity component $\mathbf{H}^0$ has no non-trivial K-character.

We fix a basis of the K-vector space $\mathbf{V}_K$ so that one can define, for any subring $\mathcal{O}$ of K, the subsets $\mathbf{V}_\mathcal{O} \subset \mathbf{V}$, $\mathbf{Z}_\mathcal{O} \subset \mathbf{Z}$ and $\mathbf{G}_\mathcal{O} \subset \mathbf{G}$ of points with coefficients in $\mathcal{O}$. For each place v of K, denote by K_v be the completion of K with respect to the absolute value $|\cdot|_v$. We write $\mathbf{V}_v$, $\mathbf{Z}_v$ and $\mathbf{G}_v$ for $\mathbf{V}_{K_v}$, $\mathbf{Z}_{K_v}$ and $\mathbf{G}_{K_v}$, respectively.

Let S be a finite set of places of K containing all archimedean (sometimes called infinite) places with $\mathbf{G}_v$ non-compact. Note that if $\operatorname{char} K$ is positive, K does not have any archimedean place. We denote by $\mathcal{O}_S$ the ring of S-integers of K, that is,

$$\mathcal{O}_S := \{k \in K \mid |k|_v \leq 1 \text{ for each finite } v \notin S\}.$$

For instance, if $K = \mathbb{Q}$, we have $\mathcal{O}_S = \mathbb{Z}_S$. We set $\mathbf{Z}_S = \prod_{v \in S} \mathbf{Z}_v$ and similarly $\mathbf{G}_S$ and $\mathbf{H}_S$.

Note that the sets $\mathbf{Z}_{\mathcal{O}_S}$, $\mathbf{G}_{\mathcal{O}_S}$ and $\mathbf{H}_{\mathcal{O}_S}$ are discrete subsets of $\mathbf{Z}_S$, $\mathbf{G}_S$, and $\mathbf{H}_S$ respectively, via the diagonal embeddings.

By a theorem of Borel and Harish-Chandra in characteristic 0 and of Behr and Harder in positive characteristic (see Theorem I.3.2.4 in [39]), the subgroups $\mathbf{G}_{\mathcal{O}_S}$ and $\mathbf{H}_{\mathcal{O}_S}$ are lattices in $\mathbf{G}_S$ and $\mathbf{H}_S$ respectively. Again, by a theorem of Borel, Harish-Chandra, Behr and Harder, the group $\mathbf{G}_{\mathcal{O}_S}$ has only finitely many orbits in $\mathbf{Z}_{\mathcal{O}_S}$ (see Theorem 10 in [24]).

Hence our counting and equidistribution question of S-integral points of $\mathbf{Z}$ reduces to counting and equidistribution of points in a single $\mathbf{G}_{\mathcal{O}_S}$-orbit, say, for instance, in $z_0\mathbf{G}_{\mathcal{O}_S}$. Set

$$Z_S := z_0\mathbf{G}_S = \prod_{v\in S} z_0\mathbf{G}_v$$

and let Γ_S be a congruence subgroup of $\mathbf{G}_{\mathcal{O}_S}$. Let μ_{X_S} be a $\mathbf{G}_S$-invariant measure on $X_S := \Gamma_S\backslash\mathbf{G}_S$ and μ_{Y_S} an $\mathbf{H}_S$-invariant measure on $Y_S := (\Gamma_S\cap\mathbf{H}_S)\backslash\mathbf{H}_S$. For each $v \in S$, we choose an invariant measure μ_{Z_v} on $Z_v := z_0\mathbf{G}_v$ so that for $\mu_{Z_S} := \prod_{v\in S}\mu_{Z_v}$, we have $\mu_{X_S} = \mu_{Z_S}\mu_{Y_S}$ locally. For a subset $S_0 \subset S$, we set $\mu_{Z_{S_0}} = \prod_{v\in S_0}\mu_{Z_v}$.

For a Borel subset B of Z_S, we set

$$\mathrm{vol}(B) := \frac{\mu_{Y_S}(Y_S)}{\mu_{X_S}(X_S)}\,\mu_{Z_S}(B).$$

We assume that $\mathbf{G}_S$ is non-compact; otherwise $\mathbf{Z}_{\mathcal{O}_S}$ is finite. By considering a finite covering of $\mathbf{G}$ by its simply connected cover, we may also assume that $\mathbf{G}$ is simply connected without loss of generality.

Before stating our main result, we summarize our set-up:

> K is a global field such that $\mathrm{char}(K) \neq 2$, $\mathbf{Z} \simeq \mathbf{H}\backslash\mathbf{G}$ is a symmetric variety in a vector space $\mathbf{V}$ defined over K where $\mathbf{G}$ is an almost K-simple simply-connected K-group acting on $\mathbf{V}$ such that the identity component $\mathbf{H}^0$ has no non-trivial K-character, and S is a finite set of places of K containing all the infinite places v with $\mathbf{G}_v$ non-compact and satisfying that $\mathbf{G}_S$ is non-compact.

Counting S-integral points. We first state our counting results. We refer to Definition 6.1 for the notion of a well-rounded sequence of subsets B_n in Z_S.

Theorem 1.1. *For any well-rounded sequence of subsets B_n of Z_S with volume tending to infinity, we have*

$$\#(z_0\Gamma_S \cap B_n) \sim \mathrm{vol}(B_n) \quad \text{as } n \to \infty.$$

As a corollary, we obtain that the number of S-integral points of size less than T is given by the volume of the corresponding ball in Z_S. A most natural way to measure the size of an S-integral point is given by a height function H_S. For

$z \in \mathbf{Z}(\mathcal{O}_S)$, it is simply

$$\mathrm{H}_S(z) := \prod_{v \in S} \|z\|_v$$

where the $\| \cdot \|_v$ are norms on $\mathbf{V}_{K_v}$ which are euclidean when v is infinite and which are max norms when v is finite. This height function H_S naturally extends to Z_S.

Corollary 1.2. *As $T \to \infty$,*

$$\#\{z \in z_0 \Gamma_S : \mathrm{H}_S(z) < T\} \sim \mathrm{vol}(B_S(T))$$

where $B_S(T) := \{z \in Z_S : \mathrm{H}_S(z) < T\}$.

When K is a number field, a version of Theorem 1.1 with a rate of convergence is given in Theorem 13.2. For instance, we get:

Theorem 1.3. *Let K be a number field. There exists $\delta > 0$ such that as $T \to \infty$*

$$\#\{z \in z_0 \Gamma_S : \mathrm{H}_S(z) < T\} = \mathrm{vol}(B_S(T))(1 + O(T^{-\delta})).$$

Our proof of Theorem 1.3 uses Denef's result on local zeta functions which is not available in positive characteristic. This explains our hypothesis on the characteristic of K.

The rate of convergence in Theorem 1.3 is new even for the integral points in the generality of symmetric varieties. In this case, as $T \to \infty$, $\mathrm{vol}(B_S(T)) \sim cT^a \log(T)^b$, for some $c > 0$ as obtained in [28].

Note for a general S that one cannot expect $\mathrm{vol}(B_S(T)) \sim cT^a \log(T)^b$, for some $c > 0$; simply because any v-adic valuation is discrete for a non-archimedean v. We prove:

Theorem 1.4. *There exist $a \in \mathbb{Q}_{>0}$, $b \in \mathbb{Z}_{\geq 0}$ and $c_1, c_2 > 0$ such that for any $T \gg 1$,*

$$c_1 T^a \log(T)^b \leq \mathrm{vol}(B_S(T)) \leq c_2 T^a \log(T)^b .$$

Equidistribution of S-integral points. To motivate, consider the case when the set $\mathbf{Z}_{\mathbb{Z}[p^{-1}]}$ of rational points in $\mathbf{Z}$ with denominators only power of p is a dense subset in $\mathbf{Z}_{\mathbb{R}}$. A natural question is when the sequence of subsets in $\mathbf{Z}_{\mathbb{Z}[p^{-1}]}$ consisting of elements of denominator precisely p^n is equidistributed as $n \to \infty$. That is, for two compact subsets Ω_1, Ω_2 of $\mathbf{Z}_{\mathbb{R}}$, as $n \to \infty$,

$$\frac{\{x \in \mathbf{Z}_{\mathbb{Q}} \cap \Omega_1 : p^n x \in \mathbf{V}_{\mathbb{Z}}, \ p \nmid p^n x\}}{\{x \in \mathbf{Z}_{\mathbb{Q}} \cap \Omega_2 : p^n x \in \mathbf{V}_{\mathbb{Z}}, \ p \nmid p^n x\}} \sim \frac{\mathrm{vol}(\Omega_1)}{\mathrm{vol}(\Omega_2)}?$$

Since $p^n x \in V_{\mathbb{Z}}$ is equivalent to the condition that the p-adic maximum norm of x is at most p^n, the above question can be rephrased as the question of equidistribution on $\mathbf{Z}_{\mathbb{R}}$ of the sets $\{z \in \mathbf{Z}_{\mathbb{Z}[p^{-1}]} : \|z\|_p = p^n\}$.

We answer this question in greater generalities:

Theorem 1.5. *Let $S = S_0 \sqcup S_1$ be a partition of S. For any well-rounded sequence B_n of subsets of Z_{S_1} with volume tending to infinity, and for any compact subset $\Omega \subset Z_{S_0}$ of positive measure and of boundary measure 0, we have*

$$\# z_0 \Gamma_S \cap (\Omega \times B_n) \sim \frac{\mu_{Y_S}(Y_S)}{\mu_{X_S}(X_S)} \, \mu_{Z_{S_0}}(\Omega) \, \mu_{Z_{S_1}}(B_n) \quad \text{as } n \to \infty.$$

Note that the special case discussed prior to Theorem 1.5 corresponds to $K = \mathbb{Q}$, $S_0 = \{\infty\}$, $S_1 = \{p\}$, and $B_n = \{z \in \mathbf{Z}_{\mathbb{Q}_p} : \|z\|_p = p^n\}$.

Note that in all the above theorems, we may replace $z_0 \Gamma_S$ by $Z_{\mathcal{O}_S} := Z_S \cap \mathbf{Z}_{\mathcal{O}_S}$ as long as we renormalize the volume form so that the volume of a subset $E \subset Z_S$ is given by

$$(1.6) \qquad\qquad \widetilde{\mathrm{vol}}(E) \;=\; \sum \frac{\mu_{Y_S}(Y_S)}{\mu_{X_S}(X_S)} \, \mu_{Z_S}(E)$$

where we sum the contributions from each Γ_S-orbit in $Z_{\mathcal{O}_S}$. Hence we obtain:

Corollary 1.7. *Assume S has at least two places.*

   (1) *For any finite $v \in S$, the sets $Z(T) := \{z \in Z_{\mathcal{O}_S} : \|z\|_v = T\}$ become equidistributed in $Z_{S-\{v\}}$ as $T \to \infty$, subject to the condition $Z(T) \neq \emptyset$.*
   (2) *For an infinite $v \in S$, the sets $Z_T := \{z \in Z_{\mathcal{O}_S} : \|z\|_v \leq T\}$ become equidistributed in $Z_{S-\{v\}}$ as $T \to \infty$, provided Z_v is non-compact.*

Again, when K is a number field, Theorem 1.5 and Corollary 1.7 are proved with a rate of convergence (see Corollary 13.3 and Proposition 14.2). For instance, we obtain:

Theorem 1.8. *Let K be a number field and $S = S_\infty \sqcup S_f$ be the partition of S into infinite and finite places. We assume that $\mathbf{G}_{S_f}$ is non compact. Set $\beta_T := \{z \in Z_{S_f} : \mathrm{H}_{S_f}(z) < T\}$. Then there exist $\delta > 0$ such that for any compact subset Ω of Z_{S_∞} with piecewise smooth boundary,*

$$\#(z_0 \Gamma_S \cap (\Omega \times \beta_T)) = \frac{\mu_{Y_S}(Y_S)}{\mu_{X_S}(X_S)} \, w_T \, \mu_{Z_\infty}(\Omega)(1 + O(T^{-\delta})) \quad \text{as } T \to \infty$$

where $w_T := \mu_{Z_{S_f}}(\beta_T)$.

Equidistribution of translates of $\mathbf{H}_S$-orbits. Set $X_S = \mathbf{G}_{\mathcal{O}_S} \backslash \mathbf{G}_S$ and $Y_S = \mathbf{H}_{\mathcal{O}_S} \backslash \mathbf{H}_S$. Let μ_{X_S} and μ_{Y_S} be invariant probability measures on X_S and Y_S respectively. The following theorem is a crucial tool in proving Theorem 1.1. It states that the translates $Y_S g$ is equidistributed in X_S as g leaves compact subsets of $\mathbf{H}_S \backslash \mathbf{G}_S$.

Theorem 1.9. *For any $\psi \in C_c(X_S)$,*

$$\int_{Y_S} \psi(yg) d\mu_{Y_S}(y) \to \int_{X_S} \psi \, d\mu_{X_S} \quad \text{as } g \text{ tends to infinity in } \mathbf{H}_S \backslash \mathbf{G}_S.$$

In the case when $K = \mathbb{Q}$ and $S = \{\infty\}$, Theorem 1.9 was proved in [21], [17], [22] and [49]. In the case when $\mathbf{H}_S$ is semisimple and non-compact, it is recently proved in [27], by extending theorems of Mozes-Shah [42] and Dani-Margulis [12] in S-algebraic settings. None of the above papers address the rate issues, while our proof gives effective version in the case when $\mathrm{char}(K) = 0$: a smooth function on X_S is a function which is smooth for each infinite place in S and invariant under a compact open subgroup of $\mathbf{G}_v$ for each finite place $v \in S$. The following effective version of theorem 1.9 is a crucial tool in proving Theorem 1.3 as well as other effective results in this paper.

Theorem 1.10. *For K number field, there exists $\kappa > 0$ such that, for any smooth function ψ on X_S with compact support, there exists $c = c_\psi > 0$ such that*

$$\left| \int_{Y_S} \psi(yg) \, d\mu_{Y_S}(y) - \int_{X_S} \psi \, d\mu_{X_S} \right| \leq c \, \mathrm{H}_S(z_0 g)^{-\kappa} \quad \textit{for all } g \in G_S.$$

Examples. Let $K = \mathbb{Q}$ and consider the following pairs $(\mathbf{V}, f)$:
- (A) $\mathbf{V}$: the affine n-space with $n \geq 3$ and f: an integral quadratic form of n-variables. If $n = 3$, we assume that f does not represent 0 over $\mathbb{Q}$.
- (B) $\mathbf{V}$: the space of symmetric $n \times n$ matrices with $n \geq 3$ and $f = \pm \det$.
- (C) $\mathbf{V}$: the space of skew-symmetric $2n \times 2n$-matrices with $n \geq 2$ and $f = \pm \mathrm{pfaff} = \pm\sqrt{\det}$.

For a positive integer m, define

$$\mathbf{V}_m := \{x \in \mathbf{V} : f(x) = m\}.$$

Consider the radial projection $\pi : \mathbf{V}_m \to \mathbf{V}_1$ given by $x \mapsto m^{-1/d}x$ where d is the degree of f. Let $\mathbf{V}(\mathbb{Z})^{\mathrm{prim}}$ be the set of primitive integral vectors in $\mathbf{V}$. For a finite set S of primes of $\mathbb{Q}$ containing the infinite prime ∞, we denote by $\langle S \rangle \subset \mathbb{Q}^*$ the multiplicative semigroup generated by the finite primes in S.

We give a partial answer to the following Linnik problem (see [36], [48], [23], [44], [40]):

Corollary 1.11. *Fix S and $(\mathbf{V}, f)$ as above. Then there exist constants $\delta > 0$ and ω_m, such that for any non-empty compact subset $\Omega \subset \mathbf{V}_1(\mathbb{R})$ with piecewise smooth boundary, we have*

$$\#\Omega \cap \pi(\mathbf{V}_m(\mathbb{Z})^{prim}) = \omega_m \, \mathrm{vol}(\Omega)(1 + O(m^{-\delta}))$$

as $m \to \infty$ in $\langle S \rangle$, subject to $\mathbf{V}_m(\mathbb{Z})^{prim} \neq \emptyset$.

A special case of (A) gives an effective equidistribution for $\{x \in \mathbb{Z}^3 : f(x) = m\}$ with $f = x^2 + y^2 + z^2$ or $f = x^2 + y^2 - 3z^2$, hence giving a different proof of partial cases (because of the restriction on m) of theorems of Iwaniec [33] and Duke [16]. Note that a special case of (B) gives an effective equidistribution for the *positive definite* integral matrices of given determinant. These cases are of special interest

since the corresponding symmetric group $\mathbf{H}$ is either compact over the reals or a torus. When $\mathbf{H}$ is semisimple without compact factors over the reals, Corollary 1.11 with no restriction on m was obtained in [23] in its non effective form and recently in [19] in its effective form, using the works of Ratner [47] as well as of Dani and Margulis [11] on unipotent flows.

Corollary 1.12. *Keep the same assumption as in Corollary 1.11 and set m_S to be the product of the finite $p \in S$. For the case (A), we further suppose that f represents 0 over $\mathbb{Q}_p$ for at least one $p \in S$. Then there exists $\delta > 0$ such that*

$$\#\{x \in \mathbf{V}(\mathbb{Z})^{prim} : \|x\|_\infty < T, \ f(x) \in \langle S \rangle\} = v_T(1 + O(T^{-\delta}))$$

where the asymptotic v_T is given by the following sum over the divisors m of m_S^{d-1}:

$$v_T = \sum_{m \mid m_S^{d-1}} \widetilde{\mathrm{vol}}(\{x \in (\mathbf{V}_m)_S : \mathrm{H}_S(x) < T\})$$

To prove corollaries 1.11 and 1.12, we will apply the effective versions of Theorems 1.1 and 1.5 to $\mathbf{V}_1(\mathbb{Z}_S)$. We list more examples in section 16.

1.3. **Guideline.** We tried to help the reader in writing "twice" the proofs: In the first half of this paper we concentrate on the main term in the counting and equidistribution statements. In the second half, we follow the same strategy but develop more technical tools to obtain the effective versions of these statements, i.e. to control the error terms.

In section 2, we recall the decay of matrix coefficients for semisimple groups G and its application on a homogeneous space $\Gamma\backslash G$ of finite volume. In section 3, we show the wavefront property for symmetric spaces $H\backslash G$ over local fields and their products. In section 4, we explain how mixing and wavefront properties imply the equidistribution properties of translates of H-orbits in $\Gamma\backslash G$ given in Theorem 1.9. In sections 5 and 6, we explain how these equidistribution properties for translates of H-orbits in $\Gamma\backslash G$ allow us to compare for *well-rounded* sequences of functions on $H\backslash G$ each sum over a Γ-orbit with the integral on $H\backslash G$.

In the section 7, we give some general estimates for the volume of balls (and its derivative) in the orbits of algebraic groups both over the real and p-adic numbers. In sections 8 and 9, we give examples of well-rounded sequences and give proofs of Theorem 1.1, Corollary 1.2, Theorem 1.5 and Corollary 1.7.

Starting from section 10, we prove the effective results listed in the introduction. Theorem 1.10 is proved in section 12, and Theorems 1.3, 1.8, Corollaries 1.11 and 1.12, among other effective applications, are proved in section 15. We list more concrete examples in section 16.

We remark that in the whole paper the assumption of $\mathbf{H}$ symmetric is used only to obtain the (effective) wave front property for $\mathbf{H}_S\backslash\mathbf{G}_S$. The methods and the arguments in this paper work equally well for any K-subgroup $\mathbf{H}$ with no non-trivial characters satisfying the wave front property.

With the exception of section 3, the arguments in sections 2-6 are not new, as already known by the work of [17] and [21] (although some clarification of the proof of Theorem 4.1 had to be made). Hence most of new contents of this paper appear in sections 7 to 16.

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2. The mixing property

We first recall the Howe-Moore property also called *decay of matrix coefficients.*

Definition 2.1. *A locally compact group G is said to have the Howe-Moore property if, for every unitary representation $(\mathcal{H}, \pi)$ of G containing no non-zero vectors invariant by a normal non-compact subgroup, we have for all v, $w \in \mathcal{H}$,*

$$\lim_{g \to \infty} \langle \pi(g)v, w \rangle = 0 \ .$$

This Howe-Moore property is related to the following mixing property.

Let G be a (unimodular) locally compact group and Γ a lattice in G, i.e. a discrete subgroup of finite covolume. Let μ_X be a G-invariant measure on $X := \Gamma \backslash G$. The group G acts on X by right-translations.

Definition 2.2. *The action of G on X is said to be mixing if for all α and $\beta \in L^2(X)$*

$$\lim_{g \to \infty} \int_X \alpha(xg)\beta(x)d\mu_X(x) = \mu_X(X) \int_X \alpha \, d\mu_X \int_X \beta \, d\mu_X.$$

The relation between these two definitions is given by the following straightforward proposition.

Definition 2.3. *A lattice Γ in a locally compact group G is called irreducible if for any non-compact normal subgroup G' of G, the subgroup $\Gamma G'$ is dense in G.*

Note that this definition is slightly stronger than the usual definition since it excludes lattices contained in a proper subgroup of G.

Proposition 2.4. *Let G be a locally compact group satisfying the Howe-Moore property and Γ an irreducible lattice in G. Then the action of G on $\Gamma \backslash G$ is mixing.*

Proof. This is well-known. We first assume that α and β belong to $\mathcal{H} := L_0^2(X)$ of square-integrable functions with zero integral. The G-action by right-translations on $\mathcal{H}$ via $(\pi(g)f)(x) = f(xg)$ is a unitary representation of G. The irreducibility hypothesis on Γ implies precisely that $\mathcal{H}$ does not contain any non-zero vector

invariant by a normal non-compact subgroup of G. Hence, by Definition 2.1, the matrix coefficients $\langle \pi(g)\alpha, \beta \rangle$ converge to 0 as g tends to infinity.

For general α, β in $L^2(X)$, we write $\alpha = \alpha_0 + \int_X \alpha \; \mathrm{d}\mu_X$, $\beta = \beta_0 + \int_X \beta \; \mathrm{d}\mu_X$ where α_0 and β_0 have zero integral, and we conclude by a direct calculation. $\square$

The main example is due to Howe-Moore.

Theorem 2.5. *For $i = 1, .., m$, let k_i be a local field and G_i the group of k_i-points of a connected semisimple k_i-group. Then the product $G := \prod_{i=1}^m G_i$ has the Howe-Moore property.*

In this paper, "local field" means "locally compact field", i.e. a completion of a global field, or, equivalently, a finite extension of $\mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

Proof. See, for instance, Proposition II.2.3 of [39] or [3]. $\square$

3. THE WAVEFRONT PROPERTY

The wavefront property was introduced by Eskin and McMullen for real symmetric spaces [21]. Let G be a locally compact group and H a closed subgroup of G.

Definition 3.1. *The group G has the wavefront property in $H\backslash G$ if there exists a Borel subset $F \subset G$ such that $G = HF$ and, for every neighborhood U of e in G, there exists a neighborhood V of e in G such that*

$$HVg \subset HgU \quad \text{for all } g \in F.$$

This property means roughly that the g-translate of a small neighborhood of the base point $z_0 := [H]$ in $H\backslash G$ remains near $z_0 g$ uniformly over $g \in F$.

This section is devoted to proving the following:

Proposition 3.2. *Let k be a local field of characteristic not 2, $\mathbf{G}$ a connected semisimple k-group, σ a k-involution of $\mathbf{G}$, $G = \mathbf{G}_k$ and H a closed subgroup of finite index in the group G^σ of σ-fixed points.*

Then the group G has the wavefront property on $H\backslash G$.

To prove the above proposition, we need the following two lemmas. A k-torus $\mathbf{S}$ of $\mathbf{G}$ is said to be (k, σ)-split if it is k-split and if $\sigma(g) = g^{-1}$ for all $g \in \mathbf{S}$. By a theorem of Helminck and Wang [32], there are only finitely many H-conjugacy classes of maximal (k, σ)-split tori of $\mathbf{G}$. Choose a set $\{\mathbf{A}_i : 1 \leq i \leq m\}$ of representatives of H-conjugacy class of maximal (k, σ)-split tori of $\mathbf{G}$ and set $A = \cup_{i=1}^m \mathbf{A}_i(k)$.

The following lemma was proved in [21] for $k = \mathbb{R}$, in [4] for all local fields of characteristic not 2 (and independently in [13] when the residual characteristic is not 2).

Lemma 3.3 (Polar decomposition of symmetric spaces). *There exists a compact subset K of G such that*

$$G = HAK \ .$$

The second lemma we need is based on the work of Helminck and Wang.

Let $\mathbf{A}$ be a maximal (k, σ)-split torus of $\mathbf{G}$ and $\mathbf{L}$ the centralizer of $\mathbf{A}$ in $\mathbf{G}$. The set of roots $\Phi = \Phi(\mathbf{G}, \mathbf{A})$ for the action of $\mathbf{A}$ on the Lie algebra of $\mathbf{G}$ is a root system. For every positive root system $\Phi^+ \subset \Phi$, let $\mathbf{N}$ (resp. $\mathbf{N}^-$) be the unipotent subgroup of $\mathbf{G}$ generated by the root groups $\mathbf{U}_\alpha$ (resp. $\mathbf{U}_{-\alpha}$), for $\alpha \in \Phi^+$, let $\mathbf{P} := \mathbf{LN}$ (resp. $\mathbf{P}^- := \mathbf{LN}^-$) and $\mathbf{A}_k^+$ the Weyl Chamber :

$$\mathbf{A}_k^+ := \{a \in \mathbf{A}_k \mid |\alpha(a)| \le 1 \text{ , for all } \alpha \in \Phi^+\} \ .$$

When Φ^+ vary, the Weyl chambers form a finite covering of $\mathbf{A}_k$. Since $\mathbf{P}^- = \sigma(\mathbf{P})$, the parabolic k-subgroups $\mathbf{P}$ are σ-split, i.e., the product $\mathbf{HP}$ is open in $\mathbf{G}$ [32, Prop. 4.6 and 13.4]. Conversely, any minimal σ-split parabolic k-subgroups of $\mathbf{G}$ containing $\mathbf{A}$ can be constructed in this way for a suitable choice of Φ^+.

Lemma 3.4.   (1) *The multiplication map $m : \mathbf{H}_k \times \mathbf{P}_k \to \mathbf{G}_k$ is an open map.*
  (2) *There exists a basis of compact neighborhoods W of e in $\mathbf{P}_k$ such that*

$$a^{-1}Wa \subset W \quad \text{for all } a \in \mathbf{A}_k^+.$$

  (3) *For every neighborhood U of e in G, there exists a neighborhood V of e in G such that*
$$HVa \subset HaU \quad \text{for all } a \in \mathbf{A}_k^+.$$

Proof. (1) When $\mathrm{char}(k) = 0$, it follows from the fact that Lie algebras of $\mathbf{P}_k$ and $\mathbf{H}_k$ generate the Lie algebra of $\mathbf{G}_k$ as a vector space. For a characteristic free argument, see [32] or Proposition I.2.5.4 in [39].

(2) When $\mathrm{char}(k) = 0$, note that the action of $\mathbf{A}_k^+$ on the Lie algebra $\mathrm{Lie}(\mathbf{P}_k)$ gives a family of commuting semisimple linear maps $\mathrm{Ad}(a)$ whose eigenvalues have bounded above by 1 in their absolute values. It follows that there exists a basis of compact neighborhoods W_0 of 0 in $\mathrm{Lie}(\mathbf{P}_k)$ which are invariant by all $\mathrm{Ad}(a)$, $a \in \mathbf{A}_k^+$. It suffices to set $W = \exp(W_0)$.

It is easy to adapt this argument in positive characteristic case; write $\mathbf{P}_k = \mathbf{L}_k \mathbf{N}_k$, and note that $\mathbf{L}_k$ contains a $\mathbf{A}_k$-invariant compact open subgroup. Now considering the linear group $\mathbf{N}_k$ as a group of upper triangular matrices in a suitable basis where elements of $\mathbf{A}_k^+$ are diagonals with increasing coefficients in absolute value, we can find a basis of compact neighborhoods W as desired.

(3) Choose W as in (2) small enough so that $W \subset U$ and choose any neighborhood V of e in G contained in HW. We then have $HVa \subset HWa \subset HaU$, as required. $\qquad\square$

Proof of Proposition 3.2.   We will prove that G has the wavefront property on $H \backslash G$ with the subset $F = AK$ defined in Lemma 3.3. By Lemma 3.3, it suffices

to show that *for every neighborhood U of e in G, there exists a neighborhood V of e in G such that $HVg \subset HgU$ for all $g \in AK$*

Recall $A = \cup_{i=1}^{m} \mathbf{A}_i(k)$ where $\mathbf{A}_i$ is a maximal (k, σ)-split torus of G. Fix i and a positive Weyl chamber C of $\mathbf{A}_i(k)$.

Since K is a compact set, there exists a neighborhood U_0 of e in G such that $k^{-1}U_0 k$ is contained in U for all k in K. By Lemma 3.4.(3), there exists a neighborhood V_C of e in G such that $V_C a \subset HaU_0$ for all $a \in C$.

Now set $V := \cap_C V_C$ where the intersection is taken over all (finitely many) positive Weyl chambers of $\mathbf{A}_i(k)$, $1 \le i \le m$. Then for $g = ak \in (\cup_C C)K = AK$ with $k \in K$ and $a \in C$, we have

$$HVg \subset HV_C ak \subset HaU_0 k \subset HakU = HgU \ .$$

This finishes the proof.

In section 9, we will use this wavefront property in the product situation, owing to the following straightforward proposition.

Proposition 3.5. *For $i = 1, \dots, m$, let G_i be a locally compact group, $H_i \subset G_i$ a closed subgroup, $G := \prod_{i=1}^{m} G_i$ and $H := \prod_{i=1}^{m} H_i$. If G_i has the wavefront property on $H_i \backslash G_i$ for each $1 \le i \le m$, then G has the wavefront property on $H \backslash G$.*

The following theorem is an immediate consequence of Propositions 2.4, 2.5, 3.2 and 3.5.

Theorem 3.6. *For $i = 1, .., m$, let k_i be a local field, G_i the group of k_i-points of a semisimple k_i-group, σ_i an involution of G_i defined over k_i, $G_i^{\sigma_i}$ its group of fixed points and H_i a closed subgroup of finite index of $G_i^{\sigma_i}$. Let $G = \prod_{i=1}^{m} G_i$ and $H := \prod_{i=1}^{m} H_i$.*
  Then the group G has the wavefront property on $H \backslash G$.
  Moreover, for any irreducible lattice Γ in G, the action of G on $\Gamma \backslash G$ is mixing.

Note that this theorem provides many natural examples of triples (G, H, Γ) which satisfy the hypothesis of Theorem and Propositions 4.1, 5.3 and 6.2.

4. Equidistribution of translates of H-orbits

In this section, let G be a locally compact group, $H \subset G$ a closed subgroup, $\Gamma \subset G$ a lattice such that $\Gamma_H := \Gamma \cap H$ is a lattice in H. Set $X = \Gamma \backslash G$ and $Y = \Gamma_H \backslash H$. Let μ_X and μ_Y be invariant measures on X and Y respectively. As in [21, Theorem 2.1], we have

Theorem 4.1. *Suppose that the action of G on X is mixing and that G has the wavefront property on $H \backslash G$. Then the translates Yg become equidistributed in X, as $g \to \infty$ in $H \backslash G$.*

This means that as the image of g in $H\backslash G$ leaves every compact subsets, the sequence of probability measures $\frac{1}{\mu_Y(Y)}g_*\mu_Y$ weakly converges to $\frac{1}{\mu_X(X)}\mu_X$, i.e., for any $\psi \in C_c(X)$, we have

$$(4.2) \qquad \frac{1}{\mu_Y(Y)}\int_Y \psi(yg)d\mu_Y(y) \to \frac{1}{\mu_X(X)}\int_X \psi\, d\mu_X .$$

Proof. The following proof is adapted from [21]; we point out that the case when Y is non-compact requires a bit more care, which was not addressed in [21]. Since $G = HF$, we may assume that g belongs to the subset F in Definition 3.1. We assume, without loss of generalities, that μ_X and μ_Y are probability measures. Let $\psi \in C_c(X)$. Fix $\varepsilon > 0$. By the uniform continuity of ψ there exists a neighborhood U of e in G such that

$$(4.3) \qquad |\psi(xu) - \psi(x)| < \varepsilon \qquad \text{for all } u \in U \text{ and } x \in X.$$

By the wavefront property of G on $H\backslash G$, there exists a compact neighborhood $V \subset U$ of e in G such that

$$(4.4) \qquad Vg \subset HgU \quad \text{for all } g \in F$$

Choose a compact subset $Y_\epsilon \subset Y$ of measure at least $\mu_Y(Y_\epsilon) \geq 1 - \varepsilon$. Choose a Borel subset $W \subset V$ in G transversal to H, i.e., a subset W of G such that the multiplication $m : H \times W \to G$ is injective with the image HW being an open neighborhood of e in G. Using the compactness of Y_ϵ and the discreteness of Γ, we may assume that the image of W in $H\backslash G$ is small enough so that the multiplication $m : Y_\epsilon \times W \to Y_\epsilon W$ is a bijection[1] onto its image $Y_\varepsilon W \subset X$.

Let μ_W be the measure on W such that $\mu_X = \mu_Y\mu_W$ locally.

Setting

$$I_g := \int_Y \psi(yg)d\mu_Y(y),$$

we need to show that

$$(4.5) \qquad I_g \to \int_X \psi\, d\mu_X \quad \text{as } g \in F \text{ goes to infinity in } G.$$

---

[1] When Y is compact, one can choose the transversal W such that the map $Y \times W \to YW$ is bijective onto an open subset of X. When Y is not compact, such a transversal does not always exist. Here is an example: let G be the orthogonal group of the quadratic form $x^2 + y^2 + z^2 - t^2$ on $\mathbb{R}^4$, $v_0 = (1,0,0,0)$, $v_1 = (1,0,2,2)$, $\Gamma = G_\mathbb{Z}$ and H the stabilizer of the point v_0. One checks easily that (a) $v_1 = \gamma v_0$ for some $\gamma \in \Gamma$ and that (b) v_0 is a limit of elements $v_1 h_n$ of the H-orbit of v_1. Hence there exists a sequence g_n converging to e in G such that $Hg_n \cap \gamma H \neq \emptyset$. To check (a), take $\gamma = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 2 \\ 2 & 0 & 2 & 3 \end{pmatrix}$. For (b), take $h_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh n & -\sinh n \\ 0 & 0 & -\sinh n & \cosh n \end{pmatrix}$.

For simplicity, set

$$J_g = \frac{1}{\mu_W(W)} \int_{Y \times W} \psi(ywg) \, d\mu_Y(y) \, d\mu_W(w) \text{ and } K_g = \frac{1}{\mu_W(W)} \int_{Y_\varepsilon W} \psi(xg) \, d\mu_X(x).$$

Roughly speaking, we will argue that I_g is close to J_g as a consequence of the wavefront property, J_g is close to K_g since the volume of $Y - Y_\varepsilon$ is small, and finally K_g is close to the average of ψ for large g because of the mixing property.

By (4.4), for each $w \in W$ and $g \in F$, we have $wg = h_{g,w} g u$ for some $h_{g,w} \in H$. Hence

$$\begin{aligned}
|I_g - \int_Y \psi(ywg) \, d\mu_Y(y)| &= |\int_Y \psi(yg) \, d\mu_Y(y) - \int_Y \psi(yh_{g,w}gu) \, d\mu_Y(y)| \\
&= |\int_Y (\psi(yg) - \psi(ygu)) \, d\mu_Y(y)| \leq \varepsilon \quad \text{by (4.3).}
\end{aligned}$$

Therefore we have

$$|I_g - J_g| \leq \varepsilon.$$

By the choice of W, we have

$$K_g = \frac{1}{\mu_W(W)} \int_W \int_{Y_\varepsilon} \psi(ywg) \, d\mu_Y(y) \, d\mu_W(w)$$

and hence

$$|J_g - K_g| \leq 2\mu_Y(Y - Y_\varepsilon) \, \|\psi\|_\infty \leq 2\|\psi\|_\infty \, \varepsilon.$$

Since $K_g = \frac{1}{\mu_W(W)} \int_X \psi(xg) \, \mathbf{1}_{Y_\varepsilon W}(x) d\mu_X(x)$ where $\mathbf{1}_{Y_\varepsilon W}$ is the characteristic function of WY_ε, the mixing property of G on $\Gamma \backslash G$ says that K_g converges to $\mu_Y(Y_\varepsilon) \int_X \psi d\mu_X$ as $g \to \infty$ in F. Hence for $g \in F$ large enough we have,

$$|K_g - \int_X \psi \, d\mu_X| \leq \varepsilon + \mu_Y(Y - Y_\varepsilon) \int_X \psi \, d\mu_X \leq (1 + \|\psi\|_\infty) \, \varepsilon.$$

Putting this together, we get

$$\begin{aligned}
|I_g - \int_X \psi \, d\mu_X| &\leq |I_g - J_g| + |J_g - K_g| + |K_g - \int_X \psi \, d\mu_X| \\
&\leq (2 + 3\|\psi\|_\infty) \, \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows the claim. □

Using Theorem 3.6, we obtain:

Corollary 4.6. *Let G, H, Γ be as in Theorem 3.6. Then the translates $Yg := \Gamma_H \backslash Hg$ become equidistributed in $X := \Gamma \backslash G$ as $g \to \infty$ in $H \backslash G$.*

5. Sums and integrals

As in Section 4, let G be a locally compact group, $H \subset G$ a closed subgroup, $\Gamma \subset G$ a lattice such that $\Gamma_H := \Gamma \cap H$ is a lattice in H. Let $x_0 := [\Gamma]$ be the base point in $X := \Gamma \backslash G$, $Y = x_0 H$ and $z_0 := [H]$ be the base point in $Z := H \backslash G$. We note that $z_0 \Gamma$ is a discrete subset of Z. There exist G-invariant measures μ_X, μ_Y and μ_Z on X, Y and Z. We normalize them so that $\mu_X = \mu_Z \mu_Y$ locally.

For a given sequence of non-negative functions φ_n on Z with compact support, we define a function F_n on X so that, for $x = x_0 g$, $F_n(x)$ is the sum of φ_n over the discrete orbit $z_0 \Gamma g$:

$$(5.1) \qquad F_n(x) := \sum_{\gamma \in \Gamma_H \backslash \Gamma} \varphi_n(z_0 \gamma g) \quad \text{for } x = \Gamma g.$$

We would like to compare the values of F_n with the space average over Z:

$$(5.2) \qquad I_n := \frac{\mu_Y(Y)}{\mu_X(X)} \int_Z \varphi_n(z) \, d\mu_Z(z)$$

We remark that this normalized integral I_n does not depend on the choices of measures.

The following proposition 5.3 says that the sum F_n is asymptotic to the normalized integral I_n, at least weakly.

Proposition 5.3. *Suppose that the translates Yg become equidistributed in X as $g \to \infty$ in Z. Then for any sequence of non-negative functions φ_n on Z with compact support such that $\max_n \|\varphi_n\|_\infty < \infty$ and $\lim_{n \to \infty} \int_Z \varphi_n \, d\mu_Z = \infty$, the ratios $F_n(x)/I_n$ converge weakly to 1 as $n \to \infty$.*

This means that, for all $\alpha \in C_c(X)$,

$$(5.4) \qquad \lim_{n \to \infty} \frac{1}{I_n} \int_X F_n(x) \, \alpha(x) d\mu_X(x) = \int_X \alpha(x) d\mu_X(x) \, .$$

Proof. Using transitivity properties for invariant integration on homogeneous spaces, we obtain that for all $\alpha \in C_c(X)$,

$$
\begin{aligned}
\int_{\Gamma \backslash G} F_n \alpha \, d\mu_{\Gamma \backslash G} &= \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma_H \backslash \Gamma} \varphi_n(H \gamma g) \alpha(\Gamma g) d\mu_{\Gamma \backslash G}(\Gamma g) \\
&= \int_{\Gamma_H \backslash G} \varphi_n(Hg) \alpha(\Gamma g) d\mu_{\Gamma_H \backslash G}(\Gamma_H g) \\
&= \int_{H \backslash G} \int_{\Gamma_H \backslash H} \varphi_n(Hg) \alpha(\Gamma_H h g) d\mu_{\Gamma_H \backslash H}(\Gamma_H h) d\mu_{H \backslash G}(Hg) \\
&= \int_{H \backslash G} \varphi_n(z) \beta(z) d\mu_{H \backslash G}(z)
\end{aligned}
$$

where β is the function on Z given by,

$$
\begin{aligned}
\beta(Hg) &= \int_{\Gamma_H \backslash H} \alpha(\Gamma_H h g) d\mu_{\Gamma_H \backslash H}(\Gamma_H h) \\
&= \int_Y \alpha(yg) d\mu_Y(y) \, .
\end{aligned}
$$

By assumption, we have

$$
\lim_{z \to \infty} \beta(z) = \frac{\mu_Y(Y)}{\mu_X(X)} \int_X \alpha(x) d\mu_X(x) \, .
$$

Since $\int_Z \varphi_n \to \infty$ and since the functions φ_n are uniformly bounded, by the dominated convergence theorem, we deduce

$$
\lim_{n \to \infty} \frac{1}{I_n} \int_Z \varphi_n(z) \beta(z) d\mu_Z(z) = \int_X \alpha \, d\mu_X.
$$

Hence we obtain the equality (5.4). $\qquad\square$

6. Counting and equidistribution

We will now improve the weak convergence in Proposition 5.3 to the pointwise convergence of the functions F_n. This requires some hypothesis on the sequence of functions φ_n which will be called *well-roundedness*. We keep the notations of section 5.

Definition 6.1. *A sequence or a family of non-negative integrable functions φ_n of Z with compact support is said to be well-rounded if for any $\varepsilon > 0$, there exists a neighborhood U of e in G, such that the following holds for all n.*

$$
(1-\varepsilon) \int_Z (\sup_{u \in U} \varphi_n(zu)) \, d\mu_Z(z) \leq \int_Z \varphi_n \, d\mu_Z \leq (1+\varepsilon) \int_Z (\inf_{u \in U} \varphi_n(zu)) \, d\mu_Z(z).
$$

A sequence of compact subsets B_n of Z is said to be well-rounded if the sequence $\mathbf{1}_{B_n}$ is well-rounded.

Sometimes we will apply the above definition to a continuous family $\{\varphi_T\}$ of functions or subsets, whose meaning should be clear.

Recall that we want to compare the orbital sum $F_n(x_0) = \sum_{\gamma \in \Gamma/\Gamma_H} \varphi_n(\gamma z_0)$ with the average $I_n = \frac{\mu_Y(Y)}{\mu_X(X)} \int_Z \varphi_n(z) d\mu_Z(z)$.

Proposition 6.2. *Keep the notations and hypothesis of Proposition 5.3, and assume that the sequence φ_n is well-rounded. Then,*

$$F_n(x_0) \sim I_n \quad as \ n \to \infty.$$

The notation $a_n \sim b_n$ means that the ratio of a_n and b_n tends to 1 as $n \to \infty$.

Proof. Once again, we may normalize the measures so that $\mu_X(X) = \mu_Y(Y) = 1$. Fix $\varepsilon > 0$ and let U be a neighborhood of e in G given by Definition 6.1. We introduce the functions $\varphi_n^{\pm}$ on Z defined by

$$\varphi_n^+(z) := \sup_{u \in U} \varphi_n(zu^{-1}) \quad \text{and} \quad \varphi_n^-(z) := \inf_{u \in U} \varphi_n(zu^{-1})$$

and their integrals $I_n^{\pm} := \int_Z \varphi_n^{\pm} d\mu_Z$. Note that for each n,

(6.3) $$(1 - \varepsilon)I_n^+ \leq I_n \leq (1 + \varepsilon)I_n^- .$$

We also introduce the functions $F_n^{\pm}$ on X:

$$F_n^{\pm}(x) = \sum_{\gamma \in \Gamma_H \backslash \Gamma} \varphi_n^{\pm}(z_0 \gamma g) \quad \text{for } x = \Gamma g.$$

It is easy to check that, for all $u \in U$ and $x \in X$

$$F_n^-(xu) \leq F_n(x) \leq F_n^+(xu) .$$

Choose a non-negative continuous function α on X with $\int_X \alpha = 1$ and with support included in $x_0 U$ so that the following holds for all n:

$$\int_X \alpha \, F_n^- \, d\mu_X \leq F_n(x_0) \leq \int_X \alpha \, F_n^+ \, d\mu_X .$$

Applying Proposition 5.3 to the sequences of functions $\varphi_n^{\pm}$, we obtain, for all n large,

(6.4) $$(1 - \varepsilon)I_n^- \leq F_n(x_0) \leq (1 + \varepsilon)I_n^+ .$$

Using the estimations (6.3) and (6.4), every cluster value of the sequence of ratios $F_n(x_0)/I_n$ is within the interval $[\frac{1-\varepsilon}{1+\varepsilon}, \frac{1+\varepsilon}{1-\varepsilon}]$. Hence this sequence converges to 1. $\square$

7. Volume of balls

In this section we study the property of the volume of balls which are needed in sections 8 and 14. These estimates will be consequences of the following two general theorems 7.1 and 7.6.

Volume of balls over the reals. We will first need a variation of a theorem on fiber integration. This theorem says that the volume of the fibers of an analytic function has a, term-by-term differentiable asymptotic expansion in the scale of functions $t^j(\log t)^k$ with j rational and k non-negative integer. More precisely,

Theorem 7.1. [34] *Let $X \subset \mathbb{R}^m$ be a smooth real analytic variety, $f : X \to \mathbb{R}$ a real analytic function and ν a C^∞ measure on X. Then, for any compact K of X, there exist $d \in \mathbb{N}$ and a set $\{A_{j,k} : j \in \frac{1}{d}\mathbb{N}, k \in \mathbb{Z}, 0 \le k < m\}$ of distributions on X supported by $f^{-1}(0)$ such that, for every C^∞ function $\varphi : X \to \mathbb{R}$ with support in K, the integral*

$$v_\varphi(t) := \int_{0 \le f(x) \le t} \varphi(x)\, d\nu(x)$$

has a term-by-term differentiable asymptotic expansion when $t > 0$ goes to 0

$$\sum_{j \in \frac{1}{d}\mathbb{N}} \sum_{0 \le k < m} A_{j,k}(\varphi)\, t^j(\log t)^k$$

This means that, for every $j_0 \ge 0$, defining v_{φ, j_0} by truncating the above sum:

$$v_{\varphi, j_0}(t) := \sum_{\substack{0 < j \le j_0 \\ j \in \frac{1}{d}\mathbb{N}}} \sum_{0 \le k < m} A_{j,k}(\varphi)\, t^j(\log t)^k,$$

one has $\left(\frac{d}{dt}\right)^\ell (v_\varphi - v_{\varphi, j_0}) = o(t^{j_0 - \ell})$ for every $\ell \ge 0$.

Remark This theorem is stated for a smooth analytic variety Z and a smooth measure μ. Its proof is based on the real version of Hironaka's resolution of singularity as in [1]. Using once more Hironaka's theorem it can be applied to a singular analytic variety X with a measure μ associated to a meromorphic differential form. Here is one example of such an application:

Proposition 7.2. *Let Z be a connected component of the real points of a smooth real affine algebraic variety $\mathbf{Z}$ and μ a measure on Z which is defined by a regular differential form of $\mathbf{Z}$. Let $F : Z \to \mathbb{R}$ be a positive proper regular function and set $v_T := \mu(\{z \in Z : F(z) \le T\})$. Then, there exist positive integers ℓ_0, d and constants $a_{j,k}$ such that v_T has a term-by-term differentiable asymptotic expansion when $T \to \infty$*

$$\sum_{\substack{j \le \ell_0 \\ j \in \frac{1}{d}\mathbb{Z}}} \sum_{0 \le k < m} a_{j,k} T^j(\log T)^k.$$

The condition F *regular* means that F is the restriction to Z of a regular function on the algebraic variety $\mathbf{Z}$.

Proof. Using the resolution of singularities we can view $\mathbf{Z}$ as an open real algebraic subvariety of a smooth projective variety $\mathbf{X}$ such that the boundary $\mathbf{Y} := \mathbf{X} - \mathbf{Z}$ is a divisor with normal crossing. Hence, in a neighborhood U_{y_0} of each real point y_0 of $\mathbf{Y}$, there is a real local coordinate system $(x_1, \ldots, x_m)$ such that $\mathbf{Y}$ is given by $x_1 \cdots x_r = 0$, for some $r \le m$. We are only interested in those points y_0 in the closure of Z. Near these points, the meromorphic function $f := 1/F$ is zero on $\mathbf{Y}$. Using a partition of unity associated to such a cover, we are reduced to a local problem. Namely proving, for every C^∞ function φ with compact support in U_{y_0}, the existence of a term-by-term differentiable asymptotic expansion for

$$v_{\varphi,\mu}(t) := \int_{f(x) \le t} \varphi(x) \, d\mu(x)$$

when $t := 1/T$ goes to 0. It is equivalent to prove the existence of such an asymptotic expansion for the derivative $v'_{\varphi,\mu}(t)$ which is called the *integral of μ on the fiber $f^{-1}(t)$*.

Using once more the resolution of singularities for the numerator and denominator of f and a new partition of unity, we can assume that f is monomial in these coordinate systems. Using the fact that f is positive on Z and zero on its boundary, we deduce that f is given by $f = s x_1^{p_1} \cdots x_r^{p_r}$ with $p_1, \ldots, p_r$ positive integers and $s = \pm 1$. Hence f is an analytic function near y_0.

The integral $v_{\varphi,\mu}$ is now very similar to the integral v_φ of Theorem 7.1 except that μ may not be smooth. However μ is defined by a regular differential form on $\mathbf{Z}$ hence there exists a positive integer ℓ_0 such that the measure $\nu := f^{\ell_0} \mu$ is smooth. According to Theorem 7.1, $v_{\varphi,\nu}$ has a term-by-term differentiable asymptotic expansion. The following equality between the *fiber integrals*

$$v'_{\varphi,\mu}(t) = t^{-\ell_0} v'_{\varphi,\nu}(t)$$

implies that $v_{\varphi,\mu}$ has also a term-by-term differentiable asymptotic expansion. $\square$

For us, the main example to which we will apply Proposition 7.2 is a closed orbit Z under the group of $\mathbb{R}$-points of a $\mathbb{R}$-algebraic group, an invariant measure μ on this orbit and the restriction F to Z of the square of an euclidean norm on $\mathbb{R}^m$. Hence we get,

Corollary 7.3. *Let Z be a closed orbit of the group G of $\mathbb{R}$-points of an $\mathbb{R}$-algebraic group acting algebraically on a $\mathbb{R}$-vector space V, μ an invariant measure on Z and $\| \cdot \|$ an euclidean norm on V. Set $B_T := \{z \in Z : \|z\| \le T\}$ and $v_T := \mu(B_T)$.*
a) Then $v_T \sim c\, T^a (\log T)^b$, as $T \to \infty$, where $a \in \mathbb{Q}_{\ge 0}$, $b \in \mathbb{Z}_{\ge 0}$ and $c > 0$.
b) Moreover $\frac{d}{dT} v_T \sim c \frac{d}{dT}(T^a (\log T)^b)$, as $T \to \infty$.

c) *For any $k_0 > 0$, there exists $\delta_0 > 0$ such that one has, as $T \to \infty$,*

$$\int_{B_T} \|z\|^{-k_0} d\mu(z) = O(v_T^{1-\delta_0}).$$

d) *If G is semisimple and Z is non compact then one has $a \neq 0$.*

Alex Gorodnik informed us that he also had similar ideas with Nevo for a proof of Corol. 7.3 based on Hironaka's resolution of singularities. Their arguments appeared in Section 7.5 of [26].

Remarks - When Z is a symmetric variety, the point a) is proven in [28, Corollary 6.10] for any norm on V and the parameters a and b are explicitly given.
- When G is a group of diagonal matrices, the constant a is zero.

Proof. a) and b) This is a special case of Proposition 7.2. Note that since v_T is an increasing function of T, one has $a \geq 0$. Moreover, note that, when $a = b = 0$, the orbit is of finite volume hence compact.

c) Set $u_T := \int_{B_T} \|z\|^{-k_0} d\mu(z)$. By a), one has $v_T = O(T^{m_0})$ for some $m_0 > 0$. Hence the derivative $u_T' = T^{-k_0} v_T'$ satisfies $u_T' = O(v_T^{-k_0/m_0} v_T')$ and, integrating, one gets $u_T = O(v_T^{1-k_0/m_0})$.

d) This is a special case of the following Proposition 7.4 □

Proposition 7.4. *With the notations of Corollary 7.3, one has the equivalence:*
All unipotent elements of G act trivially on Z $\Longleftrightarrow$ $a = 0$.

Proof. $\Longrightarrow$ By assumption the normal subgroup of G generated by the unipotent elements of G acts trivially on Z. Hence one can assume that G is a product of a compact group by a r-dimensional group of diagonal matrices. In this case one has $\mu(B_T) = O((\log T)^r)$ as $T \to \infty$.
$\Longleftarrow$ This implication is a consequence of the following Lemma 7.5. □

Lemma 7.5. *Let U be a one-parameter unipotent subgroup of $\mathrm{GL}_m(\mathbb{R})$, μ a U-invariant measure on $\mathbb{R}^m$ which is not supported by the U-fixed points and set B_T for the euclidean ball of radius T on $\mathbb{R}^m$. Then one has*

$$\liminf_{T \to \infty} \frac{\log(\mu(B_T))}{\log(T)} > 0.$$

Proof. First of all, note that all the orbits Uz of U in $\mathbb{R}^m$ are images of $\mathbb{R}$ by polynomial maps $t \mapsto u_t z$ of degree $d_z \leq m$. We may assume that this degree d_z is μ-almost everywhere non-zero constant. Set $d \geq 1$ for this degree, write $u_t z = t^d v_z + O(t^{d-1})$ for some non-zero $v_z \in \mathbb{R}^m$, and note that the constant involved in this $O(t^{d-1})$ is uniform on compact subsets of $\mathbb{R}^m$.

One can find a compact subset $C \subset \mathbb{R}^m$ transversal to the U-action such that $\mu(UC) > 0$. The pull-back on $\mathbb{R} \times C$ of the measure μ by the action $(t, z) \mapsto u_t z$ has the form $dt \otimes \nu$ where dt is the Lebesgue measure on $\mathbb{R}$ and ν is a non-zero measure on C. Choose $c > \sup_{z \in C} \|v_z\|$. Then, for R large, one has

$$u_{[0,R]}(C) \subset B_{cR^d}$$

and hence $\mu(B_{cR^d}) \geq R\nu(C)$. This proves our claim. $\qquad\square$

Volume of balls over the p-adics. We will also need Denef's theorem on p-adic integration. For that we need some notations. A subset of $\mathbb{Q}_p^m$ is said *semialgebraic* if it is obtained by boolean operations from sets $P_{f,r} := \{x \in \mathbb{Q}_p^m \,/\, \exists y \in \mathbb{Q}_p \,:\, f(x) = y^r\}$ with f a polynomial in m variables with coefficients in $\mathbb{Q}_p$ and $r \geq 2$. According to Macintyre's theorem, which is the p-adic analog of Tarski-Seidenberg theorem, those sets are exactly the definable sets of the field $\mathbb{Q}_p$ [37]. A function f between two $\mathbb{Q}_p$-vector spaces is said *semialgebraic* if its graph is semialgebraic. According to Denef's cell decomposition theorem ([15] and [10]), for every semialgebraic subset S, there exists a finite partition of S in semialgebraic sets $S_1, \ldots, S_{j_{max}}$ (called cells) such that, for each $j = 1, \ldots, j_{max}$, S_j is in semialgebraic bijection with a semialgebraic open subset O_j of a vector space $\mathbb{Q}_p^{d_j}$ (recently, R. Cluckers has shown the existence of a semialgebraic bijection between S itself and some $\mathbb{Q}_p^d$). A measure μ on S is said *semialgebraic* if there exists a cell decomposition of S on each cell of which μ is of the form $|g_j(x)|dx$ where g_j is a semialgebraic function on $\mathbb{Q}_p^{d_j}$ and dx is a Haar measure on $\mathbb{Q}_p^{d_j}$. A function $a : \mathbb{Z} \to \mathbb{Z}$ is said *simple* if there are finite partition of $\mathbb{N}$ and $-\mathbb{N}$ by finite sets and arithmetic progressions on which a is affine, see [14, §2.13, 2.14 and 4.4].

Theorem 7.6. [14, Theorem 3.1] *Let μ be a semialgebraic measure on an m-dimensional semialgebraic subset S over $\mathbb{Q}_p$ and f be a semialgebraic function on S. For $n \in \mathbb{Z}$, set*

$$I_n := \int_{|f(x)|=p^n} d\mu(x)$$

when this integral is finite and $I_n = 0$ otherwise. Then, for all $n \in \mathbb{Z}$, one has

$$I_n = \sum_{1 \leq i \leq e} \gamma_i(n) p^{\beta_i(n)}$$

where $e \in \mathbb{N}$, $\beta_i : \mathbb{Z} \to \mathbb{Z}$ is a simple function and $\gamma_i : \mathbb{Z} \to \mathbb{Z}$ is a product of at most m simple functions for each $1 \leq i \leq e$.

For instance, an orbit under the group of $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-algebraic group acting algebraically is definable and hence semialgebraic, by Macintyre's theorem, and an invariant measure on this orbit is semialgebraic. Hence one gets:

Corollary 7.7. *Let k be a finite extension of $\mathbb{Q}_p$, q the absolute value of an uniformizer, G the group of k-points of an algebraic k-group, $\rho : G \to GL(V)$ a representation of G defined over k, Z a closed G-orbit in V, μ an invariant measure on Z and $\|\cdot\|$ a max norm on V. Denote by S_T the sphere $S_T = \{z \in Z : \|z\| = T\}$ and set $v_T := \mu(S_T)$.*
a) There exists $N_0 \in \mathbb{N}$ such that, for each $0 \le j_0 < N_0$, one of the following holds:

(1) *S_{q^j} is empty, for $j \equiv j_0 \mod N_0$ large;*
(2) *there exist $a_{j_0} \in \mathbb{Q}_{\ge 0}$, $b_{j_0} \in \mathbb{Z}_{\ge 0}$, and $c_{j_0} > 0$ such that,*

$$v_{q^j} \sim c_{j_0}\, q^{a_{j_0} j}\, j^{b_{j_0}} \qquad \text{for } j \equiv j_0 \mod N_0 \;\; large.$$

b) For any $k_0 > 0$, there exists $\delta_0 > 0$ such that one has, as $T \to \infty$,

$$\int_{S_T} \|z\|^{-k_0} d\mu(z) = O(v_T^{1-\delta_0}).$$

c) If G is semisimple and Z is non compact then, for all j_0 in case (2), one has $a_{j_0} \ne 0$.

Remarks - Let us recall that a max norm is a norm given in some basis $e_1, \dots, e_m$ by $\|\sum x_i e_i\| = \max |x_i|$.
- When G is a group of diagonal matrices, all the constants a_{j_0} are zero.

Proof. Viewing V as a $\mathbb{Q}_p$ vector space, we may assume that $k = \mathbb{Q}_p$.
  a) This is a special case of Theorem 7.6.
  b) By a), there exists $m_0 > 0$ such that $v_T = O(T^{m_0})$. Hence one has $\int_{S_T} \|z\|^{-k_0} d\mu(z) = T^{-k_0} v_T = O(v_T^{1-k_0/m_0})$.
  c) This is a special case of the following Proposition 7.8 which is analogous to Proposition 7.4. $\qquad\square$

Proposition 7.8. *With the notations of Corollary 7.7, the following are equivalent:*

(i) *All unipotent elements of G act trivially on Z,*
(ii) *For all j_0 in case (2), one has $a_{j_0} = 0$,*
(iii) *Either Z is compact or, for some j_0 in case (2), one has $a_{j_0} = 0$.*

Proof. The proof is as in Proposition 7.4.
  $(i) \Rightarrow (ii)$ By assumption the normal subgroup of G generated by the unipotent elements of G acts trivially on Z. Hence one can assume that G is a product of a compact group by an r-dimensional group of diagonal matrices. In this case, one has $\mu(S_{p^j}) = O(j^r)$ as $j \to \infty$.
  $(ii) \Rightarrow (iii)$ If Z is non compact, at least one j_0 is in case (2).
  $(iii) \Rightarrow (i)$ This implication is a consequence of the following Lemma 7.9. $\quad\square$

Lemma 7.9. *Let k be a finite extension of $\mathbb{Q}_p$, U a one-parameter unipotent subgroup of $\mathrm{GL}(m,k)$, μ a U-invariant measure on k^m which is not supported by the U-fixed points and denote by S_T the sphere of radius T on k^m for the max norm. Then one has, as $T \to \infty$ subject to the condition $\mu(S_T) \neq 0$,*

$$\liminf \frac{\log(\mu(S_T))}{\log(T)} > 0.$$

Proof. The proof is as in Lemma 7.5. First of all, note that all the orbits Uz of U in k^m are images of k by polynomial maps $t \mapsto u_t z$ of degree $d_z \leq m$. We may assume that this degree d_z is μ-almost everywhere non-zero constant. Set $d \geq 1$ for this degree and write $u_t z = t^d v_z + O(t^{d-1})$ for some non-zero $v_z \in k^m$. Let q be the absolute value of an uniformizer. The set $\{j \in \mathbb{N} : Uz \cap S_{q^j} \neq \emptyset\}$ is then equal, up to finite sets, to some arithmetic progression $j_z + \mathbb{N}d$ with $0 \leq j_z < d$. We may assume that this integer j_z is μ-almost everywhere constant. Set j_0 for this integer.

One can find a compact subset $C \subset k^m$ transversal to the U-action such that $\mu(UC) > 0$ and on which $\|v_z\|$ is constant equal to some power $q^{j_0 + dm_0}$ with $m_0 \in \mathbb{N}$. The pull-back on $k \times C$ of the measure μ by the action $(t,z) \mapsto u_t z$ has the form $dt \otimes \nu$ where dt is a Haar measure on k and ν is a non-zero measure on C. Then, for $|t| = q^\ell$ large, one has

$$u_t(C) \subset S_{q^{j_0 + dm_0 + d\ell}}$$

and hence $\mu(S_{q^{j_0 + dm_0 + d\ell}}) \geq (q-1)\, q^{\ell-1}\, \nu(C)$. This proves our claim. $\qquad\square$

8. Well-roundedness

In this section, we provide explicit examples of well-rounded sequences φ_n in order to apply Proposition 6.2. We start with an observation that the product of well-rounded sequences is again well-rounded.

Example 8.1. *For each $i = 1, \ldots, m$, let G_i be a locally compact group, $H_i \subset G_i$ a closed subgroup and $\varphi_{i,n}$ be a well-rounded sequence of functions on $Z_i := H_i \backslash G_i$. Let $G := \prod_{i=1}^m G_i$, $H := \prod_{i=1}^m H_i$ and $Z := \prod_{i=1}^m Z_i$. Then the sequence φ_n defined by*

$$\varphi_n(z_1, \ldots, z_m) = \prod_{1 \leq i \leq m} \varphi_{i,n}(z_i)$$

is well-rounded.

Proof. Fix $\varepsilon > 0$. Let U_i be a neighborhood of e in G_i such that the functions $\varphi_{i,n}^{\pm}$ on Z_i defined by $\varphi_{i,n}^+(z_i) = \sup_{u_i \in U_i} \varphi_{i,n}(z_i u_i)$ and $\varphi_{i,n}^-(z_i) = \inf_{u_i \in U_i} \varphi_{i,n}(z_i u_i)$ satisfy

$$(1-\varepsilon) \int_{Z_i} \varphi_{i,n}^+ \leq \int_{Z_i} \varphi_{i,n} \leq (1+\varepsilon) \int_{Z_i} \varphi_{i,n}^- .$$

Let $U := \prod U_i$, and $\varphi_n^{\pm}$ be the functions on Z defined by

$$\varphi_n^+(z) = \sup_{u \in U} \varphi_n(zu) \quad \text{and} \quad \varphi_n^-(z) = \inf_{u \in U} \varphi_n(zu)$$

so that $\varphi_n^{\pm} = \prod \varphi_{i,n}^{\pm}$ and

$$(1 - \varepsilon)^m \int_Z \varphi_n^+ \leq \int_Z \varphi_n \leq (1 + \varepsilon)^m \int_Z \varphi_n^- .$$

Hence the sequence φ_n is well-rounded. $\qquad\square$

The next example deals with the constant sequences. It will be used both for the archimedean and the non-archimedean factors.

Example 8.2. *Let G be a locally compact group, H a closed subgroup of G, $Z = H\backslash G$, and $\varphi \in C_c(Z)$ with $\varphi \geq 0$ and $\varphi \neq 0$. Then the constant sequence $\varphi_n = \varphi$ is well-rounded.*

Proof. Use the uniform continuity of φ and the compactness of its support. $\quad\square$

The following example deals with the archimedean factors.

Example 8.3. *Let G be a real semisimple Lie group with finitely many connected components, V a finite dimensional representation of G, Z a closed G-orbit in V with an invariant measure μ and $\|.\|$ an euclidean norm on V. Then the family of balls $B_T := \{z \in Z \mid \|z\| \leq T\}$, $T \gg 1$ is well-rounded.*

Proof. By Corollary 7.3.a of section 7, we have

$$\mu(B_T) \sim_T c\,T^a(\log T)^b$$

for some $a \in \mathbb{Q}_{\geq 0}$, $b \in \mathbb{Z}_{\geq 0}$ and $c > 0$. It is easy to deduce the claim from the above asymptotic using the assumption that the action of G is linear on Z. $\quad\square$

As for the non-archimedean factors, we have:

Example 8.4. *Let k be a non-archimedean local field, G the group of k-points of a connected semisimple k-group, $\rho : G \to \mathrm{GL}(V)$ a representation of G defined over k, Z a closed G-orbit in V with an invariant measure and $\|\cdot\|$ a norm on V. Then, both the family of non empty balls $B_T := \{z \in Z \mid \|z\| \leq T\}$, and the family of non-empty spheres $S_T := \{z \in Z \mid \|z\| = T\}$, are well-rounded.*

Proof. Since the action of G on V is linear, the stabilizer in G of the norm is a compact open subgroup of G. Hence this example is a special case of the following easy assertion. $\quad\square$

Example 8.5. *Let G be a locally compact (unimodular) group, H a closed (unimodular) subgroup of G, $Z = H\backslash G$ and U a compact open subgroup of G. Then any sequence φ_n of non-negative U-invariant L^1-functions on Z is well-rounded.*

As the last example, we will show that a sequence of the height balls is well rounded. We will need the following basic lemma.

Lemma 8.6. *Let I be a finite set. For each $i \in I$, let $\tau_i > 1$ and $\lambda_i > 0$ be given. Let $\lambda : \mathbb{N}^I \to \mathbb{R}^+$ be given by $\lambda(m) = \sum_{i \in I} \lambda_i m_i$ for $m = (m_i)$, and $P : \mathbb{N}^I \to \mathbb{R}^+$ a function given by $P(m) = \prod_{i \in I} P_i(m_i)$ where P_i is a real-valued function of a variable x given by a polynomial expression in (x, τ_i^x) and which is positive on $\mathbb{N}$. Then we have*

$$w_{t+1} = O(w_t) \quad \text{for } t \text{ large}$$

where

$$w_t := \sum_{\{m \in \mathbb{N}^I, \, \lambda(m) \leq t\}} P(m) \,.$$

Proof. Since each P_i is positive on $\mathbb{N}$, there exists $C > 0$ such that for $i \in I$ and $n \in \mathbb{N}$, one has

$$P_i(n+1) \leq C \, P_i(n) \,.$$

Hence for each $m \in \mathbb{N}^I$ and each e in the basis E of $\mathbb{N}^I$, one has

$$P(m + e) \leq C \, P(m) \,.$$

Setting $t_0 := \min_i \lambda_i = \min_{e \in E} \lambda(e)$, one gets

$$w_{t+t_0} \leq \sum_{e \in E} \sum_{\lambda(m) \leq t} P(m + e) \leq rC w_t \,.$$

for $r = |I|$. Hence we conclude that $w_{t+1} \leq (rC)^k w_t$ with $k = \frac{1}{t_0} + 1$. $\qquad\square$

Remark 8.7. One can improve the conclusion of Lemma 8.6: *there exist $a \geq 0$, $b \in \mathbb{Z}_{\geq 0}$ and $c_1, c_2 > 0$ such that, for t large,*

$$c_1 e^{at} t^b \leq w_t \leq c_2 e^{at} t^b \,.$$

Moreover, setting $C_i \tau_i^{d_i x} x^{b_i}$ for the dominant term of $P_i(x)$, the exponents a and b are given respectively by $e^a = \max\limits_{i \in I} \tau_i^{d_i/\lambda_i}$ and b is given by $b = \sum_i (b_i + 1) - 1$ where the sum is taken over all i such that $e^a = \tau_i^{d_i/\lambda_i}$. The proof is a straightforward induction on $|I|$. Here is a sketch: one may assume that, for all i, $P_i(x) = C_i e^{a_i x} x^{b_i}$ and $\lambda_i = 1$. One fixes $i_0 \in I$, set $\check{I} := I - \{i_0\}$ and writes $w_t := \sum_{1 \leq n \leq t} P_{i_0}(n) \check{w}_{t-n}$ where, by induction hypothesis, $\check{w}_t$ satisfies a similar estimation, as $t \to \infty$: $\check{c}_1 e^{\check{a} t} t^{\check{b}} \leq \check{w}_t \leq \check{c}_2 e^{\check{a} t} t^{\check{b}}$ for some $\check{a} \geq 0$, $\check{b} \in \mathbb{Z}_{\geq 0}$ and $\check{c}_1, \check{c}_2 > 0$. From that, one gets the required estimation for w_t.

For the rest of this section, let I be a finite set. For each $i \in I$, let k_i be a local field of characteristic 0, G_i the group of k_i-points of an algebraic k_i-group, V_i an algebraic representation of G_i, and $Z_i \subset V_i$ a non-zero closed G_i-orbit with an invariant measure μ_i.

We set

$$G := \prod_{i \in I} G_i, \ Z := \prod_{i \in I} Z_i, \ \mu := \otimes_{i \in I} \mu_i.$$

Let $I_\infty \subset I$ be the set of indices with k_i archimedean, and $I_f := I \setminus I_\infty$. The partition $I = I_\infty \sqcup I_f$ induces decompositions $G = G_\infty \times G_f$ of the group, $Z = Z_\infty \times Z_f$ of the orbit, and $\mu = \mu_\infty \otimes \mu_f$ of the invariant measure.

Let $\| \cdot \|_i$ be a norm on V_i. We assume that $\| \cdot \|_i$ is euclidean if $i \in I_\infty$ and a max norm otherwise. These norms define a height function $h : Z \to \mathbb{R}^+$

$$h(z) = \prod_{i \in I} \|z_i\|_i.$$

Since each Z_i is a closed non-zero subset in V_i, we have $\min_{z \in Z_i} \|z\|_i > 0$ and hence h is a proper function on Z. We can also write $h = h_\infty \otimes h_f$ where $h_\infty := \prod_{i \in I_\infty} \| \cdot \|_i$ and $h_f := \prod_{i \in I_f} \| \cdot \|_i$.

Set

$$b_T := \{z \in Z_\infty \mid h_\infty(z) \leq T\} \ , \ \beta_T := \{z \in Z_f \mid h_f(z) \leq T\} \,,$$

$$V_T := \mu(B_T) \ , \ v_T = \mu_\infty(b_T) \ \text{and} \ w_T = \mu_f(\beta_T) \,.$$

Lemma 8.8. *Assume that h_∞ is not constant on Z_∞.*

(1) *There exist $a \in \mathbb{Q}_{\geq 0}$, $b \in \mathbb{Z}_{\geq 0}$, and $c > 0$ such that as $T \to \infty$,*

$$v_T \sim c\, T^a (\log T)^b \quad \text{and} \quad \tfrac{d}{dT} v_T \sim c \tfrac{d}{dT} (T^a (\log T)^b).$$

(2) *There exist constants $\kappa > 0$ and $C_1 > 0$ such that for all $\varepsilon \in]0, 1[$ and all $T \geq 0$*

$$v_{(1+\varepsilon)T} - v_T \leq C_1(v_T + 1)\varepsilon^\kappa.$$

(3) *For T large, one has $w_{2T} = O(w_T)$.*

(4) *There exist $\kappa > 0$ such that uniformly for T large and $\varepsilon \in]0, 1[$,*

$$V_{(1+\varepsilon)T} = (1 + O(\varepsilon^\kappa))V_T \,.$$

If $I = I_\infty$ is a singleton, then κ can be taken to be 1.

Proof. (1): Apply Proposition 7.2, to the regular function $F := h_\infty^2$ on the orbit Z_∞. Note that since v_T is an increasing function of T, one has $a \geq 0$. Moreover, note that, when $a = b = 0$, the orbit is of finite volume and hence compact.

(2): First note that, since h_∞ is not constant on Z_∞, the function v_T is continuous.

When T is large, we use (1) to get the following bound $v_{(1+\varepsilon)T} - v_T = O(\varepsilon v_T)$ which is uniform in $\varepsilon \in]0, 1[$.

When T is bounded, we use the fact that the function v_T is differentiable except at the critical values τ of h_∞. Since h_∞^2 is a regular function, there are only finitely many such critical values τ. Around these points, there exists a

constant κ, $0 < \kappa < 1$ such that, for $\varepsilon > 0$ small, one has the following bound for the derivative:

$$(8.9) \qquad\qquad v'_{\tau \pm \varepsilon} \; = \; O(\varepsilon^{\kappa - 1}).$$

This assertion is a consequence of Theorem 7.1. More precisely, set $f := \pm(h_\infty - \tau)$. Since Z_∞ is smooth, one can choose $\varepsilon_0 > 0$ and an open covering U_j of $f^{-1}(]-\varepsilon_0, \varepsilon_0[)$ by open sets bianalytically homeomorphic to balls. A partition of unity gives us C^∞ functions φ_j with compact support in U_j such that $\sum_j \varphi_j = 1$ on $f^{-1}(]-\varepsilon_0/2, \varepsilon_0/2[)$. We simply apply Theorem 7.1 to these functions f and φ_j to get (8.9).

Integrating v'_t on the interval $[T, (1+\varepsilon)T]$, and using (8.9) near the critical values in this interval, one gets, uniformly for ε small and T bounded,

$$v_{(1+\varepsilon)T} - v_T = O(\varepsilon^\kappa).$$

Putting these together proves the claim.

(3): We will assume, as we may, that $\inf_{z \in Z_i} \|z\|_i \geq 1$ for each $i \in I_f$. For any $|I_f|$-tuple $m = (m_i)_{i \in I_f} \in \mathbb{N}^{I_f}$, we set

$$S(m) = \prod_{i \in I_f} S_i(m_i) \quad \text{where} \quad S_i(m_i) := \{z \in Z_i : \|z\|_i = m_i\}.$$

Letting $\omega_m := \mu_f(S(m))$ and $\pi_m := \prod_i m_i$, one has

$$w_T = \sum_{\{m \in \mathcal{M}, \pi_m \leq T\}} \omega_m$$

where $\mathcal{M} \subset \mathbb{N}^{I_f}$ consists of $m \in \mathbb{N}^{I_f}$ with non-empty $S(m)$. The main point of the proof is to use the formula for ω_m given by Theorem 7.6. According to this formula, there is a finite partition of $\mathcal{M}$ in finitely many pieces $\mathcal{M}_\alpha$ such that
- each piece $\mathcal{M}_\alpha$ is a product of subsets $\mathcal{M}_{\alpha,i}$ of $\mathbb{N}$ which are either points or of the form $\{m_i = c_{\alpha,i} \, q_{\alpha,i}^{n_i} : n_i \in \mathbb{N}\}$ for some positive integers $c_{\alpha,i}, q_{\alpha,i}$,
- on each piece $\mathcal{M}_\alpha$, the volume ω_m is given by a formula $\prod_{i \in I_f} P_{\alpha,i}(n_i, q_{\alpha,i}^{n_i/d})$ where $P_{\alpha,i}$ is a polynomial and d a positive integer.

According to Lemma 8.6 with $T = 2^t$, the volume

$$w_{\alpha, T} := \sum_{\{m \in \mathcal{M}_\alpha, \pi_m \leq T\}} \omega_m$$

satisfy the bound $w_{\alpha, 2T} = O(w_{\alpha, T})$. Hence one has $w_{2T} = O(w_T)$ as required.

(4): Let $T_0 := \inf_{z \in Z_\infty} h_\infty(z) > 0$. According to (2), there exists $C > 0$ such that for T large

$$\begin{aligned}
V_{(1+\varepsilon)T} - V_T \; &= \; \sum_m (v_{(1+\varepsilon)T/\pi_m} - v_{T/\pi_m}) \, \omega_m \\
&\leq \; C\varepsilon^\kappa \left(\sum_m v_{T/\pi_m} \, \omega_m + \sum_m \omega_m \right) \\
&\leq \; C\varepsilon^\kappa (V_T + w_{2T/T_0})
\end{aligned}$$

where the above sums are over all the multi-indices $m \in \mathbb{N}^{I_f}$ with $T_0 \pi_m \leq 2T$. Then, applying (3) twice, there exists $C' > 0$ such that for T large

$$
\begin{aligned}
V_{(1+\varepsilon)T} - V_T &\leq C\varepsilon^\kappa (V_T + C' w_{T/2T_0}) \\
&\leq C(1 + C' v_{2T_0}^{-1})\varepsilon^\kappa V_T,
\end{aligned}
$$

as required, since $v_{2T_0} > 0$. $\hfill\square$

Remark 8.10. One has the following estimate for the volume V_T of the height ball : *there exist $a \in \mathbb{Q}_{\geq 0}$, $b \in \mathbb{Z}_{\geq 0}$ and $c_1, c_2 > 0$ such that for T large,*

$$c_1 T^a \log(T)^b \leq V_T \leq c_2 T^a \log(T)^b \ .$$

This is a straightforward consequence of the formula $V_T := \int_0^\infty w_{T/t} v'_t dt$ and of the estimation of w_T and v'_T given in Remark 8.7 and Lemma 8.8 (1).

Proposition 8.11. (Height ball) *The family of height balls $B_T := \{z \in Z \mid h(z) \leq T\}$, $T \gg 1$, is well rounded.*

Proof. We will assume as we may that all the orbits Z_i have positive dimension. When $I_\infty = \emptyset$, the well-roundedness of B_T is a consequence of Example 8.5. Hence we will assume that $I_\infty \neq \emptyset$. When the height function h_∞ is constant on Z_∞, the well-roundedness of B_T is a consequence of Example 8.1.

When the height function h_∞ is not constant on Z_∞, the well-roundedness of B_T follows from Lemma 8.8 (4) and of the linearity of the action of each G_i on V_i. $\hfill\square$

Although we stated the above proposition only for characteristic 0 fields, when all the k_i have positive characteristic, the height balls are also well-rounded by Example 8.5.

9. Applications

We will be applying the following theorem and corollary to the above examples of well rounded sequences.

Theorem 9.1. *Let I be a finite set. For each $i \in I$, let k_i be a local field of characteristic not 2, G_i the group of k_i-points of a semisimple algebraic k_i-group, $H_i \subset G_i$ the k_i-points of a symmetric k_i-subgroup. Set $G_I := \prod_{i \in I} G_i$, $H_I := \prod_{i \in I} H_i$, $Z_I := H_I \backslash G_I$ and $z_0 = [H_I]$. Let Γ be an irreducible lattice of G_I such that $\Gamma_H := \Gamma \cap H_I$ is a lattice in H_I. Then for any sequence B_n of well-rounded subsets of Z_I with volume tending to infinity, we have, as $n \to \infty$,*

$$\#(z_0\Gamma \cap B_n) \sim \frac{\mu_Y(Y)}{\mu_X(X)} \mu_{Z_I}(B_n),$$

where $X = \Gamma \backslash G_I$, $Y = \Gamma_H \backslash H_I$ and the volumes are computed using invariant measures as in (5.2).

Proof. Use Corollary 4.6 and Proposition 6.2 with $\varphi_n := \mathbf{1}_{B_n}$. $\qquad\qquad\qquad\square$

In the product situation of $Z_I = Z_{I_0} \times Z_{I_1}$, we will be taking a well-rounded sequences of Z_I which are products of a fixed compact subset in one factor Z_{I_1} and a well-rounded sequence of subsets in the other factor Z_{I_0}. This will give us equidistribution results in the space Z_{I_1} when Z_{I_0} is non-compact.

Corollary 9.2. *Keeping the same hypothesis as in Theorem 9.1, let $I = I_0 \sqcup I_1$ be a partition of I. Letting B_n be a well-rounded sequence of subsets of Z_{I_1} with volume going to infinity, consider the following discrete multisets $Z(n)$ of Z_{I_0}:*

$$Z(n) := \{z \in Z_{I_0} \mid (z, z') \in z_0\Gamma \cap (Z_{I_0} \times B_n) \quad \text{for some } z' \in Z_{I_1}\}.$$

Then, as $n \to \infty$, the sets $Z(n)$ become equidistributed in Z_{I_0} with respect to a suitably normalized invariant measure. In fact, for any $\varphi \in C_c(Z_{I_0})$,

$$\lim_{n\to\infty} \frac{1}{\mu_{Z_{I_1}}(B_n)} \sum_{z \in Z(n)} \varphi(z) = \frac{\mu_Y(Y)}{\mu_X(X)} \int_{Z_{I_0}} \varphi \, d\mu_{Z_{I_0}}.$$

In particular, $Z(n)$ is non-empty for all large n.

Multiset means that the points of $Z(n)$ are counted with multiplicity according to the cardinality of the fibers of the projection $z_0\Gamma \cap (Z_{I_0} \times B_n) \to Z(n)$. Since $z_0\Gamma$ is discrete and B_n is relatively compact, we note that these fibers are finite and that $Z(n)$ is discrete in Z_{I_0}.

Proof. It suffices to prove the claim for non-negative functions $\varphi \in C_c(Z_{I_0})$. Define a sequence of functions φ_n on Z by

$$\varphi_n(z, z') := (\varphi \otimes \mathbf{1}_{B_n})(z, z') = \varphi(z)\mathbf{1}_{B_n}(z') \quad \text{for } (z, z') \in Z_{I_0} \times Z_{I_1}.$$

By Example 8.1, this sequence φ_n is well-rounded and

$$\sum_{z \in Z(n)} \varphi(z) = \sum_{z \in z_0\Gamma} \varphi_n(z)$$

since $Z(n)$ is a multiset.

By Corollary 4.6, we can apply Proposition 6.2 to the sequence φ_n and obtain

$$\lim_{n\to\infty} \frac{1}{\mu_{Z_{I_1}}(B_n)} \sum_{z \in Z(n)} \varphi(z) = \lim_{n\to\infty} \frac{1}{\mu_{Z_{I_1}}(B_n)} \sum_{z \in z_0\Gamma} \varphi_n(z) = \frac{\mu_Y(Y)}{\mu_X(X)} \int_{Z_{I_0}} \varphi \, d\mu_{Z_{I_0}} \, . \qquad \square$$

Remark 9.3. In Propositions 5.3 and 6.2, one can replace the hypothesis "the L^1-norm of φ_n goes to infinity" by the hypothesis that "the support of φ_n is non-empty and goes to infinity" i.e. for every compact C of Z, $\varphi_n|_C$ is null for all n large. The proof is exactly the same. A similar remark applies to Theorem 1.1, 9.1 and Corollary 9.2. This remark is useful for the non-empty spheres in

Example 8.4, since it avoids to check that their volume goes to infinity with the radius.

Proof of Theorems 1.1, 1.5, 1.9 and Corollary 1.2. We are now ready to prove the non-effective statements in the introduction.

Theorem 1.1 is a consequence of Theorem 9.1 with $I = S$, $G_S = \mathbf{G}_S$, $H_S = \mathbf{H}_S$ and $\Gamma = \Gamma_S$. The only thing we have to check is that Γ_S is an irreducible lattice in $\mathbf{G}_S$. This is the following classical lemma 9.4.

Theorem 1.5 is a consequence of Corollary 9.2 with $I = S$ and $I_0 = S_0$. Note that the projection $z_0 \Gamma \to Z_{I_0}$ is injective and hence the multiset $Z(n)$ is a set.

Theorem 1.9 is an immediate consequence of Corollary 4.6.

Corollary 1.2 is a consequence of Theorem 1.1 and Proposition 8.11. $\square$

Let $\mathbf{G}$ be a connected semisimple group defined over a global field K, and let S be a finite set of places of K containing all archimedean places v such that $\mathbf{G}_v$ is non-compact. Recall that these conditions assure that the subgroup $\mathbf{G}_{\mathcal{O}_S}$ is a lattice in $\mathbf{G}_S := \prod_{v \in S} \mathbf{G}_v$.

Lemma 9.4. *Let Γ_S be a subgroup of finite index in $\mathbf{G}_{\mathcal{O}_S}$. Suppose that $\mathbf{G}$ is simply connected, almost K-simple and that $\mathbf{G}_S$ is non-compact. Then Γ_S is an irreducible lattice in $\mathbf{G}_S$ (see Definition 2.3).*

Proof. Since $\mathbf{G}$ is simply connected and $\mathbf{G}_S$ is non-compact, the group $\mathbf{G}$ has the strong approximation property with respect to S, that is, the diagonal embedding of $\mathbf{G}_K$ is dense in the S-adeles $\mathbf{G}_{\mathbb{A}_S}$, i.e., the adeles without S-component (see [46, Th. 7.12] for characteristic 0 cases and [45] for the positive characteristic case). Since $\mathbf{G}$ is K-simple, it follows that Γ_S is an irreducible lattice in $\mathbf{G}_S$ [39, Cor. I.2.3.2 & Th. II.6.8]. $\square$

> For the rest of this paper, we will transform the proofs explained in the above chapters into effective proofs. For that we need to control precisely all the error terms appearing in these proofs. There are mainly four error terms to control. The first three come from the mixing property, the wave front property and the approximation of μ_Y by a smooth function. Their control will give the equidistribution speed of the translates of μ_Y. The last error term comes from the well roundedness of the balls B_T. We will dedicate one section to each of these terms.

10. Effective mixing

In this section, we introduce notations which will be used through the section 15 and we describe an effective version (Theorem 10.2) of the mixing property based on the uniform decay of matrix coefficients.

We let K be a number field, $\mathbf{G}$ a connected simply connected almost K-simple group and $\mathbf{H}$ a K-subgroup of $\mathbf{G}$ with no non-trivial K-character. Let S be a finite set of places of K containing all the infinite places v such that $\mathbf{G}_v$ is non-compact. We write S_∞ and S_f for the sets of infinite and finite places in S respectively. We assume that $\mathbf{G}_S := \prod_{v \in S} \mathbf{G}_v$ is non-compact. Let Γ_S be a congruence subgroup of $\mathbf{G}(\mathcal{O}_S)$. Note that $\mathbf{H}_S \cap \Gamma_S$ is a lattice in $\mathbf{H}_S$.

Set $X_S := \Gamma_S \backslash \mathbf{G_S}$ and $Y_S = \Gamma_S \cap \mathbf{H}_S \backslash \mathbf{H}_S$. Let μ_{X_S} and μ_{Y_S} denote the invariant probability measures on X_S and Y_S respectively. Set $Z_S := \mathbf{H}_S \backslash \mathbf{G_S}$. For each $v \in S$, choose an invariant measure μ_{Z_v} on $\mathbf{H}_v \backslash \mathbf{G}_v$ so that the invariant measure $\mu_{Z_S} := \prod_{v \in S} \mu_{Z_v}$ on Z_S satisfies $\mu_{X_S} = \mu_{Y_S} \mu_{Z_S}$ locally. For $S_0 \subset S$, we set $\mu_{Z_{S_0}} := \prod_{v \in S_0} \mu_{Z_v}$.

By a smooth function on X_S we mean a function which is smooth on each $\mathbf{G}_\infty$-orbit and which is invariant under a compact open subgroup of $\mathbf{G}_f$. The notation $C_c^\infty(\Gamma_S \backslash \mathbf{G_S})$ denotes the set of smooth functions with compact support on $\mathbf{G_S}$.

For each $v \in S$, recall the "Cartan" decomposition due to Bruhat and Tits in [5] and [6]: one has $\mathbf{G}_v = M_v \Omega_v B_v^+ M_v$ where M_v is a good maximal compact subgroup, B_v^+ a positive Weyl chamber of a maximal K_v-split torus and Ω_v is a finite subset in the centralizer of B_v.

For simplicity, we set $\mathbf{G}_\infty = \mathbf{G}_{S_\infty}$ and $\mathbf{G}_f = \mathbf{G}_{S_f}$. We also set $M_\infty := \prod_{v \in S_\infty} M_v$ and $M_f := \prod_{v \in S_f} M_v$.

Let $X_1, \cdots, X_d$ be an orthonormal basis of the Lie algebra of M_∞ with respect to an Ad-invariant scalar product. We denote by $\mathcal{D}$ the elliptic operator $\mathcal{D} := 1 - \sum_{i=1}^d X_i^2$.

Fix any closed embedding of $\mathbf{Z} = \mathbf{H} \backslash \mathbf{G}$ into a finite dimensional vector space $\mathbf{V}$ defined over K; such an embedding always exists by the well known theorem of Chevalley. To measure how far an element $z \in \mathbf{Z}_S$ is from the base point $z_0 = [\mathbf{H}_S] \in Z_S$, we may use a height function

$$\tag{10.1} \mathrm{H}_S(z) := \prod_{v \in S} \|z_v\|_v$$

where $\| \cdot \|_v$ is a norm on $\mathbf{V}_v$. This norm is assumed to be euclidean when v is an infinite place and a max norm when v is a finite place. Note that, since $\inf_{z_v \in Z_v} \|z_v\|_v > 0$, the height function $\mathrm{H}_S : Z_S \to \mathbb{R}^+$ is a proper function.

Theorem 10.2. *There exists $\kappa > 0$ and $m \in \mathbb{N}$ such that for any open compact subgroup U_f of $\mathbf{G}_f$, there exists $C_{U_f} > 0$ satisfying that for any $\psi_1, \psi_2 \in C_c^\infty(X_S)^{U_f}$ and any $g \in \mathbf{G}_S$,*

$$\left| \langle g\psi_1, \psi_2 \rangle - \int_{X_S} \psi_1 d\mu_{X_S} \int_{X_S} \psi_2 d\mu_{X_S} \right| \leq C_{U_f} \mathrm{H}_S(z_0 g)^{-\kappa} \|\mathcal{D}^m(\psi_1)\|_{L^2} \|\mathcal{D}^m(\psi_2)\|_{L^2}.$$

Proof. The above claim is a straightforward consequence of Theorem 2.20 of [25] based on the results of [8] and [43]. [25, Theorem 2.20] relies on the following hypothesis: "the only character appearing in $L^2(\Gamma_S\backslash \mathbf{G}_S)$ is the trivial one". This hypothesis is satisfied here since the non compactness of $\mathbf{G}_S$ and the simply-connectedness of $\mathbf{G}$ imply the irreducibility of Γ_S by Lemma 9.4.

The conclusion of [25, Theorem 2.20] is the above claim where $\mathrm{H}_S^{-\kappa}$ is replaced by a function $\widetilde{\xi}_{\mathbf{G}}$ which is a product over $v \in S$ of bi-M_v-invariant functions ξ'_v satisfying

$$\xi'_v(a) \leq \prod_{\alpha \in Q_v} \alpha(a)^{-1/2+\varepsilon} \quad \text{for all } a \in B_v^+$$

where Q_v is a maximal strongly orthogonal system of the root system of $(\mathbf{G}_v, B_v)$.

We only have to check that this function $\widetilde{\xi}_{\mathbf{G}}$ is bounded by a multiple of $\mathrm{H}_S^{-\kappa}$. For that, denote by ρ the representation of $\mathbf{G}$ into $\mathbf{GL}(\mathbf{V})$ such that the stabilizer of $z_0 \in \mathbf{V}_K$ is $\mathbf{H}$ and choose a weight λ larger on B_v^+ than any weight of ρ. Then there exists a positive integer k such that, for all $a \in B_v^+$,

$$\|z_0\rho(a)\|_v \leq \|z_0\|_v \|\rho(a)\|_v \leq \|z_0\|_v |\lambda(a)|_v \leq \|z_0\|_v \prod_{\alpha \in Q_v} \alpha(a)^k.$$

Since M_v and Ω_v are compact subsets, by the continuity, this implies that there exists $\kappa > 0$ and $c > 0$ such that

$$\xi'_v(g) \leq c \|z_0\rho(g)\|_v^{-\kappa}$$

for all $g \in \mathbf{G}_v$. This implies our claim. $\square$

11. Injective radius and the approximation by smooth functions

The aim of this section is to get an effective upper bound on the volume of the set of points in Y_S with small injectivity radius in X_S and approximate the characteristic function

Fix a closed embedding $\mathbf{G} \hookrightarrow \mathbf{GL}_N$. We may consider each element g of $\mathbf{G}_S$ as an $|S|$-tuples of $N \times N$ matrices g_v. We also fix a norm $\|.\|_v$ on each of these K_v-vector spaces of matrices.

For $x \in X_S$, consider the projection map $p_x : \mathbf{G}_S \to X_S$ given by $g \mapsto xg$. The injectivity radius r_x is defined to be

$$r_x = \sup\{r > 0 : p_x|_{B_r \times M_f} \text{ is injective}\}$$

where $B_r = \{g \in \mathbf{G}_\infty : \max_{v \in S_\infty} \|g_v - e\|_v \leq r \}$.

Of course, this definition makes sense only when S_∞ is non-empty. This does not matter since, when S_∞ is empty, X_S is compact.

Lemma 11.1. *Suppose $S_\infty \neq \emptyset$. For any $x \in X_S$, one has $r_x > 0$.*

Proof. Since Γ_S does not meet $\mathbf{G}_f$ and $\mathbf{G}_f$ is normal in $\mathbf{G}_S$, the group $\mathbf{G}_f$ acts freely on X_S. Hence $p_z|_{\{e\}\times M_f}$ is injective. Since M_f is compact and p_x is locally injective, $p_x|_{B_r\times M_f}$ is still injective for some small $r > 0$. $\qquad\square$

Moreover we have a quantitative version of the above lemma.

Lemma 11.2. *Suppose $S_\infty \neq \emptyset$. There exist $c_1 > 0$, $p_1 > 0$ such that for all sufficiently small $\varepsilon > 0$, $\mu_{Y_S}(\{y \in Y_S \mid r_y < \varepsilon\}) \leq c_1\varepsilon^{p_1}$.*

Proof. We use the reduction theory for $\mathbf{H}_S$ (cf. [46]). We first recall what a Siegel set is. Let $\mathbf{A}$ be a maximal K-split torus of $\mathbf{H}$ and $\mathbf{P}$ a minimal parabolic subgroup containing $\mathbf{A}$. Then $\mathbf{P} = \mathbf{N}\mathbf{R}\mathbf{A}$ where $\mathbf{R}$ is a $\mathbb{Q}$-anisotropic reductive subgroup and $\mathbf{N}$ the unipotent radical of $\mathbf{P}$. Set $\mathbf{A}_\infty := \prod_{v\in S_\infty}\mathbf{A}(K_v)$ and similarly $\mathbf{N}_\infty$ and $\mathbf{R}_\infty$. Denoting by Δ the system of simple roots of $\mathbf{H}_\infty$ determined by the choice of $\mathbf{P}$, we set for $t > 0$,

$$A_t = \{a \in \mathbf{A}_\infty : \alpha(a) \geq t \quad \text{for all } \alpha \in \Delta\}.$$

Then for a compact subset $\omega \subset \mathbf{N}_\infty\mathbf{R}_\infty$ and a maximal compact subgroup K_0 of $\mathbf{H}_S$, the set $\Sigma_t := \omega A_t K_0$ is called a Siegel set. Now the reduction theory says that there exist $h_1, \cdots, h_r \in \mathbf{H}_S$, and a Siegel set $\Sigma_{t_0} = \omega A_{t_0}K_0$ such that

$$\mathbf{H}_S = \cup_{i=1}^r (\mathbf{H}_S \cap \Gamma_S)h_i\Sigma_{t_0}.$$

Let $M'_f := \cup_{i=1}^r h_i M_f h_i^{-1}$. As in Lemma 11.1, there exists $\varepsilon_0 > 0$ such that

(11.3) $$\Gamma_S \cap B_{\varepsilon_0}M'_f = \{e\}.$$

Set

$$C_\varepsilon := \cup_{i=1}^r h_i\{wak \in \Sigma_0 : t_0 \leq \alpha(a) \leq \varepsilon^{-r_0} \text{ for each } \alpha \in \Delta\ \},$$

where $r_0 > 0$ is chosen independent of ε, so that, for all g in C_ε and $v \in S_\infty$, one has

$$\|g_v\|_v \leq \varepsilon^{-1/4} \quad \text{and} \quad \|g_v^{-1}\|_v \leq \varepsilon^{-1/4}\ .$$

Let Y'_ε denote the image of C_ε in Y_S under the projection $\mathbf{H}_S \to Y_S$. The integration formula [46, p. 213] shows that for some constant $c_1 > 0$ and $p_1 > 0$,

$$\mu_{Y_S}(Y_S - Y'_\varepsilon) \leq c_1\varepsilon^{p_1}.$$

Hence it is enough to show that for all $z \in Y'_\varepsilon$, one has $r_z \geq \varepsilon$. Suppose $p_z(x) = p_z(y)$ with $x, y \in B_\varepsilon M_f$ and write $z = \Gamma_S g$ for some $g \in C_\varepsilon$. We want to prove that $x = y$.

The element $\gamma := gxy^{-1}g^{-1}$ belongs to Γ_S. Moreover, for some fixed constant $c > 1$, one has, for all $v \in S_\infty$,

$$\begin{aligned}
\|\gamma - e\|_v &= \|g_v(x_v - y_v)y_v^{-1}g_v^{-1}\|_v \\
&\leq c\varepsilon^{-1/2}\|x_v - y_v\|_v\,\|y_v^{-1}\|_v \\
&\leq c^2\varepsilon^{1/2}.
\end{aligned}$$

But the finite component of γ is in M_f', hence, γ is in $B_{c^2\varepsilon^{1/2}} \times M_f$ and one gets from (11.3) that, for $\varepsilon < c^{-4}\varepsilon_0^2$, one has $\gamma = e$. Therefore $x = y$ as well. $\qquad \square$

For all $v \in S$, we choose a small neighborhood $\mathfrak{s}_v$ of 0 in a supplementary subspace of the Lie algebra $\mathfrak{h}_v$ in $\mathfrak{g}_v$ and set $\mathfrak{s} := \prod_{v \in S} \mathfrak{s}_v$. The set $W := exp(\mathfrak{s})$ is then a transversal to $\mathbf{H}_S$ in $\mathbf{G}_S$. We set μ_W the measure on W such that $d\mu_{X_S} = d\mu_{Y_S} d\mu_W$ locally.

Recall that B_ε denotes the ball of center e and radius ε in $\mathbf{G}_\infty$ and let U_ε be the ball of center e and radius ε in $\mathbf{G}_f$:

$$(11.4) \qquad B_\varepsilon \;=\; \{g \in \mathbf{G}_\infty \mid \max_{s \in S_\infty} \|g - e\|_v \leq \varepsilon\},$$

$$(11.5) \qquad U_\varepsilon \;=\; \{g \in \mathbf{G}_f \mid \max_{s \in S_f} \|g - e\|_v \leq \varepsilon\}.$$

We fix $\varepsilon_0 > 0$ small. For ε small we let

$$H_\varepsilon := \mathbf{H}_S \cap B_\varepsilon U_{\varepsilon_0} \quad \text{and} \quad W_\varepsilon := W \cap B_\varepsilon U_{\varepsilon_0}$$

so that the multiplication $H_\varepsilon \times W_\varepsilon \to H_\varepsilon W_\varepsilon$ is an homeomorphism onto a neighborhood of e. Fix $m > \dim \mathbf{G}_\infty$ and $\kappa > 0$ satisfying Theorem 10.2 and fix $l \in \mathbb{N}$ as in Lemma 12.3. We can assume that $U_{l\varepsilon_0} \subset U_f \cap M_f$ and let U_f' be an open subgroup of $\mathbf{G}_f$ such that $U_{\varepsilon_0} U_f' = U_{\varepsilon_0}$.

Lemma 11.6. *Let $Y_\varepsilon := \{y \in Y_S \mid$ the map $g \mapsto yg$ is injective on $H_\varepsilon W_\varepsilon\}$. There exist $c_1 > 0$ and $p_1 > 0$ such that for all small $\varepsilon > 0$,*

$$\mu_{Y_S}(Y_S - Y_\varepsilon) \leq c_1 \varepsilon^{p_1}.$$

Proof. When $S_\infty \neq \emptyset$, since $B_\varepsilon B_\varepsilon \subset B_{\varepsilon^{1/2}}$ for ε small, the set Y_ε contains the set of points y such that $r_y \geq \varepsilon^{1/2}$. Just apply then Lemma 11.2.

When $S_\infty = \emptyset$, X_S is compact, hence Y_ε is equal to Y_S for ε_0 and ε small. $\quad \square$

The following proposition provides the approximation of the characteristic function $\mathbf{1}_{Y_\varepsilon W}$ by a smooth function φ_ε with the controlled Sobolev norm.

We first recall the Sobolev norm $\mathcal{S}_m(\psi)$ of a function $\psi \in C_c^\infty(X_S)$. Choose a basis $X_1, ..., X_n$ of the Lie algebra of $\mathbf{G}_\infty$. For each k-tuple of integers $a := (a_1, ..., a_k)$ with $1 \leq a_i \leq n$, the product $X_a := X_{a_1} \ldots X_{a_k}$ defines a left-invariant differential operator on $\mathbf{G}_S$, hence a differential operator on X_S. By definition $\mathcal{S}_m(\psi)^2 = \sum_a \|X_a \psi\|_{L^2}^2$ where the sum is over all the k-tuples a with $0 \leq k \leq m$ and where $X_\emptyset \psi$ stands for ψ.

Proposition 11.7. *There exist $p_2 > 0$ such that, for all sufficiently small ε, one can choose*
- a non-negative smooth function ρ_ε on W with support in W_ε such that $\int_W \rho_\varepsilon = 1$,
- a non-negative smooth function τ_ε on Y_S with support in Y_ε such that $\tau_\varepsilon \leq 1$ on

Y_S and $\tau_\varepsilon|_{Y_{4\varepsilon}} = 1$.
- *Moreover, let φ_ε be the function on X_S defined by*

$$\varphi_\varepsilon(x) = \sum_{\{(y,w) \in Y_\varepsilon \times W_\varepsilon \,|\, yw = x\}} \tau_\varepsilon(y)\rho_\varepsilon(w).$$

The choices can be made so that φ_ε is U'_f-invariant and $\mathcal{S}_m(\varphi_\varepsilon) \leq \varepsilon^{-p_2}$.

We remark that the sum defining φ_ε is a finite sum and hence φ_ε is well defined.

To prove Proposition 11.7, we first need a lemma which constructs some test functions α_ε near e.

Lemma 11.8. *For a given $m \geq 0$, there exists $p \in \mathbb{N}$, such that, for all sufficiently small $\varepsilon > 0$, one can choose smooth non-negative functions β_ε on H_ε and smooth non-negative functions ρ_ε on W_ε satisfying the following:*
- one has $\beta_\varepsilon \geq 1$ on H_{ε^2}.
- one has $\int_W \rho_\varepsilon d\mu_W = 1$
- if α_ε denotes the smooth function on $H_\varepsilon W_\varepsilon$ given by $\alpha_\varepsilon(hw) = \beta_\varepsilon(h)\rho_\varepsilon(w)$, then α_ε is U'_f-invariant and $\mathcal{S}_m(\alpha_\varepsilon) \leq \varepsilon^{-p}$.

Proof. The general case reduces to the case of $S = S_\infty$, by considering tensor products with characteristic functions of $U_{\varepsilon_0} \cap H$ and of $U_{\varepsilon_0} \cap W$. Hence, we can assume that $S = S_\infty$ so that $\mathbf{G}_S$ is a real Lie group. Set $d = \dim W$. Fix some smooth non-negative functions β on $\mathfrak{h} := \oplus_{v \in S}\mathfrak{h}_v$ and ρ on $\mathfrak{s}$ with support in a sufficiently small neighborhood of 0 such that $\beta(0) > 1$ and $\int_\mathfrak{s} \rho = 1$. Then, for suitable constants $c_\varepsilon > 0$ converging to 1, the functions given by

$$\beta_\varepsilon(exp(X)) = \beta(X/\varepsilon) \quad \text{and} \quad \rho_\varepsilon(exp(Y)) = c_\varepsilon \varepsilon^{-d}\rho(Y/\varepsilon),$$

for X (resp. Y) in a fixed compact neighborhood of 0 in $\mathfrak{h}$ (resp. $\mathfrak{s}$), satisfy the properties listed above. $\qquad\square$

Proof of Proposition 11.7. We choose the function ρ_ε from Lemma 11.8. To construct the function τ_ε, consider a maximal family $\mathcal{G}_\varepsilon$ of points $y \in Y_\varepsilon$ such that the subsets yH_{ε^3} of Y_S are disjoint and meet $Y_{2\varepsilon}$ and let $\mathcal{F}_\varepsilon \subset \mathcal{G}_\varepsilon$ the subfamily for which yH_{ε^3} meets $Y_{4\varepsilon}$. For all $y \in \mathcal{G}_\varepsilon$ the volumes $\mu_{Y_S}(yH_{\varepsilon^3})$ are equal and of order ε^{3d} with $d = \dim(\mathbf{H}_\infty)$. Since $\mu_{Y_S}(Y_S) = 1$, the cardinality of $\mathcal{G}_\varepsilon$ is at most $O(\varepsilon^{-3d})$.

For $y \in \mathcal{G}_\varepsilon$ we define a test function $\beta_{y,\varepsilon}$ on Y_S with support on yH_ε by $\beta_{y,\varepsilon}(yh) = \beta_\varepsilon(h)$ and let $\beta_{\mathcal{G},\varepsilon} := \sum_{y \in \mathcal{G}_\varepsilon} \beta_{y,\varepsilon}$. Since $B_{\varepsilon^3}B_{\varepsilon^3} \subset B_{\varepsilon^2}$, the sets yH_{ε^2}, $y \in \mathcal{G}_\varepsilon$, cover $Y_{2\varepsilon}$. Hence $\beta_{\mathcal{G},\varepsilon} \geq 1$ on $Y_{2\varepsilon}$.

For each $y \in \mathcal{F}_\varepsilon$, consider the test function $\tau_{y,\varepsilon}$ on Y_S with support in $Y_{2\varepsilon}$ given by $\tau_{y,\varepsilon} := \beta_{y,\varepsilon}/\beta_{\mathcal{G},\varepsilon}$ on $Y_{2\varepsilon}$ and set

$$\tau_\varepsilon := \sum_{y \in \mathcal{F}_\varepsilon} \tau_{y,\varepsilon}.$$

Note that $0 \le \tau_\varepsilon \le 1$ on Y_S, $\tau_\varepsilon|_{Y_{4\varepsilon}} = 1$ and $\tau_\varepsilon|_{Y_S - Y_\varepsilon} = 0$. For $y \in \mathcal{F}_\varepsilon$, we also define the test function $\varphi_{y,\varepsilon}$ on X_S with support on $yH_\varepsilon W_\varepsilon$ by

$$\varphi_{y,\varepsilon}(yhw) := \tau_{y,\varepsilon}(yh)\rho_\varepsilon(w) = \alpha_\varepsilon(hw)/\beta_{\mathcal{G},\varepsilon}(yh).$$

These functions $\varphi_{y,\varepsilon}$ are well-defined since y belongs to the set Y_ε defined in Lemma 11.6. By construction, we have

$$\varphi_\varepsilon = \sum_{y \in \mathcal{F}_\varepsilon} \varphi_{y,\varepsilon}.$$

It follows from $\mathcal{S}_m(\alpha_\varepsilon) \le \varepsilon^{-p}$ that there exists $p_0 > 0$ such that

$$\max_{y \in \mathcal{F}_\varepsilon} \mathcal{S}_m(\varphi_{y,\varepsilon}) = O(\varepsilon^{-p_0})$$

and hence $\mathcal{S}_m(\varphi_\varepsilon) = O(\varepsilon^{-3d-p_0})$. $\qquad\square$

12. EFFECTIVE EQUIDISTRIBUTION OF TRANSLATES OF $\mathbf{H}_S$-ORBITS

The goal of this section is to prove Theorem 1.10, or its stronger version Theorem 12.5 below. This is an effective version of Theorem 4.1 on the equidistribution of translates of $\mathbf{H}_S$-orbits in X_S.

Definition 12.1. *We say that the translates $Y_S g$ are effectively equidistributed in X_S as $g \to \infty$ in Z_S if there exists $m \in \mathbb{N}$ and $r > 0$ such that, for any compact open subgroup U_f of $\mathbf{G}_f$ and any compact subset C of X_S, there exists $c = c(U_f, C) > 0$ satisfying that for any smooth function $\psi \in C_c^\infty(X_S)^{U_f}$ with support in C, one has for all $g \in \mathbf{G}_S$*

$$(12.2) \qquad |\int_{Y_S} \psi(yg)\, d\mu_{Y_S}(y) - \int_{X_S} \psi\, d\mu_{X_S}| \le c\, \mathcal{S}_m(\psi)\, \mathrm{H}_S(z_0 g)^{-r}.$$

Assume further that $\mathbf{H}$ is a symmetric K-subgroup of $\mathbf{G}$. Taking the product of the polar decompositions $\mathbf{G}_v = \mathbf{H}_v A_v K_v$ given in Lemma 3.3 over $v \in S$, we obtain a polar decomposition of the shape

$$\mathbf{G}_S = \mathbf{H}_S A_S K_S \quad \text{and we set} \quad F_S := A_S K_S.$$

The following effective version of the wavefront property 3.1 is a main technical reason why our proof of Theorem 12.5 works for $\mathbf{H}$ a symmetric subgroup.

Lemma 12.3. *There exists $l \in \mathbb{N}$ such that for all small $\varepsilon, \varepsilon' > 0$ and all $g \in F_S$,*

$$(12.4) \qquad\qquad \mathbf{H}_S\, B_{\varepsilon/l} U_{\varepsilon'/l}\, g \subset \mathbf{H}_S\, g\, B_\varepsilon U_{\varepsilon'}.$$

Proof. We only have to check this separately at each place v. This statement is then a strengthening of Proposition 3.2 on the wavefront property and is an output of the proof of this Proposition. $\qquad\square$

Theorem 12.5. *If $\mathbf{H}$ is a symmetric K-subgroup of $\mathbf{G}$, then the translates $Y_S g$ are effectively equidistributed in X_S as $g \to \infty$ in Z_S.*

Proof. Since $\mathbf{G}_S = \mathbf{H}_S F_S$, it suffices to prove the above claim for $g \in F_S$. We may also assume that $\int_{X_S} \psi \, d\mu_{X_S} = 1$. We want to bound $|I_g - 1|$ where

$$I_g := \int_{Y_S} \psi(yg) d\mu_{Y_S}(y).$$

We follow the proof of Theorem 4.1. The main modification will be to replace the characteristic function $\mathbf{1}_{Y_\varepsilon W}$ by the test function φ_ε constructed in Proposition 11.7. By the same argument as in section 4, but using the stronger version 12.3 of the wavefront lemma, we have that for all small $\varepsilon > 0$ and for any $w \in W_\varepsilon$

$$(12.6) \qquad |I_g - \int_{Y_S} \psi(ywg) d\mu_{Y_S}(y)| \leq l\varepsilon \, C_\psi.$$

Here C_ψ is the Lipschitz constant at ∞, i.e. the smallest constant such that for all $\varepsilon > 0$, $|\psi(xu) - \psi(x)| \leq C_\psi \, \varepsilon$ for all $x \in X_S$ and $u \in B_\varepsilon$.

Set τ_ε, ρ_ε and φ_ε the functions constructed in Proposition 11.7 and

$$J_{g,\varepsilon} := \int_{W_\varepsilon} \int_{Y_S} \psi(ywg) \rho_\varepsilon(w) d\mu_{Y_S}(y) d\mu_W(w).$$

By integrating (12.6) against ρ_ε, we obtain

$$|I_g - J_{g,\varepsilon}| \leq l\varepsilon \, C_\psi.$$

Set also

$$\begin{aligned} K_{g,\varepsilon} &:= \int_{X_S} \psi(xg) \varphi_\varepsilon(x) \, d\mu_{X_S}(x) \\ &= \int_{W_\varepsilon} \int_{Y_S} \psi(ywg) \tau_\varepsilon(y) \rho_\varepsilon(w) d\mu_{Y_S}(y) \, d\mu_W(w). \end{aligned}$$

Noting that $\tau_\varepsilon(y) = 1$ for $y \in Y_{4\varepsilon}$, we have for some $c_1, p_1 > 0$

$$\begin{aligned} |J_{g,\varepsilon} - K_{g,\varepsilon}| &= \left| \int_{W_\varepsilon} \int_{Y_S} \psi(ywg)(1 - \tau_\varepsilon(y)) \rho_\varepsilon(w) d\mu_{Y_S}(y) \, d\mu_W(w) \right| \\ &\leq 2\mu_{Y_S}(Y_S - Y_{4\varepsilon})(\int_{W_\varepsilon} \rho_\varepsilon) \|\psi\|_\infty \\ &\leq c_1 \, \varepsilon^{p_1} \|\psi\|_\infty \, . \end{aligned}$$

Note that $K_{g,\varepsilon} = \langle g.\psi, \varphi_\varepsilon \rangle$. Since φ_ε and ψ are U_f'-invariant, by Theorem 10.2 and Proposition 11.7 , we deduce for some $c', c_2, p_2 > 0$

$$\begin{aligned} |K_{g,\varepsilon} - \int_{X_S} \varphi_\varepsilon d\mu_{X_S}| &\leq c' \|\mathcal{D}^m(\psi)\|_{L^2} \|\mathcal{D}^m(\varphi_\varepsilon)\|_{L^2} \, \mathrm{H}_S(z_0 g)^{-\kappa} \\ &\leq c_2 \varepsilon^{-p_2} \|\mathcal{D}^m(\psi)\|_{L^2} \, \mathrm{H}_S(z_0 g)^{-\kappa}. \end{aligned}$$

Moreover, one has

$$\left| \int_{X_S} \varphi_\varepsilon d\mu_{X_S} - 1 \right| = \left| \int_{Y_S} \tau_\varepsilon(y) \, d\mu_{Y_S}(y) - 1 \right| \le \mu_{Y_S}(Y - Y_{4\varepsilon}) \le c_1 \, \varepsilon^{p_1} .$$

Since C is compact, the C^1-norm of a U'_f-invariant function ψ supported on C is bounded above by a uniform multiple of a suitable Sobolev norm as in [2, Theorem 2.20] i.e., one has an inequality

$$\max(\|\psi\|_\infty, C_\psi) \le c'' \, \mathcal{S}_{2m}(\psi)$$

with $c'' = c''(U_f, C) > 0$ independent of ψ. Hence, putting all these upper bounds together and using the inequality $1 \le \|\psi\|_\infty$, we get

$$
\begin{aligned}
|I_g - 1| &\le |I_g - J_{g,\varepsilon}| + |J_{g,\varepsilon} - K_{g,\varepsilon}| + |K_{g,\varepsilon} - \int \varphi_\varepsilon| + |\int \varphi_\varepsilon - 1| \\
&\le l\varepsilon \, C_\psi + c_1 \varepsilon^{p_1} \|\psi\|_\infty + c_2 \varepsilon^{-p_2} \|\mathcal{D}^m(\psi)\|_{L^2} \, \mathrm{H}_S(z_0 g)^{-\kappa} + c_1 \varepsilon^{p_1} \\
&\le (c'_1 \varepsilon^{p_1} + c'_2 \varepsilon^{-p_2} \, \mathrm{H}_S(z_0 g)^{-\kappa}) \mathcal{S}_{2m}(\psi).
\end{aligned}
$$

Note in the above that the positive constants c'_i, p_i, $i = 1, 2$, are independent of ψ.

Now by taking $\varepsilon = \mathrm{H}_S(z_0 g)^{-r/p_1}$ with $r = \frac{\kappa p_1}{p_1 + p_2}$, we obtain as required

$$|I_g - 1| \le c \, \mathcal{S}_{2m}(\psi) \, \mathrm{H}_S(z_0 g)^{-r} .$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remarks

(1) One could also, as an output of our proof, compute explicitly m and r and describe how the constant c depends on the compact sets U_f and C.

(2) Note that the above theorem 12.5 is precisely the effective version of Theorem 4.1, since we have shown that the effective mixing theorem 10.2 together with the effective wave front lemma 12.3 imply the effective equidistribution of $Y_S g$.

13. Effective counting and equidistribution

The following definition is an effective version of Definition 6.1. Recall that $B_\varepsilon = B(e, \varepsilon)$ is the ball of center e and radius ε in $\mathbf{G}_\infty$ (11.4) and that H_S is a height function on Z_S as defined in (10.1).

Definition 13.1. *A sequence of subsets B_n in Z_S is said to be effectively well-rounded if*

(1) *it is invariant under a compact open subgroup of $\mathbf{G}_f$,*

(2) *there exists $\kappa > 0$ such that, uniformly for all $n \ge 1$ and all $\varepsilon \in]0, 1[$,*

$$\mu_{Z_S}(B_{n,\varepsilon}^+ - B_{n,\varepsilon}^-) = O(\varepsilon^\kappa \mu_{Z_S}(B_n))$$

where $B_{n,\varepsilon}^+ = B_n B_\varepsilon$ and $B_{n,\varepsilon}^- = \cap_{u \in B_\varepsilon} B_n u$,

(3) *for any $k > 0$, there exists $\delta > 0$ such that, uniformly for all $n \gg 1$ and all $\varepsilon \in]0, 1[$, one has*

$$\int_{B_{n,\varepsilon}^+} \mathrm{H}_S^{-k}(z) \, d\mu_{Z_S}(z) = O(\mu_{Z_S}(B_n)^{1-\delta}).$$

If S_∞ is empty, then the assumption (2) is void.

A subset Ω of Z_S is said to be *effectively well-rounded* if the constant sequence $B_n = \Omega$ is effectively well-rounded. This means that Ω is of non-empty interior and that the volume $\mu_{Z_S}(\partial_\varepsilon \Omega)$ of the ε-neighborhood of the boundary of Ω is a $O(\varepsilon^\kappa)$ for ε small. For instance, a compact subset of Z_{S_∞} with piecewise smooth (or even piecewise C^1) boundary is effectively well-rounded in Z_{S_∞}.

Theorem 13.2. *Suppose that the translates $Y_S g$ become effectively equidistributed in X_S as $g \to \infty$ in Z_S. Then for any effectively well-rounded sequence of subsets B_n in Z_S such that $\mathrm{vol}(B_n) \to \infty$ there exists a constant $\delta_0 > 0$ such that*

$$\# z_0 \Gamma_S \cap B_n = \mathrm{vol}(B_n)(1 + O(\mathrm{vol}(B_n)^{-\delta_0})).$$

Proof. Set $\Gamma_{S,H} := \Gamma_S \cap \mathbf{H}_S$. As in sections 5 and 6, we define a function F_n on $X_S = \Gamma_S \backslash \mathbf{G}_S$ by

$$F_n(x_0 g) = \sum_{\gamma \in \Gamma_{S,H} \backslash \Gamma_S} \mathbf{1}_{B_n}(z_0 \gamma g) \ , \text{ for } g \in \mathbf{G}_S.$$

For instance, one has

$$F_n(x_0) = \# z_0 \Gamma_S \cap B_n.$$

Let m and r be the integers given by Theorem 12.5 and U_f a compact open subgroup of $\mathbf{G}_f$. By Lemma 11.8, there exists $p > 0$, a smooth U_f-invariant function α_ε on $\mathbf{G}_S$, supported on $B_\varepsilon U_f$ such that $\int_{\mathbf{G}_S} \alpha_\varepsilon = 1$ and $\mathcal{S}_m(\alpha_\varepsilon) \leq \varepsilon^{-p}$. Here we take ε and U_f small enough so that $B_\varepsilon U_f$ injects to X_S, and hence we may consider α_ε as a function on X_S.

We also introduce the functions $F_n^\pm$ on X_S:

$$F_{n,\varepsilon}^\pm(x_0 g) = \sum_{\gamma \in \Gamma_{S,H} \backslash \Gamma_S} \mathbf{1}_{B_{n,\varepsilon}^\pm}(z_0 \gamma g) \ , \text{ for } g \in \mathbf{G}_S.$$

Then

$$F_{n,\varepsilon}^-(x_0 g) \leq F_n(x_0) \leq F_{n,\varepsilon}^+(x_0 g) \quad \text{for all } g \in B_\varepsilon \times U_f$$

and hence

$$\langle F_{n,\varepsilon}^-, \alpha_\varepsilon \rangle \leq F_n(x_0) \leq \langle F_{n,\varepsilon}^+, \alpha_\varepsilon \rangle.$$

Note that

$$\langle F_{n,\varepsilon}^\pm, \alpha_\varepsilon \rangle = \int_{B_{n,\varepsilon}^\pm} \left(\int_{Y_S} \alpha_\varepsilon(yg) d\mu_{Y_S}(y) \right) d\mu_{Z_S}(z_0 g).$$

Set $v_n := \operatorname{vol}(B_n)$ and $v_{n,\varepsilon}^{\pm} := \operatorname{vol}(B_{n,\varepsilon}^{\pm})$. Then by Theorem 12.5 and the assumptions (2) and (3) of the definition 13.1, there exist positive constants κ, δ and c_i such that for all $n \gg 1$ and small $\varepsilon > 0$,

$$|\langle F_{n,\varepsilon}^{\pm}, \alpha_\varepsilon \rangle - v_n| \leq \int_{B_{n,\varepsilon}^+} \left| \int_{Y_S} (\alpha_\varepsilon(yg) - 1) d\mu_{Y_S}(y) \right| d\mu_{Z_S}(z_0 g) + (v_{n,\varepsilon}^+ - v_{n,\varepsilon}^-)$$

$$\leq c\, \mathcal{S}_m(\alpha_\varepsilon) \int_{B_{n,\varepsilon}^+} \mathrm{H}_S(z)^{-k} d\mu_{Z_S}(z) + c_2 \varepsilon^\kappa v_n$$

$$\leq c_1 \varepsilon^{-p} v_n^{1-\delta} + c_2 \varepsilon^\kappa v_n.$$

Setting $\delta_0 := \frac{\delta}{1+p/\kappa}$ and choosing $\varepsilon = v_n^{-\delta_0/\kappa}$ we get $F_n(x_0) = v_n(1 + O(v_n^{-\delta_0}))$. $\square$

Corollary 13.3. *Let $S = S_0 \sqcup S_1$ be a partition of S. There exist $\delta_0, c > 0$ such that for any effectively well-rounded sequence of subsets B_n in Z_{S_1} whose volumes $v_n := \mu_{Z_{S_1}}(B_n)$ tend to ∞ and for any compact effectively well-rounded subset Ω of Z_{S_0}, we have*

$$\# z_0 \Gamma_S \cap (\Omega \times B_n) = \frac{\mu_{Y_S}(Y_S)}{\mu_{X_S}(X_S)} \, v_n \, \mu_{Z_{S_0}}(\Omega) \, (1 + O(v_n^{-\delta_0})).$$

Proof. To apply Theorem 13.2, we only have to check that the sequence $A_n := \Omega \times B_n$ of subsets of Z_S is effectively well-rounded, which is straightforward. $\square$

14. Effective well-roundedness

In this section, we give explicit examples of effectively well-rounded families (see Definition 13.1).

We keep the notations for K, S, $\mathbf{G}$, $\mathbf{H}$, Z, $\mathrm{H}_S(z) = \prod_{v \in S} \|z\|_v$ etc., from the beginning of section 10.

We also set for $T > 0$,

(14.1) $B_v(T) := \{z \in Z_v \mid \|z\|_v \leq T\}$ and $S_v(T) := \{z \in Z_v \mid \|z\|_v = T\}.$

Proposition 14.2.   (1) *Fix a subset $S_0 \subset S$ containing S_∞. For an $|S|$-tuple $m = (m_v)$ of positive numbers, define*

$$Z(m) := \prod_{v \in S_0} B_v(m_v) \times \prod_{v \in S \setminus S_0} S_v(m_v).$$

Then the family of sets $Z(m)$, $m_v \gg 1$, provided non-empty, is effectively well-rounded.

(2) *The family of height balls $B_T = \{z \in Z_S : \mathrm{H}_S(z) \leq T\}$ is effectively well-rounded.*

Proof. The proof relies heavily on the section 7. For (1), we may assume that S contains only one place v. When v is infinite, the condition 13.1 (2) is Lemma 8.8 (2) and 13.1 (3) is Corollary 7.3.c. When v is finite, the condition 13.1 (2) is empty and the condition 13.1 (3) is Corollary 7.7.b.

For (2), 13.1 (2) is Lemma 8.8 (4), and 13.1 (3) is a combination of the following lemma 14.3 with the facts that, on one hand one has $B_{T,\varepsilon} \subset B_{kT}$ for some fixed $k > 0$ and, on the other hand, one has $V_{kT} = O(V_T)$ again by Lemma 8.8 (4). $\square$

Lemma 14.3. *Let* $B_T = \{z \in Z_S : \mathrm{H}_S(z) \leq T\}$ *and* $V_T := \mu_{Z_S}(B_T)$. *Then, for any* $k > 0$, *there exists* $\delta > 0$ *such that*

$$\int_{B_T} \mathrm{H}_S(z)^{-k} d\mu_{Z_S}(z) = O(V_T^{1-\delta}).$$

Proof. We may assume that, for all v in S and z in Z_v, one has $\|z\|_v \geq 1$. Set $b_T = \{z \in Z_{S_\infty} : \mathrm{H}_{S_\infty}(z) \leq T\}$ and $v_T = \mu_{Z_{S_\infty}}(b_T)$.

We first claim that there exists $\delta > 0$ and $C > 0$ such that for any $T > 0$,

(14.4) $$\int_{b_T} \mathrm{H}_{S_\infty}^{-k} d\mu_{Z_{S_\infty}} < C\, v_T^{1-\delta}.$$

Set u_T to be the left hand side of the above inequality. For T large, by Lemma 8.8 (1) one has $v_T = O(T^{m_0})$ for some $m_0 > 0$, hence the derivative $u'_T = T^{-k} v'_T$ satisfies $u'_T = O(v_T^{-k/m_0} v'_T)$ and, integrating, one gets $u_T = O(v_T^{1-k/m_0})$. For T bounded, since H_{S_∞} is bounded below, one gets $u_T = O(v_T)$. Putting this together, one gets (14.4).

Now, for any tuple $m = (m_v) \in \mathbb{N}^{S_f}$, set $S(m) := \prod S_v(m_v) \subset Z_{S_f}$ where $S_v(m_v) := \{z \in Z_v : \|z\|_v = m_v\}$.

Also set $\pi_m := \prod_v m_v \in \mathbb{N}$ and $\omega_m = \mu_{Z_{S_f}}(S(m))$.

Then we have, where the following sums are taken over the tuples $m \in \mathbb{N}^{S_f}$ for which $S(m)$ is non empty,

$$
\begin{aligned}
\int_{B_T} \mathrm{H}_S(z)^{-k} d\mu_{Z_S}(z) &= \sum_m \left(\int_{b_{T/\pi_m}} \mathrm{H}_{S_\infty}^{-k} d\mu_{Z_{S_\infty}} \right) \left(\int_{S(m)} \mathrm{H}_{S_f}^{-k} d\mu_{Z_{S_f}} \right) \\
&= \sum_m \pi_m^{-\frac{k}{2}} \left(\int_{b_{T/\pi_m}} \mathrm{H}_{S_\infty}^{-k} d\mu_{Z_{S_\infty}} \right) \left(\int_{S(m)} \mathrm{H}_{S_f}^{-\frac{k}{2}} d\mu_{Z_{S_f}} \right) \\
&\leq C \sum_m \pi_m^{-\frac{k}{2}} \left(v_{T/\pi_m}\, \omega_m \right)^{1-\delta} \\
&\leq C \left(\sum_m \pi_m^{-\frac{k}{2\delta}} \right)^{\delta} \left(\sum_m v_{T/\pi_m} \omega_m \right)^{1-\delta} \\
&\leq C \left(\prod_{v \in S_f} (1 - q_v^{-\frac{k}{2\delta}}) \right)^{-\delta} V_T^{1-\delta},
\end{aligned}
$$

with positive constants C and δ given by (14.4) and Corollary 7.7.b. $\square$

15. Effective applications

In this section, assuming K is a number field and $\mathbf{H}$ is a symmetric K-subgroup, we give proofs of effective versions of our main theorems listed in the introduction, keeping the notations therein.

Corollary 15.1. *Assume that $Z_{\mathcal{O}_S} \neq \emptyset$. Then for any finite $v \in S$, there exists $\delta > 0$ such that for any effectively well-rounded subset $\Omega \subset Z_{S-\{v\}}$,*

$$\#\{z \in Z_{\mathcal{O}_S} \cap \Omega : \|z\|_v = T\} = \widetilde{\mathrm{vol}}(\Omega \times S_v(T))\,(1 + O(T^{-\delta}))$$

as $T \to \infty$ subject to $S_v(T) \neq \emptyset$.

Recall that the normalized volume $\widetilde{\mathrm{vol}}$ has been defined in (1.6). This corollary is an equidistribution statement since one has

$$\widetilde{\mathrm{vol}}(\Omega \times S_v(T)) = C\,\mu_{Z_{S-\{v\}}}(\Omega)\,\mu_{Z_v}(S_v(T))$$

with a constant C independent of n and Ω.

Proof of Corollary 15.1: The same claim with the error term $O(T^{-\delta})$ replaced by $O((\mu_{Z_v}(S_v(T))^{-\delta})$ follows immediately from Theorems 12.5, 13.2 and Proposition 14.2. Now by Corollary 7.7.c, we have a constant $a > 0$ such that for T large and $S_v(T) \neq \emptyset$,

$$\mu_{Z_{v_0}}(S_v(T))^{-1} = O(T^{-a/2}).$$

This proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For $z \in \mathbf{V}_{\mathbb{Q}}$, the condition $\|z\|_p \leq p^n$ is equivalent to $z \in p^{-n}\mathbf{V}_{\mathbb{Z}}$. Hence we obtain:

Corollary 15.2. *Assume $K = \mathbb{Q}$ and fix a prime p such that Z_p is non compact and $Z_{\mathbb{Z}[p^{-1}]}$ is non empty. Then there exists $\delta > 0$ such that, for any non-empty compact subset $\Omega \subset Z_{\mathbb{R}}$ with piecewise smooth boundary,*

$$\#\,\Omega \cap p^{-n}\mathbf{V}_{\mathbb{Z}} = \widetilde{\mathrm{vol}}(\Omega \times B_p(p^n))\,(1 + O(p^{-\delta n}))$$

as $n \to \infty$, where $B_p(p^n)$ is defined in (14.1).

Note again that $\widetilde{\mathrm{vol}}(\Omega \times B_p(p^n)) = C\,\mu_{Z_{\mathbb{R}}}(\Omega)\,\mu_{Z_p}(B_p(p^n))$ for some $C > 0$ independent of n and Ω.

Proof of Theorems 1.3 and 1.8 Letting $v_T := \mathrm{vol}(B_S(T))$, Theorem 1.3 with $O(T^{-\delta})$ replaced by $O(v_T^{-\delta})$ immediately follows from Theorems 12.5 and 13.2 and Proposition 14.2. To obtain the given error term, note that at least one of the factors Z_{v_0} is non-compact. Fix $R_0 > 1$ such that the volume of $B_{S\setminus\{v_0\}}(R_0) = \{z \in Z_{S-v_0} : \mathrm{H}_{S\setminus\{v_0\}}(z) \leq R_0\}$ is positive. Since $B_S(T)$ contains the product $B_{v_0}(TR_0^{-1})B_{S\setminus\{v_0\}}(R_0)$, we have $v_T^{-1} = O(\mu_{Z_{v_0}}(B_{v_0}(TR_0^{-1}))^{-1})$. By Corollary 7.3.d for v_0 archimedean and Corollary 7.7.c for v_0 non-archimedean, we have a constant $a > 0$ satisfying

$$\mu_{Z_{v_0}}(B_{v_0}(T))^{-1} = O(T^{-a/2}).$$

This proves the claim. The same proof works for Theorem 1.8 applying Corollary 13.3 in place of Theorem 13.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Corollary 1.11. All three cases fit in our setting as in the introduction.

For (A), if f has signature (r, s), $\mathbf{V}_1(\mathbb{R})$ can be identified with $\mathrm{Spin}(r - 1, s) \backslash \mathrm{Spin}(r, s)$ where the $\mathrm{Spin}(r, s)$-action on $\mathbb{R}^{r+s}$ is given through the projection $\mathrm{Spin}(r, s) \to \mathrm{SO}(r, s)$.

For (B): we have the action of $\mathbf{G} = \mathbf{SL}_n$ on $\mathbf{V}$ by $(g, v) \mapsto g^t v g$. And $\mathbf{V}_1(\mathbb{R})$ is a finite disjoint union of $\mathrm{SO}(r, s) \backslash \mathrm{SL}_n(\mathbb{R})$ for $r + s = n$, each of them being the variety consisting of symmetric matrices of signature (r, s).

For (C), we have $\mathbf{V}_1(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}) \backslash \mathrm{SL}_{2n}(\mathbb{R})$ with the action $(g, v) \mapsto g^t v g$.

Note for (A), if $n = 3$, $\mathbf{H}_{\mathbb{R}} = \mathrm{Spin}(1, 1)$ may arise and the additional assumption that f does not represent 0 over $\mathbb{Q}$ implies that $\mathbf{H}$ does not allow any non-trivial $\mathbb{Q}$-character. In all other cases, $\mathbf{H}$ is semisimple and hence has no character.

Now we give a uniform proof assuming that $S = \{\infty, p\}$ for the sake of simplicity. It is easy to generalize the argument for a general S. Also note that this proof works equally well for any homogeneous integral polynomial f whose level set can be identified with a symmetric variety in our set-up.

Let $d = \deg f$. For each $0 \leq j \leq d - 1$, consider the radial projection $\pi_j : V_{p^{kd+j}} \to V_{p^j}$ given by $x \mapsto p^{-k}x$. Then since the degree of f is d and the radial projection is bijective,

$$\# \Omega \cap \pi(\mathbf{V}_{p^{kd+j}}(\mathbb{Z})^{\mathrm{prim}}) = \# \left\{ z \in \mathbf{V}_{p^j}(\mathbb{Z}[p^{-1}]) \cap \Omega_j : \|z\|_p = p^k \right\}$$

where $\Omega_j := p^{j/d}\Omega \subset \mathbf{V}_{p^j}$. Since $\mathbf{V}_{p^j}(\mathbb{Z}[p^{-1}])$ is a finite union of $\mathbf{G}(\mathbb{Z}[p^{-1}])$-orbits, we obtain by Corollary 13.3 with $S_0 = \{\infty\}$ and $S_1 = \{p\}$

$$\# \Omega \cap \pi(\mathbf{V}_{p^{kd+j}}(\mathbb{Z})^{\mathrm{prim}}) \sim \omega_{p^{kd+j}} \mathrm{vol}(\Omega)(1 + O(\omega_{p^{kd+j}}^{-\delta}))$$

where $\omega_{p^{kd+j}} = \mu(\{x \in \mathbf{V}_{p^j}(\mathbb{Q}_p) : \|x\|_p = p^k\})$. Note that, by Remark 9.3, ω_m go to infinity with m when it is non zero. $\square$

Proof of Corollary 1.12. As before, we assume $S = \{\infty, p\}$ for simplicity. We use the same notation as in the above proof. Then for each fixed $0 \leq j \leq d - 1$, $f(x) = p^{kd+j}$ is equivalent to $f(p^{-k}x) = p^j$ and, if $z = p^{-k}x$ with $x \in \mathbf{V}(\mathbb{Z})^{\mathrm{prim}}$, one has $\|z\|_\infty \|z\|_p = \|x\|_\infty$. Therefore

$$N_{j,T} := \#\{x \in \mathbf{V}(\mathbb{Z})^{\mathrm{prim}} : \|x\|_\infty < T, \ f(x) = p^{kd+j} \text{ for some integer } k \geq 0\}$$
$$= \#\{z \in \mathbf{V}_{p^j}(\mathbb{Z}[p^{-1}]) : \|z\|_\infty \|z\|_p < T\}.$$

By Theorem 1.3, one has $N_{j,T} = v_{j,T}(1 + O(v_{j,T}^{-\delta_j}))$ where

$$v_{j,T} = \widetilde{\mathrm{vol}}(\{(z_\infty, z_p) \in \mathbf{V}_{p^j}(\mathbb{R}) \times \mathbf{V}_{p^j}(\mathbb{Q}_p) : \|z_\infty\|_\infty \|z_p\|_p < T\}).$$

Since $\#\{x \in \mathbf{V}(\mathbb{Z})^{\mathrm{prim}} : \|x\|_\infty < T, \ f(x) \in p^{\mathbb{Z}}\} = \sum_j N_{j,T}$, and $v_T = \sum_j v_{j,T}$, this proves the claim. $\square$

16. More examples

Here are a few concrete examples of applications of Theorem 1.5 to emphasize the meaning of our results. For each of them, we have selected a specific global field K with sets S_0, S_1 (most often $K = \mathbb{Q}$, $S_0 = \{v_0\}$ and $S_1 = \{v_1\}$) and we have selected a classical symmetric space Z defined over K. We look at the repartition of S-integral points z in Z_{v_0} when imposing conditions on the v_1-norm of z.

Symmetric matrices with two real places. This example is very classical. Let τ be the non trivial automorphism of the real quadratic field $K := \mathbb{Q}[\sqrt{2}]$ and set, for $d \geq 2$,

$$Z_{\{\infty\}} := \{M \in \mathrm{M}_d(\mathbb{R}) \ \text{positive definite symmetric matrix of determinant 1}\}$$

and

$$Z_n := \{M \in Z_{\{\infty\}} \cap \mathrm{M}_d(\mathbb{Z}[\sqrt{2}]) \mid \textstyle\sum_{i,j} \tau(m_{i,j})^2 \leq n^2\} .$$

Lemma 16.1. *As $n \to \infty$, these discrete sets Z_n become equidistributed in the non-compact Riemannian symmetric space $Z_{\{\infty\}}$.*

Proof. Let v_0 and v_1 be the two infinite places of K : for $\lambda \in K$, $|\lambda|_{v_0} = |\lambda|$ and $|\lambda|_{v_1} = |\tau(\lambda)|$. Apply Theorem 1.5 to

$$(16.1) \qquad K = \mathbb{Q}[\sqrt{2}] , \quad S_0 = \{v_0\} , \quad S_1 = \{v_1\} , \quad \mathbf{Z} = \mathbf{SL}_d/\mathbf{SO}_d ,$$

and to the group $\mathbf{G} = \mathbf{SL}_d$ which acts by $M \to gM\,{}^t g$ on the vector space $\mathbf{V}$ of symmetric $d \times d$-matrices, with $\mathbf{Z} \sim \{M \in \mathbf{V} \mid \det(M) = 1 \}$ as a $\mathbf{G}$-orbit.

Note that the group $\mathrm{SL}(d, \mathbb{R})$ acts transitively on $Z_{\{\infty\}}$. $\square$

Orthogonal projections with one real and one finite place. This example is also quite classical. Let p be a prime number, $d = d_1 + d_2 \geq 3$,

$$Z_{\{\infty\}} := \{\pi \in \mathrm{M}_d(\mathbb{R}) \ \pi^2 = {}^t\pi = \pi \ \text{and} \ \mathrm{tr}(\pi) = d_1\}$$

be the set of orthogonal projections of rank d_1 in $\mathbb{R}^d$, and

$$Z_n := \{\pi \in Z_{\{\infty\}} \mid p^n \pi \in \mathrm{M}_d(\mathbb{Z})\} .$$

Lemma 16.2. *As $n \to \infty$, these discrete sets Z_n become equidistributed in the compact Riemannian symmetric space $Z_{\{\infty\}}$.*

Proof. Apply Theorem 1.5 and Remark 9.3 with

$$(16.2) \qquad K = \mathbb{Q} , \quad S_0 = \{\infty\} , \quad S_1 = \{p\} , \quad \mathbf{Z} = \mathbf{O}_d/\mathbf{O}_{d_1} \times \mathbf{O}_{d_2} ,$$

and to the group $\mathbf{G} = \mathbf{Spin}_d$ which acts, by conjugation via $\mathbf{SO}_d$, on the vector space $\mathbf{V}$ of $d \times d$-matrices, with $\mathbf{Z} \sim \{\pi \in \mathbf{V} \mid \pi^2 = {}^t\pi = \pi \ \text{and} \ \mathrm{tr}(\pi) = d_1\}$ as a $\mathbf{G}$-orbit.

Note that the group $\mathrm{Spin}(d, \mathbb{R})$ acts transitively on $Z_{\{\infty\}}$. $\square$

Complex structures with one finite and one real place. In this example, one chooses a prime number p and set, for $d \geq 1$,

$$Z_{\{p\}} := \{J \in \mathrm{M}_{2d}(\mathbb{Q}_p) \mid J^2 = -\mathrm{Id} \ \text{and} \ tr(J) = 0 \}$$

and

$$Z_R := \{J \in Z_{\{p\}} \cap \mathrm{M}_{2d}(\mathbb{Z}[\tfrac{1}{p}]) \mid \textstyle\sum_{i,j} J_{i,j}^2 \leq R^2 \} .$$

Lemma 16.3. *As $R \to \infty$, these discrete sets Z_R become equidistributed in the p-adic symmetric space $Z_{\{p\}}$.*

Proof. Apply Theorem 1.5 to

(16.3) $K = \mathbb{Q}$, $S_0 = \{p\}$, $S_1 = \{\infty\}$, $\mathbf{Z} = \mathbf{GL}_{2d}/\mathbf{GL}_d \times \mathbf{GL}_d$,

and to the group $\mathbf{G} = \mathbf{SL}_{2d}$ which acts by conjugation on the vector space $\mathbf{V}$ of $2d \times 2d$-matrices, with $\mathbf{Z} \sim \{J \in \mathbf{V} \mid J^2 = -\mathrm{Id} \ \text{and} \ tr(J) = 0 \}$ as a $\mathbf{G}$-orbit.

Note that the group $\mathrm{SL}(2d, \mathbb{Q}_p)$ acts transitively on $Z_{\{p\}}$. $\qquad\square$

Antisymmetric matrices with two finite places in characteristic zero. In this example, one chooses two distinct prime numbers p and ℓ, and set, for $d \geq 1$, $n \geq 0$, and $R > d$,

$$Z_{\{p\}} := \{A \in \mathcal{M}_{2d}(\mathbb{Q}_p) \mid A = -{}^t A \ \text{and} \ \det(A) = 1 \}$$

and

$$Z_{n,R} := \{A \in Z_{\{p\}} \mid \textstyle\sum_{i,j} A_{i,j}^2 < R \ \text{and} \ \ell^n A \in \mathrm{M}_{2d}(\mathbb{Z}[\tfrac{1}{p}]) \} .$$

Lemma 16.4. *As $n + R \to \infty$, these discrete sets $Z_{n,R}$ become equidistributed in the p-adic symmetric space $Z_{\{p\}}$.*

Proof. Apply Theorem 1.5 and Remark 9.3 with

(16.4) $K = \mathbb{Q}$, $S_0 = \{p\}$, $S_1 = \{\infty, \ell\}$, $\mathbf{Z} = \mathbf{SL}_{2d}/\mathbf{Sp}_d$,

and to the group $\mathbf{G} = \mathbf{SL}_{2d}$ which acts by $g \to gA\,{}^t g$ on the vector space $\mathbf{V}$ of antisymmetric $2d \times 2d$-matrices, with $\mathbf{Z} \sim \{A \in \mathbf{V} \mid \det(A) = 1 \}$ as a $\mathbf{G}$-orbit.

Note that the group $\mathrm{SL}_{2d}(\mathbb{Q}_p)$ acts transitively on $Z_{\{p\}}$. $\qquad\square$

Quadrics with two places in positive characteristic. In this example, p is an odd prime, and one set, for $d \geq 3$,

$$Z_{\{0\}} := \{P \in \mathbb{F}_p((t))^d \mid P_1^2 + \cdots + P_d^2 = 1 \}$$

and, for $n_1 \geq n_2 \geq \ldots \geq n_d \geq 0$,

$$Z_{n_1,\ldots,n_d} := \{P \in Z_{\{0\}} \cap \mathbb{F}_p[t, t^{-1}]^d \mid \deg(P_i) = n_i \ \forall i \} ,$$

where $\deg(\sum a_i t^i) := \max\{i \mid a_i \neq 0\}$.

Lemma 16.5. *If $n_1 = n_2 = n_3$ goes to infinity or if $p \equiv 1 \bmod 4$ and $n_1 = n_2$ goes to infinity, these discrete sets Z_n become equidistributed in the sphere $Z_{\{0\}}$.*

Proof. Let 0 and ∞ be the two (finite) places of the field $\mathbb{F}_p(t)$ associated to the two points 0 and ∞ of $\mathbb{P}^1(\mathbb{F}_p)$. Apply Theorem 1.5 and Remark 9.3 with

$$(16.5) \qquad K = \mathbb{F}_p(t) \ , \quad S_0 = \{0\} \ , \quad S_1 = \{\infty\} \ , \quad \mathbf{Z} = \mathbf{SO}_d/\mathbf{SO}_{d-1} \ ,$$

and to the group $\mathbf{G} = \mathbf{Spin}_d$ which acts naturally, via $\mathbf{SO}_d$, on the d-dimensional vector space $\mathbf{V}$, with the sphere $\mathbf{Z} \sim \{v \in \mathbf{V} \ | \ v_1^2 + \ldots v_d^2 = 1 \ \}$ as a $\mathbf{G}$-orbit.

The corresponding two completions are the fields of Laurent series $K_0 = \mathbb{F}_p((t))$ and $K_\infty = \mathbb{F}_p((t^{-1}))$, and the ring of S-integers is $\mathcal{O}_S = \mathbb{F}_p[t, t^{-1}]$. Set

$$Z_{\{\infty\}} := \{P \in \mathbb{F}_p((t^{-1}))^d \ | \ P_1^2 + \cdots + P_d^2 = 1 \ \} \ ,$$

note that the well-rounded subset

$$B_{n_1,\ldots,n_d} := \{P \in Z_{\{\infty\}} \ | \ \deg(P_i) = n_i \ \forall i \ \}$$

is non-empty if and only if $n_1 = n_2 = n_3$ or $p \equiv 1 \bmod 4$ and $n_1 = n_2$.

Note also that the group $\mathrm{Spin}(d, \mathbb{F}_p((t)))$ acts transitively on $Z_{\{0\}}$. $\square$

Other examples. The reader may construct easily many similar examples choosing other triples (K, S, Z). For instance, "Quadrics with three infinite places", "Lagrangian decompositions with two infinite and three finite places", "Hermitian matrices with four places in positive characteristic", and so on....

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