ON THE DISTRIBUTION OF ORBITS OF GEOMETRICALLY FINITE HYPERBOLIC GROUPS ON THE BOUNDARY (WITH APPENDIX BY FRANÇOIS MAUCOURANT)

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ABSTRACT. We investigate the distribution of orbits of a non-elementary discrete hyperbolic subgroup Γ acting on $\mathbb{H}^n$ and its geometric boundary $\partial_\infty(\mathbb{H}^n)$. In particular, we show that if Γ admits a finite Bowen-Margulis-Sullivan measure (for instance, if Γ is geometrically finite), then every Γ-orbit in $\partial_\infty(\mathbb{H}^n)$ is equidistributed with respect to the Patterson-Sullivan measure supported on the limit set $\Lambda(\Gamma)$. The appendix by Maucourant is the extension of a part of his thesis where he obtains the same result as a simple application of Roblin's theorem.

Our approach is via establishing the equidistribution of solvable flows on the unit tangent bundle of $\Gamma \backslash \mathbb{H}^n$, which is of independent interest.

1. INTRODUCTION

Let G be the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^n$ and $\Gamma < G$ a torsion-free non-elementary (=not virtually abelian) discrete subgroup. The action of Γ extends to $\overline{\mathbb{H}^n} := \mathbb{H}^n \cup \partial_\infty(\mathbb{H}^n)$ where $\partial_\infty(\mathbb{H}^n)$ denotes the geometric boundary of $\mathbb{H}^n$, and we define the limit set $\Lambda(\Gamma)$ as the set of accumulation points of a Γ-orbit in $\overline{\mathbb{H}^n}$.

If we denote by δ_Γ the critical exponent of Γ, then there exists a Γ-invariant conformal density $\{\nu_x : x \in \mathbb{H}^n\}$ of dimension δ_Γ on $\Lambda(\Gamma)$ by Patterson [14] for $n = 2$ and Sullivan [19] for n general. We consider the Bowen-Margulis-Sullivan measure m_Γ^{BMS} on the unit tangent bundle $\mathrm{T}^1(\Gamma \backslash \mathbb{H}^n)$ associated to the density $\{\nu_x\}$ (Def. 2.1). When the total mass $|m_\Gamma^{\mathrm{BMS}}|$ finite, the geodesic flow is ergodic on $\mathrm{T}^1(\Gamma \backslash \mathbb{H}^n)$ [19].
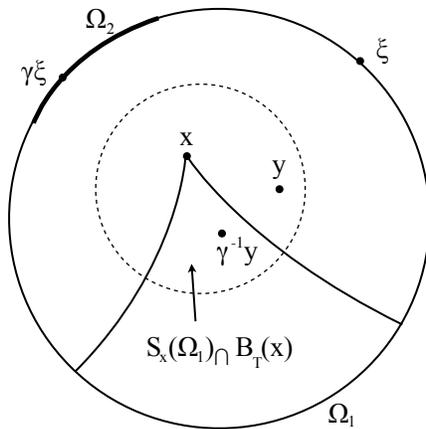
For a subset $\Omega \subset \partial_\infty(\mathbb{H}^n)$ and $x \in \mathbb{H}^n$, we denote by $S_x(\Omega) \subset \mathbb{H}^n$ the set of all points lying in geodesics emanating from x toward Ω, and by $B_T(x) \subset \mathbb{H}^n$ the hyperbolic ball of radius T centered at x.

Our main theorem is the following:

Theorem 1.1. *Suppose that the total mass $|m_\Gamma^{\mathrm{BMS}}|$ is finite. Let Ω_1 and Ω_2 be Borel subsets of $\partial_\infty(\mathbb{H}^n)$ whose boundaries are of zero Patterson-Sullivan measure. Then, for any $x, y \in \mathbb{H}^n$ and $\xi \in \partial_\infty(\mathbb{H}^n)$, as $T \to \infty$,*

$$\#\{\gamma^{-1}(y) \in S_x(\Omega_1) \cap B_T(x) : \ \gamma(\xi) \in \Omega_2\} \sim \frac{\nu_x(\Omega_1)\nu_y(\Omega_2)}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot e^{\delta_\Gamma T}.$$

FIGURE 1. Orbits of Γ on $\mathbb{H}^n \times \partial_\infty(\mathbb{H}^n)$

If Γ is geometrically finite, that is, if the unit neighborhood of the convex core[1] $\mathcal{C}_\Gamma$ has finite volume, then $|m_\Gamma^{\mathrm{BMS}}| < \infty$ [20]. However the above theorem is not restricted only to those groups as there are geometrically infinite groups with $|m_\Gamma^{\mathrm{BMS}}| < \infty$ (see [15]).

We remark that the assumption of $|m_\Gamma^{\mathrm{BMS}}| < \infty$ implies that the conformal density $\{\nu_x\}$ is determined uniquely up to homothety (see [17, Coro.1.8]).

When $\Omega_1 = \Omega_2 = \partial_\infty(\mathbb{H}^n)$, the above counting problem is simply the non-Euclidean lattice point counting problem, and was solved by Lax and Phillips [10] for geometrically finite groups with $\delta_\Gamma > (n-1)/2$. Theorem 1.1 for $\Omega_2 = \partial_\infty(\mathbb{H}^n)$ is due to Roblin [17]. When Γ is a lattice, the same type of orbital counting result for $\Omega_2 = \partial_\infty(\mathbb{H}^n)$ was obtained in a much more general setting of Riemannian symmetric spaces (see [11], [2], [5], [6], etc.). Theorem 1.1 for general Ω_1, Ω_2 was proved in [7] for all lattices in semisimple Lie groups (see also [9] for the case when $\Omega_1 = \partial_\infty(\mathbb{H}^n)$).

We highlight Theorem 1.1 for the Möbius transformation action of $\mathrm{PSL}_2(\mathbb{C})$, that is, the action on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$ and $z \in \widehat{\mathbb{C}}$. In the upper half-space model $\mathbb{H}^3 = \{(x, y, r) : r > 0\}$ of the hyperbolic 3-space with the metric $d = \frac{\sqrt{dx^2 + dy^2 + dr^2}}{r}$, the Möbius transformations by elements of $\mathrm{PSL}_2(\mathbb{C})$ give rise to all orientation preserving isometries of $\mathbb{H}^3$.

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[1]The *convex core* $\mathcal{C}_\Gamma \subset \Gamma\backslash\mathbb{H}^n$ is defined to be the minimal convex set which contains all geodesics connecting any two points in $\Lambda(\Gamma)$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{C})$, we have

$$\cosh(d(g(j), j)) = \frac{|a|^2 + |b|^2 + |c|^2 + |d|^2}{2},$$

where $j = (0, 0, 1)$. Hence the following follows from Theorem 1.1:

Corollary 1.2. Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary geometrically finite discrete subgroup. For any Borel subset Ω of $\widehat{\mathbb{C}}$ with $\nu_j(\partial(\Omega)) = 0$, we have, as $T \to \infty$,

$$\# \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : |a|^2 + |b|^2 + |c|^2 + |d|^2 < 2\cosh T, \ \frac{az+b}{cz+d} \in \Omega \right\} \sim \frac{|\nu_j| \cdot \nu_j(\Omega)}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot e^{\delta_\Gamma T}.$$

A similar result holds for the linear fractional transformation action of non-virtually cyclic and finitely generated subgroups of $\mathrm{PSL}_2(\mathbb{R})$ on $\widehat{\mathbb{R}}$.

After the submission, we were pointed out by the referee that in F. Maucourant's thesis [12], Theorem 1.1 was already proved in the case when the sector is taken to be the whole ball (i.e., $\Omega_1 = \partial_\infty(\mathbb{H}^n)$) and that his approach which elegantly uses a theorem of Roblin [17, Theorem 4.11] can be extended to obtain Theorem 2 of the Appendix. As Maucourant's result is not published, Maucourant agreed to write an appendix on his result.

Our approach is different from his, as we do not rely on the aforementioned theorem of Roblin but on a recent result of Oh and Shah (see Theorem 2.3). In section 2, we obtain the main ergodic theorem which is the equidistribution of solvable flows (Theorem 2.7) which is of independent interest. In section 3, we relate the counting function in Theorem 1.1 with an average over a solvable flow of a certain function on $\mathrm{T}^1(\Gamma \backslash \mathbb{H}^n)$ (Lemma 3.1) and then apply the results in section 2 to conclude Theorem 1.1. Some computations such as Lemma 3.3 are a bit tricky due to the fact that the Burger-Roblin measure m_Γ^{BR} is not an invariant measure in general.

This approach of establishing the equidistribution of Γ-orbits on the boundary via the study of solvable flows on $\mathrm{T}^1(\Gamma \backslash \mathbb{H}^n)$ was first used in [7].

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2. Equidistribution of solvable flows

For $x, y \in \mathbb{H}^n$ and $\xi \in \partial_\infty(\mathbb{H}^n)$, the *Busemann function* β is defined as follows:

$$\beta_\xi(x, y) = \lim_{t \to \infty} \{ d(x, \xi_t) - d(y, \xi_t) \}.$$

where ξ_t is a geodesic ray toward ξ.

For a unit tangent vector $u \in \mathrm{T}^1(\mathbb{H}^n)$, we denote by $\pi(u)$ the base point of u and by u^+ (resp. u^-) the forward (resp. backward) endpoint of the geodesic determined by u.

Let Γ be a non-elementary discrete subgroup of $G = \mathrm{Isom}^+(\mathbb{H}^n)$. Let $\{\nu_x : x \in \mathbb{H}^n\}$ denote a Patterson-Sullivan density for Γ, i.e., each ν_x is a finite measure supported on $\partial_\infty(\mathbb{H}^n)$ satisfying: for any $x, y \in \mathbb{H}^n$, $\xi \in \partial_\infty(\mathbb{H}^n)$ and $\gamma \in \Gamma$,

$$\gamma_* \nu_x = \nu_{\gamma x}; \quad \text{and} \quad \frac{d\nu_y}{d\nu_x}(\xi) = e^{-\delta_\Gamma \beta_\xi(y, x)},$$

where $\gamma_* \nu_x(R) = \nu_x(\gamma^{-1}(R))$.

Definition 2.1. The *Bowen-Margulis-Sullivan measure* m_Γ^{BMS} associated to $\{\nu_x\}$ ([3], [11], [20]) is defined as the measure induced on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ of the following Γ-invariant measure $\widetilde{m}^{\mathrm{BMS}}$ on $\mathrm{T}^1(\mathbb{H}^n)$:

$$d\widetilde{m}^{\mathrm{BMS}}(u) = e^{\delta_\Gamma \beta_{u^+}(x,\pi(u))} e^{\delta_\Gamma \beta_{u^-}(x,\pi(u))} d\nu_x(u^+) d\nu_x(u^-) dt.$$

We denote by $\{m_x : x \in \mathbb{H}^n\}$ a G-invariant conformal density of dimension $n-1$, which is unique up to homothety.

Definition 2.2. The *Burger-Roblin measure* m_Γ^{BR} associated to $\{\nu_x\}$ and $\{m_x\}$ ([4], [17]) is defined as the measure induced on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ of the following Γ-invariant measure $\widetilde{m}^{\mathrm{BR}}$ on $\mathrm{T}^1(\mathbb{H}^n)$:

$$d\widetilde{m}^{\mathrm{BR}}(u) = e^{(n-1)\beta_{u^+}(x,\pi(u))} e^{\delta_\Gamma \beta_{u^-}(x,\pi(u))} dm_x(u^+) d\nu_x(u^-) dt.$$

The measure $\widetilde{m}^{\mathrm{BR}}$ is supported on the set of unit tangent vectors u such that u^- belongs to the limit set Λ_Γ.

We fix $x \in \mathbb{H}^n$ and $\xi \in \partial_\infty(\mathbb{H}^n)$ in the rest of this section. Let K be the stabilizer of x in G and P denote the stabilizer of $\xi \in \partial_\infty(\mathbb{H}^n)$. The subgroup P is a minimal parabolic subgroup of G and is the normalizer of its unipotent radical N. Without loss of generality, we may assume that m_x is the probability measure.

Denote by $X_0 \in \mathrm{T}^1(\mathbb{H}^n)$ the unit vector based at x such that $X_0^- = \xi$. We set

$$\xi_x := X_0^+.$$

Setting $A = \{a_t := \exp(tX_0) : t \in \mathbb{R}\}$, we have (cf. [7, Lem. 4.1])

- $G = KA^+K$ where $A^+ := \{a_t : t \geq 0\}$;
- $P = MAN$ where M is the centralizer of A in K and $M = K \cap P$;
- N is the expanding horospherical subgroup of G with respect to A^+, i.e., $N = \{g \in G : a_t g a_{-t} \to e \quad \text{as } t \to \infty\}$.

The above Cartan decomposition $G = KA^+K$ says that for any $g \in G$, there exists a unique element $a \in A^+$ such that $g = k_1 a k_2$, for $k_1, k_2 \in K$. Moreover, $k_1 a k_2 = k_1' a k_2'$ implies that $k_1 = k_1' m$ and $k_2 = m^{-1} k_2'$ for some $m \in M$.

We may identify G/K with $\mathbb{H}^n$ where gK corresponds to $g(x)$ and G/M with $\mathrm{T}^1(\mathbb{H}^n)$ where gM corresponds to $g(X_0)$.

Let B_0 be the maximal split solvable subgroup of G given by

$$B_0 = AN.$$

For $T > 0$ and a subset $\Omega \subset K$ with $\Omega M = \Omega$, set

$$B_0(T, \Omega) := B_0 \cap \Omega A_T^+ K$$

where $A_T^+ := \{a_t : 0 \leq t \leq T\}$. Our aim in this section is to prove an equidistribution of $B_0(T, \Omega)$ on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$: Theorem 2.7.

The following is the main ergodic ingredient we use.

Theorem 2.3. [13] *Suppose that* $|m_\Gamma^{\mathrm{BMS}}| < \infty$. *Let* Ω *be a Borel subset of* K *with* $\Omega M = \Omega$ *and with* $\nu_x(\partial(\Omega(\xi_x))) = 0$. *For any* $\varphi \in C_c(\Gamma\backslash G)^M$,

$$e^{(n-1-\delta_\Gamma)t} \int_{s \in \Omega/M} \varphi(sa_t) dm_x(s) \sim \frac{\nu_x(\Omega(\xi_x))}{|m_\Gamma^{\mathrm{BMS}}|} \cdot m_\Gamma^{\mathrm{BR}}(\varphi) \quad \text{as } t \to +\infty.$$

By the Iwasawa decomposition $G = ANK$, the map
$$K \longrightarrow B_0 \backslash G : k \mapsto B_0 k$$
is a diffeomorphism, say, ι. Let N^- be the contracting horospherical subgroup of G with respect to A^+: $N^- = \{g \in G : a_{-t} g a_t \to e \text{ as } t \to \infty\}$.

The map $M \times N^- \to B_0 \backslash G$, $mn \mapsto B_0 mn$, composed with ι^{-1}, is an M-equivariant map $M \times N^- \to K$ which is a diffeomorphism onto its image, which is a Zariski open subset. Let S be the image of $\{e\} \times N^-$ under this map. We note that the complement of $M \backslash MS$ in $M \backslash K$ is a point.

Lemma 2.4. *Let $s \in S$. If $V \subset S$ is a neighborhood of s and S_0 is a compact subset of S, there exists $C = C(S_0) > 1$ such that for any $m \in M$,*
$$MS_0 m \subset MV s^{-1} m a_{-t} \quad \text{for all } t > C.$$

Proof. Since $e \in V s^{-1}$, the conjugation by a_t expands $V s^{-1} \subset S$ by the factor of e^t, and hence we can find $C > 1$ such that
$$S_0 \subset a_t V s^{-1} a_{-t}$$
for all $t > C$. Hence
$$B_0 M S_0 m \subset B_0 M a_t V s^{-1} a_{-t} m = B_0 M V s^{-1} m a_{-t}$$
as $a_t \in B_0$.

By the uniqueness of the decomposition $G = B_0 K$, we have the desired inclusion. $\square$

We denote by dh the Haar measure on G such that for $h = k_1 a_t k_2 \in KA^+ K$,
$$dh = 2^{n-1} (\sinh t \cosh t)^{(n-1)/2} dk_1 dt dk_2$$
where dk denotes the probability Haar measure on K.

We denote by ρ_ℓ the left-invariant Haar measure on B_0 given by the relation:
$$dh = d\rho_\ell(b) dk$$
where $h = bk \in B_0 K$.

In the rest of this section, we assume that $|m_\Gamma^{\mathrm{BMS}}| < \infty$.

The following lemma is a special case of [16, Prop. 3.1]:

Lemma 2.5. *Any sphere centered at $\xi_0 \in \Lambda(\Gamma)$ has measure zero with respect to ν_x.*

Proposition 2.6. *Let V be an open neighborhood of e in K such that $MV = V$. Let Ω be a Borel subset of K with $\Omega M = \Omega$ and with $\nu_x(\partial(\Omega(\xi_x))) = 0$. Then for any $\psi \in C_c(\Gamma \backslash G)$,*
$$\int_V \int_{B_0(T,\Omega)} \psi(bk) d\rho_\ell(b) dk \sim \frac{e^{\delta_\Gamma T} \nu_x(\Omega(\xi_x))}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot m_\Gamma^{\mathrm{BR}}(\psi * \chi_V),$$
as $T \to \infty$, where $\psi * \chi_V(h) = \int_{k \in V} \psi(hk) dk$ and $B_0(T,\Omega) := B_0 \cap \Omega A_T^+ K$.

Proof. Note that
$$B_0(T,\Omega)V = B_0 V \cap \Omega A_T^+ K$$
$$= \{k_1 a_t k_2 : k_1 \in \Omega_{k_2}(t), 0 < t < T, k_2 \in K\},$$
where $\Omega_{k_2}(t) = \Omega \cap B_0 V k_2^{-1} a_{-t}$.

Setting $\Xi(t) = 2^{n-1}(\sinh t \cosh t)^{(n-1)/2}$, we have

$$\int_{B_0(T,\Omega)} \psi(bk)d\rho_\ell(b)dk$$

$$= \int_{h \in B_0(T,\Omega)} \psi(h)dh$$

$$= \int_{k_1 a_t k_2 \in B_0(T,\Omega)} \psi(k_1 a_t k_2)\Xi(t)dk_2 dt dk_1$$

$$= \int_{k_2 \in K} \int_{0<t<T} \int_{k_1 \in \Omega_{k_2}(t)} \psi(k_1 a_t k_2)\Xi(t)dk_2 dt dk_1.$$

Set for $m \in M$,

$$\Omega_m := \Omega \cap MSm^{-1}$$

where S is the image of N^- in K. We note that since $S \subset M \backslash K$ is an open Zariski dense subset whose complement is a point and ν_x is atom free, $\nu_x(\Omega) = \nu_x(\Omega_m)$ and $\nu_x(\partial(\Omega)) = \nu_x(\partial(\Omega_m))$. Write $V = MV_0$ for $V_0 \subset S$. Let $k_2 = ms \in MS$ with $s \in V_0$.

By Lemma 2.5, for any fixed $\epsilon > 0$, we can take a compact subset $S_\epsilon \subset S$ such that $\nu_x(\Omega(\xi_x) - S_\epsilon(\xi_x)) < \epsilon$ and $\nu_x(\partial(S_\epsilon(\xi_x))) = 0$. If we set $\Omega_m(S_\epsilon) := \Omega \cap MS_\epsilon m^{-1}$, then $\nu_x(\Omega_m(\xi_x) - \Omega_m(S_\epsilon)(\xi_x)) < \epsilon$ and $\nu_x(\partial(\Omega_m(S_\epsilon)(\xi_x))) = 0$ since $\partial(\Omega_m(S_\epsilon)(\xi_x)) \subset \partial(\Omega(\xi_x)) \cup \partial(S_\epsilon(\xi_x)))$.

By Lemma 2.4, there exists $C_\epsilon > 1$ such that for all $t > C_\epsilon$,

$$\Omega_m(S_\epsilon) \subset \Omega_{ms}(t).$$

On the other hand, as $a_t \in B_0$,

$$\Omega_{ms}(t) \subset \Omega \cap B_0 MVs^{-1}m^{-1}a_{-t} = \Omega \cap B_0 M(a_t Vs^{-1}a_{-t})m^{-1} \subset \Omega \cap MSm^{-1} = \Omega_m.$$

Without loss of generality we assume below that ψ is non-negative. Hence for all $t > C_\epsilon$,

$$\int_{k_1 \in \Omega_m(S_\epsilon)} \psi(k_1 a_t sm)dk_1 \leq \int_{k \in \Omega_{ms}(t)} \psi(k_1 a_t ms)dk_1 \leq \int_{k_1 \in \Omega_m} \psi(k_1 a_t ms)dk_1.$$

Note that by applying Theorem 2.3

$$\int_{k_1 \in \Omega_m(S_\epsilon)} \psi(k_1 a_t ms)dk_1$$

$$= \int_{s \in \Omega_m(S_\epsilon)/M} \int_{m_1 \in M} \psi(sa_t m_1 ms)dm_1 dm_x(s)$$

$$= \int_{s \in \Omega_m(S_\epsilon)/M} \psi_{ms}(sa_t)dm_x(s)$$

$$\sim e^{-(n-1-\delta_\Gamma)t} \frac{1}{|m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_{ms})\nu_x(\Omega_m(S_\epsilon)(\xi_x))$$

where $\psi_{ms}(h) := \int_{m_1 \in M} \psi(hm_1 ms)dm_1$.

Hence

$$\liminf_t e^{(n-1-\delta_\Gamma)t} \int_{k_1 \in \Omega_{ms}(t)} \psi(k_1 a_t ms) dk_1$$

$$\geq \liminf_t e^{(n-1-\delta_\Gamma)t} \int_{k_1 \in \Omega_m(S_\epsilon)} \psi(k_1 a_t ms) dk_1$$

$$= \frac{1}{|m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_{ms}) \nu_x(\Omega_m(S_\epsilon)(\xi_x))$$

$$\geq \frac{1}{|m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_{ms})(\nu_x(\Omega_m(\xi_x)) - \epsilon)$$

and similarly

$$\limsup_t e^{(n-1-\delta_\Gamma)t} \int_{k_1 \in \Omega_{ms}(t)} \psi(k_1 a_t ms) dk_1 \leq \frac{1}{|m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_{ms}) \nu_x((\Omega_m(\xi_x)) + \epsilon).$$

As $\epsilon > 0$ is arbitrary and $\Omega_m(\xi_x) = \Omega(\xi_x)$, we deduce

$$\lim_{t \to \infty} e^{(n-1-\delta_\Gamma)t} \int_{k \in \Omega_{ms}(t)} \psi(k_1 a_t ms) dk_1 = \frac{1}{|m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_{ms}) \nu_x(\Omega(\xi_x)).$$

Using that $\Xi(t) \sim_t e^{(n-1)t}$, we obtain that for any $ms \in MV_0$, as $T \to \infty$,

$$\int_{A_T^+} \int_{\Omega_{ms}(t)} \psi(k_1 a_t ms) \Xi(t) dk_1 dt$$

$$\sim \frac{e^{\delta_\Gamma T}}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_{ms}) \nu_x(\Omega(\xi_x)).$$

Now for $ms \notin MV_0$, we claim that

$$\limsup_T e^{-\delta_\Gamma T} \int_{A_T^+} \int_{\Omega_{ms,t}} \psi(k_1 a_t ms) \Xi(t) dk_1 dt = 0.$$

Consider the set

$$\Omega_{ms,t}^c := \Omega \cap B_0(MS - MV_0)s^{-1}m^{-1}a_{-t}.$$

As $s \in MS - MV_0$, we have by the previous case that

$$\lim_T e^{-\delta_\Gamma T} \int_{A_T^+(C)} \int_{\Omega_{ms,t}^c} \psi(k_1 a_t ms) \Xi(t) dk_1 dt = \frac{1}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_{ms}) \nu_x(\Omega_m(\xi_x)).$$

Since $\Omega_{ms,t}^c \subset \Omega_m$ and

$$\lim_T e^{-\delta_\Gamma T} \int_{A_T^+(C)} \int_{\Omega_m} \psi(k_1 a_t ms) \Xi(t) dk_1 dt = \frac{1}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_{ms}) \nu_x(\Omega_m(\xi_x)),$$

the claim follows.

Since the image of S is an open Zariski dense subset of $M \backslash K$, we may replace K by MS in the integration over K and hence

$$\int_{k_2 \in K} \int_{A_T^+} \int_{k_1 \in \Omega_{k_2}(t)} \psi(k_1 a_t ms) \Xi(t) dk_1 dt dk_2$$

$$\sim \int_{ms \in MV_0} \int_{A_T^+} \int_{k_1 \in \Omega_{k_2}(t)} \psi(k_1 a_t ms) \Xi(t) dk_1 dt dk_2$$

$$\sim \frac{e^{\delta_\Gamma T}}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi * \chi_V) \nu_x(\Omega(\xi_x)).$$

This completes the proof of Proposition 2.6.   □

Theorem 2.7. *Let Ω be a Borel subset of K/M with $\nu_x(\partial(\Omega(\xi_x))) = 0$. Then for any $\psi \in C_c(\Gamma\backslash G)$, as $T \to \infty$,*

$$\int_{B_0(T,\Omega)} \psi(b)d\rho_\ell(b) \sim \frac{e^{\delta_\Gamma T}}{\delta_\Gamma} \cdot \frac{\nu_x(\Omega(\xi_x))}{|m_\Gamma^{\mathrm{BMS}}|} \cdot m_\Gamma^{\mathrm{BR}}(\psi).$$

Proof. Let V_ϵ be an ϵ-neighborhood of e in K such that $V_\epsilon M = V_\epsilon$. For any $\psi \in C_c(\Gamma\backslash G)^M$ and $\epsilon > 0$, define functions $\psi_\varepsilon^\pm \in C_c(\Gamma\backslash G)^M$ as follows:

$$\psi_\varepsilon^+(h) := \sup_{k\in V_\varepsilon} \psi(hk), \text{ and } \psi_\varepsilon^-(h) := \inf_{k\in V_\varepsilon} \psi(hk).$$

Let $\eta > 0$. By the uniform continuity of ψ and the M-invariance, there exists $\epsilon = \epsilon(\eta)$ such that $|\psi_\varepsilon^+(h) - \psi_\varepsilon^-(h)| < \eta$ for all $h \in G$.

Without loss of generality we may assume $\psi \geq 0$. Note that, by applying Proposition 2.6,

$$\limsup_T e^{-\delta_\Gamma T} \int_{B_0(T,\Omega)} \psi(b)d\rho_\ell(b)$$

$$\leq \limsup e^{-\delta_\Gamma T} \mathrm{Vol}(V_\epsilon)^{-1} \int_{k\in V_\epsilon} \int_{B_0(T,\Omega)} \psi_\epsilon^+(bk)d\rho_\ell(b)dk$$

$$= \mathrm{Vol}(V_\epsilon)^{-1} \frac{1}{\delta_\Gamma |m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_\epsilon^+ * \chi_{V_\epsilon})\nu_x(\Omega(\xi_x)).$$

Similarly

$$\liminf_T e^{-\delta_\Gamma T} \int_{B_0(T,\Omega)} \psi(b)d\rho_\ell(b)$$

$$\geq \liminf_T e^{-\delta_\Gamma T} \mathrm{Vol}(V_\epsilon)^{-1} \int_{k\in V_\epsilon} \int_{B_0(T,\Omega)} \psi_\epsilon^-(bk)d\rho_\ell(b)dk$$

$$= \mathrm{Vol}(V_\epsilon)^{-1} \frac{1}{\delta_\Gamma |m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi_\epsilon^- * \chi_{V_\epsilon})\nu_x(\Omega(\xi_x)).$$

Since

$$m_\Gamma^{\mathrm{BR}}(\psi_\epsilon^\pm * \chi_{V_\epsilon}) = m_\Gamma^{\mathrm{BR}}(\psi) \mathrm{Vol}(V_\epsilon) + O(\eta),$$

we deduce that

$$\limsup_T e^{-\delta_\Gamma T} \int_{B_0(T,\Omega)} \psi(b)d\rho_\ell(b) = \frac{1}{\delta_\Gamma |m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi)\nu_x(\Omega(\xi_x)) + O(\eta)$$

and

$$\liminf_T e^{-\delta_\Gamma T} \int_{B_0(T,\Omega)} \psi(b)d\rho_\ell(b) = \frac{1}{\delta_\Gamma |m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi)\nu_x(\Omega(\xi_x)) + O(\eta).$$

As $\eta > 0$ is arbitrary, this proves the claim.   □

3. Proof of Theorem 1.1

Fix $x \in \mathbb{H}^n$ and $\xi \in \partial_\infty(\mathbb{H}^n)$. We keep the same notation from the previous section. Let $y \in \mathbb{H}^n$ and choose $g \in G$ such that $g(x) = y$.

For a subset W of G, we denote by W^g the conjugate gWg^{-1}. Note that K^g is the stabilizer of y and that $B := B_0^g$ stabilizes $g(\xi) = g(X_0^-)$.

For $\hat{\Omega}_1, \hat{\Omega}_2 \subset \partial_\infty(\mathbb{H}^n)$, we set

$$\Omega_1 := \{k \in K/M : k\xi_x \in \hat{\Omega}_1\}, \quad \Omega_2 := \{k \in K^g/M^g : k(g(\xi)) \in \hat{\Omega}_2\}$$

so that $\hat{\Omega}_1 = \Omega_1(\xi_x)$ and $\hat{\Omega}_2 = \Omega_2(g(\xi))$. We assume that the boundaries of $\hat{\Omega}_i$ have measure zero with respect to the Patterson-Sullivan density.

In this notation, we have

$$\{z \in S_x(\hat{\Omega}_1), d(z, x) < T\} = \Omega_1 A_T^+(x)$$

and hence the condition $\gamma^{-1}y \in S_{x,T}(\hat{\Omega}_1)$ becomes $\gamma \in gKA_T^-\Omega_1^{-1}$. And $\gamma(\xi) \in \hat{\Omega}_2$ is equivalent to $\gamma g^{-1}(g(\xi)) \in \Omega_2(g(\xi))$ and hence to $\gamma g^{-1} \in \Omega_2 B$.

For $h \in G$, we write

$$h = h_{K^g} h_B g$$

where $h_{K^g} \in K^g$ and $h_B \in B$ are uniquely determined.

Hence setting

$$B(T, \Omega_1) = B \cap g\Omega_1 A_T^+ K g^{-1},$$

the number we want to count is the following:

$$\begin{aligned}
N_T(\hat{\Omega}_1, \hat{\Omega}_2) &:= \#\{\gamma \in \Gamma : \gamma(g(\xi)) \in \hat{\Omega}_2, \gamma^{-1}(y) \in S_x(\hat{\Omega}_1)\} \\
&= \#\Gamma \cap gKA_T^-\Omega_1^{-1} \cap \Omega_2 Bg \\
&= \{\gamma \in \Gamma : \gamma_{K^g} \in \Omega_2, \gamma_B^{-1} \in B(T, \Omega_1)\}.
\end{aligned}$$

Let V_ϵ be an ϵ-neighborhood of e in K such that $MV_\epsilon M = V_\epsilon$.

For the ϵ-neighborhood $A_\varepsilon = \{a_t : |t| < \epsilon\}$ of e in A, by the strong wavefront Lemma (see [8] or [7]) there exists a symmetric neighborhood $\mathcal{O}_\varepsilon'$ of e in G and $C > 1$ such that for all $k \in K$ and all $t > C$,

$$(3.1) \qquad gka_t Kg^{-1}\mathcal{O}_\varepsilon' \subset gkV_\epsilon a_t A_\epsilon Kg^{-1}.$$

Choose a symmetric neighborhood $\tilde{V}_\epsilon \subset V_\epsilon$ so that

$$(3.2) \qquad \mathrm{Vol}(V_\epsilon^+ - V_\epsilon^-) < \eta \, \mathrm{Vol}(V_\epsilon)$$

where $V_\epsilon^+ := V_\epsilon \tilde{V}_\epsilon$ and $V_\epsilon^- := \cap_{u \in \tilde{V}_\epsilon} V_\epsilon u$.

We may assume without loss of generality that $\mathcal{O}_\epsilon'$ satisfies

$$a_t nk(g^{-1}\mathcal{O}_\epsilon' g) \subset a_t A_\epsilon Nk\tilde{V}_\epsilon$$

for all $a_t nk \in ANK$.

We set

$$\mathcal{O}_\epsilon := \mathcal{O}_\epsilon' \cap B$$

and note that $\mathcal{O}_\epsilon^{-1} = \mathcal{O}_\epsilon$.

Fix $\eta > 0$. Then there exists $0 < \epsilon(\eta) < \eta$ such that for all $0 < \epsilon < \epsilon(\eta)$,

$$(3.3) \qquad \nu_x(\Omega_{1,\epsilon}^+(\xi_x) - \Omega_{1,\epsilon}^-(\xi_x)) < \eta$$

where $\Omega_{1,\epsilon}^+ = \Omega_1 V_\epsilon^+$ and $\Omega_{1,\epsilon}^- = \cap_{k \in V_\epsilon^-} \Omega_1 k$. This is possible since the boundary of $\hat{\Omega}_1$ has measure zero with respect to ν_x.

Similarly, we may assume that

$$(3.4) \qquad \nu_y(\Omega_{2,\epsilon}^+(g(\xi)) - \Omega_{2,\epsilon}^-(g(\xi))) < \eta$$

where $\Omega_{2,\epsilon}^+ = \Omega_2 U_\epsilon^+$ and $\Omega_{2,\epsilon}^- = \cap_{k \in U_\epsilon^-} \Omega_2 k$ where $U_\epsilon^\pm := gV_\epsilon^\pm g^{-1}$. We also set $U_\epsilon = gV_\epsilon g^{-1}$.

We choose $\phi_\epsilon^\pm = \phi_{\Omega_2,\epsilon}^\pm \in C_c(K^g)^{M^g}$ such that $0 \leq \phi_{\Omega_2,\epsilon}^- \leq \phi_{\Omega_2,\epsilon}^+ \leq 1$, $\phi_\epsilon^+(k) = 1$ for $k \in \Omega_2$, $\phi_\epsilon^+(k) = 0$ for $k \notin \Omega_2 U_\epsilon$, $\phi_\epsilon^-(k) = 1$ for $k \in \cap_{u \in U_\epsilon} \Omega_2 u$, and $\phi_\epsilon^-(k) = 0$ for $k \notin \Omega_2$.

We denote by ρ the left invariant Haar measure on B given by: for $\psi \in C_c(B)$,

$$\int_B \psi(b) d\rho(b) := \int_{b_0 \in B_0} \psi(g^{-1} b_0 g) d\rho_\ell(b_0).$$

Choosing a non-negative function $\psi_\epsilon \in C_c(B)$ supported on $\mathcal{O}_\epsilon$ and with $\int_B \psi_\epsilon(b) d\rho(b) = 1$, we define a function $f_{\Omega_2,\eta}^\pm$ on $G = K^g B g$ by

$$f_{\Omega_2,\eta}^\pm(h) = \phi_{\Omega_2,\epsilon(\eta)}^\pm(h_{K^g}) \psi_{\epsilon(\eta)}(h_B)$$

where $h = h_{K^g} h_B g \in G$ with $h_K \in K^g$ and $h_B \in B$ uniquely determined and $\epsilon = \epsilon(\eta)$. Define

$$F_{\Omega_2,\eta}^\pm(h) = \sum_{\gamma \in \Gamma} f_{\Omega_2,\eta}^\pm(\gamma h),$$

which is an integrable function defined on $\Gamma \backslash G$.

We set

$$B_0^C(T, \Omega_1) := B_0 \cap \Omega_1 A_T^+(C) K;$$
$$B^C(T, \Omega_1) := B \cap g\Omega_1 A_T^+(C) K g^{-1};$$
$$N_T^C(\hat{\Omega}_1, \hat{\Omega}_2) := \#\Gamma \cap gK A_T^-(C)\Omega_1^{-1} \cap \Omega_2 B g$$

where $A_T^-(C) = \{a_{-t} : C < t < T\}$ and $A_T^+(C) = \{a_t : C < t < T\}$. When $C = 0$, we simply omit the superscript 0 from the above notation.

Note that

$$N_T^C(\hat{\Omega}_1, \hat{\Omega}_2) = \{\gamma \in \Gamma : \gamma_{K^g} \in \Omega_2, \gamma_B^{-1} \in B^C(T, \Omega_1)\}.$$

Lemma 3.1. *Let $C > 1$ be taken so that (3.1) holds. For any $T > 1$ and small $\eta > 0$, we have*

(1)
$$N_T^C(\hat{\Omega}_1, \hat{\Omega}_2) \leq \int_{B_0(T+\varepsilon, \Omega_{1,\epsilon}^+)} F_{\Omega_2,\eta}^+(b_0) d\rho_\ell(b_0);$$

(2)
$$\int_{B_0^C(T-\varepsilon, \Omega_{1,\epsilon}^-)} F_{\Omega_2,\eta}^-(b_0) d\rho_\ell(b_0) \leq N_T(\hat{\Omega}_1, \hat{\Omega}_2)$$

where $\Omega_{1,\epsilon}^+ = \Omega_1 V_\epsilon$ and $\Omega_{1,\epsilon}^- = \cap_{k \in V_\epsilon} \Omega_1 k$ and $\epsilon = \epsilon(\eta)$.

Proof. For simplicity, we set $F^\pm := F_{\Omega_2,\eta}^\pm$ and $\Omega_1^\pm := \Omega_{1,\epsilon}^\pm$. We have

$$\int_{B_0(T+\varepsilon, \Omega_{1,\epsilon}^+)} F^+(b_0) d\rho_\ell(b_0)$$

$$= \int_{B(T+\epsilon, \Omega_1^+)} F^+(g^{-1} bg) d\rho(b)$$

$$\geq \int_{B(T+\epsilon, \Omega_1^+)} \sum_{\gamma \in \Gamma} \chi_{\Omega_2}(\gamma_{K^g}) \psi_\epsilon(\gamma_B b) d\rho(b)$$

$$= \sum_{\gamma \in \Gamma, \ \gamma_{K^g} \in \Omega_2} \int_{\gamma_B B(T+\epsilon, \Omega_1^+) \cap \mathcal{O}_\epsilon} \psi_\epsilon(b) d\rho(b)$$

since ρ is left-invariant. Since we have chosen $\mathcal{O}_\varepsilon$ so that $B^C(T, \Omega_1)\mathcal{O}_\epsilon \subset B(T + \epsilon, \Omega_1^+)$, for any $\gamma \in \Gamma$ such that $\gamma_B^{-1} \in B^C(T, \Omega_1)$,

$$B(T + \epsilon, \Omega_1^+) \cap \gamma_B^{-1}\mathcal{O}_\epsilon = \gamma_B^{-1}\mathcal{O}_\varepsilon,$$

and hence

$$\int_{\gamma_B B^C(T+\epsilon, \Omega_1^+)\cap \mathcal{O}_\epsilon} \psi_\epsilon(b)d\rho(b) = \int_{\mathcal{O}_\epsilon} \psi_\epsilon(b)d\rho(b) = 1.$$

It follows that

$$\int_{B(T+\epsilon, \Omega_1^+)} F^+(b)d\rho_\ell(b) \geq \#\{\gamma \in \Gamma : \gamma_{K^g} \in \Omega_2, \gamma_B^{-1} \in B^C(T, \Omega_1)\}$$

$$= N_T^C(\hat{\Omega}_1, \hat{\Omega}_2).$$

Similarly, we have

$$\int_{B_0^C(T-\epsilon, \Omega_1^-)} F^-(b_0)d\rho_\ell(b_0)$$

$$= \int_{B^C(T-\epsilon, \Omega_1^-)} F^-(g^{-1}bg)d\rho(b)$$

$$\leq \int_{B^C(T-\epsilon, \Omega_1^-)} \sum_{\gamma \in \Gamma} \chi_{\Omega_2}(\gamma_{K^g})\psi_\epsilon(\gamma_B b)d\rho(b)$$

$$= \sum_{\gamma \in \Gamma,\ \gamma_{K^g} \in \Omega_2} \int_{\gamma_B B^C(T-\epsilon, \Omega_1^-)\cap \mathcal{O}_\epsilon} \psi_\epsilon(b)d\rho(b).$$

Since $\Omega_1^- V_\epsilon \subset \Omega_1$, we have

$$B^C(T - \epsilon, \Omega_1^-)\mathcal{O}_\epsilon \subset B(T, \Omega_1).$$

Therefore for $\gamma \in \Gamma$ such that $\gamma_B^{-1} \notin B(T, \Omega_1)$, we have $\rho_\ell(B^C(T-\epsilon, \Omega_1^-)\cap\gamma_B^{-1}\mathcal{O}_\varepsilon) = 0$.

Hence it follows that

$$\int_{B^C(T-\epsilon, \Omega_1^-)} F(b)d\rho(b) \leq \#\{\gamma \in \Gamma : \gamma_{K^g} \in \Omega_2, \gamma_B^{-1} \in B(T, \Omega_1)\} = N_T(\hat{\Omega}_1, \hat{\Omega}_2).$$

$\square$

Lemma 3.2. *Let $k \in K$ and $k_2 \in K^g$. Writing $k^{-1}k_2g = a_r n k_0 \in ANK$, we have*

$$r = \beta_{k\xi}(y, x).$$

Proof. Since $\xi = \lim_{t\to\infty} a_{-t}x$, we compute that

$$\beta_{k\xi}(y, x) = \beta_{k\xi}(k_2 y, x)$$

$$= \beta_\xi(k^{-1}k_2 y, x)$$

$$= \lim_{t\to\infty} d(a_r n k_0 x, a_{-t}x) - t$$

$$= \lim_{t\to\infty} d(a_r(a_t n a_{-t})a_t k_0 x, x) - t = r.$$

$\square$

For simplicity, we set $F_\eta^\pm := F_{\Omega_2, \eta}^\pm$ and $f_\eta^\pm := f_{\Omega_2, \eta}^\pm$.

Lemma 3.3. *We have*
$$\limsup_{\eta} m_\Gamma^{\mathrm{BR}}(F_\eta^+) = \liminf_{\eta} m_\Gamma^{\mathrm{BR}}(F_\eta^-) = \nu_y(\hat\Omega_2).$$

Proof. We use the formula for $\tilde m^{\mathrm{BR}}$: for any $\Psi \in C_c(G)^M$,
$$\widetilde m^{\mathrm{BR}}(\Psi) = \int_{KAN} \Psi(ka_rn)e^{-\delta r}dndrd\nu_x(k(\xi)).$$

Define functions $\mathfrak{R}_\epsilon, \mathfrak{R}_\epsilon^+, \mathfrak{R}_\epsilon^-$ on G: for $h = a_rnk \in ANK$,
$$\mathfrak{R}_\epsilon(h) = e^{-\delta_\Gamma r}\chi_{V_\epsilon}(k), \ \mathfrak{R}_\epsilon^+(h) = e^{-\delta_\Gamma r}\chi_{V_\epsilon^+}(k), \ \mathfrak{R}_\epsilon(h) = e^{-\delta_\Gamma r}\chi_{V_\epsilon^-}(k).$$

Note that
$$\int_B \psi_\epsilon(b^{-1})d\rho(b) = \int_{AN}\psi_\epsilon(ga_tng^{-1})e^{-(n-1)t}dtdn$$

and hence
$$e^{-(n-1)\epsilon} \le \int_B \psi_\epsilon(b^{-1})d\rho(b) \le e^{(n-1)\epsilon}.$$

We then have
$$m_\Gamma^{\mathrm{BR}}(F_\eta^+ * \chi_{V_\epsilon}) = \widetilde m^{\mathrm{BR}}(f_\eta^+ * \chi_{V_\epsilon})$$
$$= \int_{KAN}\int_{k_1 \in V_\epsilon} f_\eta^+(k(a_rnk_1))\chi_{V_\epsilon}(k_1)e^{-\delta r}dk_1dndrd\nu_x(k(\xi))$$
$$= \int_{k \in K}\int_{h \in G} f_\eta^+(kh)\mathfrak{R}_\epsilon(h)dhd\nu_x(k(\xi))$$
$$= \int_{k \in K}\int_{h \in G} f_\eta^+(h)\mathfrak{R}_\epsilon(k^{-1}h)dhd\nu_x(k(\xi))$$
$$= \int_{k \in K}\int_{k_2 \in K^g}\int_{b \in B} f_\eta^\pm(k_2b^{-1}g)\mathfrak{R}_\epsilon(k^{-1}k_2b^{-1}g)d\rho(b)dk_2d\nu_x(k(\xi))$$
$$= \int_{k \in K}\int_{k_2 \in K^g}\int_{b \in B} \phi_{\Omega_2,\epsilon(\eta)}^+(k_2)\psi_{\epsilon(\eta)}(b^{-1})\mathfrak{R}_\epsilon(k^{-1}k_2b^{-1}g)d\rho(b)dk_2d\nu_x(k(\xi))$$
$$= (1 + O(\eta))\int_{k_2 \in K^g}\int_{k \in K} \phi_{\Omega_2,\epsilon(\eta)}^+(k_2)\mathfrak{R}_\epsilon^+(k^{-1}k_2g)dk_2d\nu_x(k(\xi)).$$

For $h \in G$, define $\hat k_h \in K$ to be the unique element such that
$$h \in B_0\hat k_h.$$

We note that
$$\hat k_{k^{-1}k_2g} = \hat k_{k^{-1}g}(g^{-1}k_2g).$$

Hence together with Lemma 3.2,
$$\mathfrak{R}_\epsilon^\pm(k^{-1}k_2g) = \chi_{V_\epsilon^\pm}(\hat k_{k^{-1}g}(g^{-1}k_2g)) \cdot \epsilon^{-\delta_\Gamma\beta_{k\xi}(y,x)}.$$

Define functions $\tilde\phi_{\Omega_2,\epsilon}^\pm \in C(K^g)^{M^g}$ by
$$\tilde\phi_{\Omega_2,\epsilon}^+(k_2) := \sup_{k \in U_\epsilon^+} \phi_{\Omega_2,\epsilon}^+(k_2k) \quad \text{and} \quad \tilde\phi_{\Omega_2,\epsilon}^-(k_2) := \inf_{k \in U_\epsilon^-} \phi_{\Omega_2,\epsilon}^-(k_2k).$$

Note that $0 \le \tilde\phi_{\Omega_2,\epsilon}^+ \le 1$ vanishes outside $\Omega_2U_\epsilon^+$ and is 1 on Ω_2.

Therefore, using the conformal property of $\{\nu_x : x \in \mathbb{H}^n\}$:
$$e^{-\delta_\Gamma\beta_{k\xi}(y,x)}d\nu_x(k\xi) = d\nu_y(k\xi),$$

we have

$$\int_{k_2 \in K^g} \int_{k \in K} \phi_{\Omega_2, \epsilon(\eta)}^+(k_2) \chi_{V_\epsilon^+}(\hat{k}_{k^{-1}g}(g^{-1}k_2 g)) \cdot e^{-\delta_\Gamma \beta_{k\xi}(y,x)} dk_2 d\nu_x(k(\xi))$$

$$= \int_{k_2 \in K^g} \int_{k \in K} \phi_{\Omega_2, \epsilon(\eta)}^+(g\hat{k}_{k^{-1}g}^{-1} g^{-1} k_2) \chi_{V_\epsilon^+}(g^{-1}k_2 g) dk_2 d\nu_y(k(\xi))$$

$$\leq \int_{k_2 \in K^g} \int_{k \in K} \tilde{\phi}_{\Omega_2, \epsilon(\eta)}^+(g\hat{k}_{k^{-1}g}^{-1} g^{-1}) \chi_{V_\epsilon^+}(g^{-1}k_2 g) dk_2 d\nu_y(k(\xi))$$

$$= (1 + O(\eta)) \operatorname{Vol}(V_\epsilon) \int_{k \in K} \tilde{\phi}_{\Omega_2, \epsilon(\eta)}^+(g\hat{k}_{k^{-1}g}^{-1} g^{-1}) d\nu_y(k(\xi)).$$

Since $kB_0 = (g\hat{k}_{k^{-1}g}^{-1} g^{-1})(gB_0)$ and B_0 stabilizes ξ, we have

(3.5) $$k(\xi) = (g\hat{k}_{k^{-1}g}^{-1} g^{-1})(g\xi).$$

Therefore we have

$$m_\Gamma^{\mathrm{BR}}(F_\eta^+ * \chi_{V_\epsilon})$$

$$= (1 + O(\eta)) \operatorname{Vol}(V_\epsilon) \int_{k' \in K^g} \tilde{\phi}_{\Omega_2, \epsilon(\eta)}^+(k') d\nu_y(k'(g(\xi)))$$

$$= (1 + O(\eta)) \operatorname{Vol}(V_\epsilon) \nu_y(\Omega_2(g(\xi))) \quad \text{by (3.4)}.$$

Hence we conclude

$$\limsup_\epsilon \frac{m_\Gamma^{\mathrm{BR}}(F_\eta^+ * \chi_{V_\epsilon})}{\operatorname{Vol}(V_\epsilon)} = (1 + O(\eta)) \nu_y(\Omega_2(g(\xi))).$$

Similarly we can deduce

$$\liminf_\epsilon \frac{m_\Gamma^{\mathrm{BR}}(F_\eta^- * \chi_{V_\epsilon})}{\operatorname{Vol}(V_\epsilon)} = (1 + O(\eta)) \nu_y(\Omega_2(g(\xi))).$$

On the other hand, it is not hard to deduce from the continuity of $F_\eta^\pm$ that

$$m_\Gamma^{\mathrm{BR}}(F_\eta^\pm) = \lim_\epsilon \frac{m_\Gamma^{\mathrm{BR}}(F_\eta^\pm * \chi_{V_\epsilon})}{\operatorname{Vol}(V_\epsilon)}.$$

Hence $\limsup_\eta m_\Gamma^{\mathrm{BR}}(F_\eta^+) = \nu_y(\Omega_2(g(\xi))) = \liminf_\eta m_\Gamma^{\mathrm{BR}}(F_\eta^-)$. $\qquad \square$

Proof of Theorem 1.1.

Since $\nu_x(\partial(\Omega_1)) = 0$ and any circle with center in $\Lambda(\Gamma)$ has measure zero by Lemma 2.5, we may choose V_ϵ so that $\nu_x(\partial(\Omega_{1,\epsilon}^+(\xi_x))) = \nu_x(\partial(\Omega_{1,\epsilon}^-(\xi_x))) = 0$.

By Lemma 3.1, Theorem 2.7 and Lemma 3.3, we have

$$\limsup_T \frac{N_T^C(\hat{\Omega}_1, \hat{\Omega}_2)}{e^{\delta_\Gamma T}} \leq \limsup_{T,\eta} \frac{1}{e^{\delta_\Gamma T}} \int_{B_0(T+\epsilon, \Omega_{1,\epsilon}^+)} F_\eta^+(b_0) d\rho_\ell(b_0)$$

$$= \limsup_\eta \frac{(1 + O(\eta)) \nu_x(\Omega_1(\xi_x))}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot \limsup_\eta m_\Gamma^{\mathrm{BR}}(F_\eta^+)$$

$$= \frac{\nu_x(\hat{\Omega}_1) \nu_y(\hat{\Omega}_2)}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|}.$$

Similarly,

$$\liminf_{T,} \frac{N_T(\hat{\Omega}_1, \hat{\Omega}_2)}{e^{\delta_\Gamma T}} \geq \liminf_{T,\eta} \frac{1}{e^{\delta_\Gamma T}} \int_{B_0(T-\epsilon, \Omega_{1,\epsilon}^-)} F_\eta^-(b_0) d\rho_\ell(b_0)$$

$$= \liminf_\eta \frac{(1 + O(\eta))\nu_x(\Omega_1(\xi_x))}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot \liminf_\eta m_\Gamma^{\mathrm{BR}}(F_\eta^-)$$

$$= \frac{\nu_x(\hat{\Omega}_1)\nu_y(\hat{\Omega}_2)}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|}.$$

Since $|N_T - N_T(C)| \leq \#\Gamma \cap K\{a_t : 0 \leq t \leq C\}K$ is a finite number independent of T, the above proves that

$$N_T(\hat{\Omega}_1, \hat{\Omega}_2) \sim e^{\delta T} \cdot \frac{\nu_x(\hat{\Omega}_1)\nu_y(\hat{\Omega}_2)}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|}.$$

$\square$

A. Appendix by François Maucourant

ABSTRACT. The purpose of this note is to show how one can recover a result in the spirit of Lim and Oh from a Theorem of Roblin. The following is part of the author's PhD Thesis [12], with some minor modifications, and some of these ideas have also been used in [9], but in the case of lattices in higher rank Lie groups.

Let (X, d) be a CAT(-1) space, and Γ a discrete, non-elementary subgroup of isometries of X. Denote by ∂X the visual boundary of X, $\overline{X} = X \cup \partial X$, δ the critical exponent of Γ, which is assumed finite, $\{\nu_x\}$ the Patterson-Sullivan density for Γ, and m_Γ^{BMS} the associated Bowen-Margulis-Sullivan measure. We shall assume that the length spectrum is non-arithmetic, and that m_Γ^{BMS} is of finite mass; remark that all these hypotheses are satisfied in the case of geometrically finite groups on hyperbolic spaces. First, let us state Roblin's Theorem.

Theorem 1. [17, Theorem 4.1.1] Let f a continuous function from $\overline{X}^2$ to $\mathbf{R}$, and $(x, y) \in X^2$. Then

$$\lim_{T \to +\infty} \frac{\delta \| m_\Gamma^{\mathrm{BMS}} \|}{e^{\delta T}} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} f(\gamma y, \gamma^{-1} x) = \int_{\partial X^2} f(\xi, \eta) d\nu_x(\xi) d\nu_y(\eta).$$

We shall prove here:

Theorem 2. Let f be a continuous function on $\overline{X}^2$, and $\zeta \in \partial X$.

$$\lim_{T \to +\infty} \frac{\delta \| m_\Gamma^{\mathrm{BMS}} \|}{e^{\delta T}} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} f(\gamma \zeta, \gamma^{-1} x) = \int_{\partial X^2} f(\xi, \eta) d\nu_x(\xi) d\nu_y(\eta).$$

A simplified version (where f does not depend on the second coordinate) appeared in [12]. The argument divides into three steps: first, we show a quantitative estimate for the recurrence of the action of Γ on the set of geodesics, of independent interest. Second, we show that the quantity on the left-hand side above does not depend too much on ζ. Third, we integrate over ζ to be able to apply Roblin's Theorem.

1: At most linear recurrence on the set of geodesics

Define $\mathcal{G} = (\partial X)^2 - diag$ to be the set of bi-infinite oriented geodesics on X, and let SX be the set of isometric embedding of $\mathbf{R}$ to X. The geodesic flow $(g^t)_{t \in \mathbf{R}}$ is the time-shift $g^t f(s) = f(s + t)$, and the canonical projection $\pi : SX \to X$ is the map $\pi(f) = f(0)$. We shall make the usual identification

$$SX = \mathcal{G} \times \mathbf{R},$$

and this can be done in such a way that $\pi((\xi, \eta), 0))$ is the point of the geodesic from ξ to η closest to a fixed reference point $o \in X$. In such coordinates, the geodesic flow is just $g^t((\xi, \eta), s) = ((\xi, \eta), s + t)$, whereas the action of Γ on SX defines a cocyle $c : \Gamma \times \mathcal{G} \to \mathbf{R}$, such that for any $((\xi, \eta), t) \in SX = \mathcal{G} \times \mathbf{R}$, we have

$$\gamma((\xi, \eta), t) = ((\gamma\xi, \gamma\eta), t + c(\gamma, (\xi, \eta))).$$

Note that $|c(\gamma, (\xi, \eta))|$ is the distance between the projections of o and $\gamma^{-1} o$ on the geodesic from ξ to η, and recall (see [1, Corollary 5.6]) that in CAT(0)

spaces, projection on a closed convex set is uniquely defined and 1-Lipschitz, so the following inequality holds for any $\gamma \in \Gamma$, $(\xi, \eta) \in \mathcal{G}$:

$$|c(\gamma, (\xi, \eta))| \leq d(o, \gamma o).$$

Proposition 1. Let K be a compact subset of $\mathcal{G}$, and $(x, y) \in X^2$. Then there exists $C_K > 0$ and $T_{x,y} > 0$ such that for any $(\xi, \eta) \in \mathcal{G}$, and any $T \geq T_{x,y}$,

$$|\{\gamma \in \Gamma \, : \, d(\gamma x, y) \leq T, (\gamma \xi, \gamma \eta) \in K\}| \leq C_K T.$$

Proof. For $v \in \Gamma \backslash SX$, define

$$f(\Gamma v) = \sum_{\gamma \in \Gamma} 1_{K \times [0,1]}(\gamma v).$$

Since Γ is discrete and acts properly on SX, and $K \times [0, 1]$ is a compact subset of SX, it follows that f is uniformly bounded by some constant C_0 depending only on K. Choose $T_{x,y} = d(o, x) + d(o, y) + 1$, $T \geq T_{x,y}$ and let $\gamma \in \Gamma$ such that $(\gamma \xi, \gamma \eta) \in K$, and $d(y, \gamma x) \leq T$. Define $v = ((\xi, \eta), 0)$, then $g^{-c(\gamma, (\xi, \eta))}(\gamma v) \in K \times \{0\}$. Thus,

$$\int_{-c(\gamma, (\xi, \eta))}^{-c(\gamma, (\xi, \eta))+1} 1_{K \times [0,1]}(\gamma g^t v) dt = 1,$$

and so, since $|c(\gamma, (\xi, \eta))| \leq d(o, \gamma o) \leq d(x, \gamma y) + d(o, x) + d(o, y) \leq T + T_{x,y} - 1$,

$$1 \leq \int_{-T - T_{x,y}}^{T + T_{x,y}} 1_{K \times [0,1]}(\gamma g^t v) dt.$$

Summing over all such γ, we obtain

$$|\{\gamma \in \Gamma \, : \, d(\gamma x, y) \leq T, (\gamma \xi, \gamma \eta) \in K\}| \leq \int_{-T - T_{x,y}}^{T + T_{x,y}} f(g^t \Gamma v) dt,$$

and the right hand side is bounded by $2(T + T_{x,y})C_0 \leq 4C_0 T$. $\qquad \square$

2: Second and third steps

Let f be a continuous function on $\overline{X}^2$. Define

$$F(\zeta, x, T) = \frac{1}{|\Gamma x \cap B_T(y)|} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} f(\gamma \zeta, \gamma^{-1} x).$$

Let $\epsilon > 0$, then since f is uniformly continuous, there exists a neighborhood U of the diagonal in ∂X^2 such that for any $(\xi, \eta) \in U$ and any $z \in X$, $|f(\xi, z) - f(\eta, z)| \leq \epsilon$. Let K be the complement of U, which is a compact subset of $\mathcal{G}$. So

$$|F(\xi, x, T) - F(\eta, x, T)| \leq \frac{1}{|\Gamma x \cap B_T(y)|}$$

$$* \left(\sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T, (\gamma \xi, \gamma \eta) \in U} \epsilon + \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T, (\gamma \xi, \gamma \eta) \in K} 2||f||_\infty \right),$$

By Proposition 1, the last sum contains at most $O(T)$ terms, so for sufficiently large T,

$$|F(\xi, x, T) - F(\eta, x, T)| \leq 2\epsilon.$$

This proves that $F(\zeta, x, T)$ does not depend too much on ζ for large T, so for any ζ, its value is close to the integral with respect to any probability measure. Fix y, it will then be sufficient to prove that the function

$$L(T, x, y) = \int_{\partial X} F(\zeta, x, T) \frac{d\nu_y(\zeta)}{||\nu_y||},$$

has limit $\frac{1}{||\nu_x||.||\nu_y||} \int f d\nu_x \nu_y$ as $T \to +\infty$; indeed, recall [17] that the orbital function satisfies

$$|\Gamma x \cap B_T(y)| \sim \frac{||\nu_x||.||\nu_y||}{\delta||m_\Gamma^{\mathrm{BMS}}||} e^{\delta T}.$$

Define the map g for any $z \in \Gamma y$ and any $x \in \overline{X}$ by:

$$g(z, x) = \frac{1}{||\nu_y||} \int_{\partial X} f(\zeta, x) d\nu_z(\zeta),$$

and extend g when z is in the limit set Λ_Γ, to be equal to $f(z, x)$. Then g is continuous on $\overline{\Gamma y} \times \overline{X}$. By Tietze-Urysohn's Theorem, g can be extended to a continuous function, still denoted by g, on $\overline{X}^2$, and moreover $\int g d\nu_x d\nu_y = \int f d\nu_x d\nu_y$. Then

$$L(T, x, y) = \frac{1}{|\Gamma x \cap B_T(y)|.||\nu_y||} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} \int_{\partial X} f(\zeta, \gamma^{-1} x) d\nu_{\gamma y}(\zeta),$$

$$= \frac{1}{|\Gamma x \cap B_T(y)|} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} g(\gamma y, \gamma^{-1} x),$$

and by Roblin's Theorem applied to the function g, we conclude that $L(T, x, y)$ has limit $\frac{1}{||\nu_x||.||\nu_y||} \int f d\nu_x d\nu_y$ as $T \to +\infty$, as desired.

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