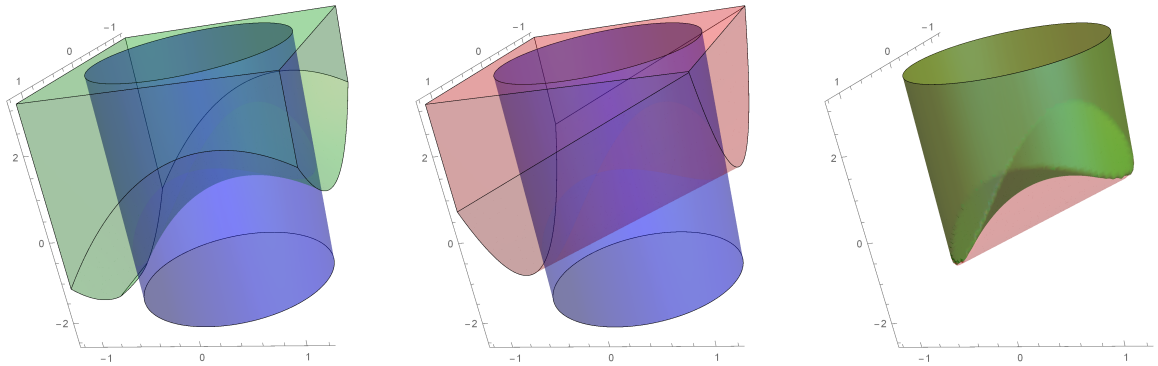

(a) $X = \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1$ (b) $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$ (c) X vs $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$

Figure 3: Plots of the epigraph of Example 3.5 without the linear inequality.

Now suppose $x \in \operatorname{relint}(\mathcal{H}^+)$. Since $x \in \operatorname{conv}(F)$, we can write x as a finite convex combination $x = \sum_k \lambda_k x^k$ where $\lambda_k > 0$, $\sum_k \lambda_k = 1$ and $x^k \in F$. Let $I = \{k : x^k \in \mathcal{H}^+\}$ and $J = \{k : x^k \notin \mathcal{H}^+\}$. If $J = \emptyset$, then we are done since $x \in \operatorname{conv}(F \cap \mathcal{H}^+)$. If not, for each $j \in J$, the line segment between x and x^j must intersect $\operatorname{bd}(\mathcal{H}^+)$ because $x \in \operatorname{relint}(\mathcal{H}^+)$ and $x^j \notin \mathcal{H}^+$. Thus, there exists a point $z^j = \alpha_j x + (1-\alpha_j)x^j \in \operatorname{bd}(\mathcal{H}^+)$ for some $\alpha_j \in (0,1)$. Note that $z^j \in \operatorname{conv}(F)$ because it is a convex combination of $x, x^j \in \operatorname{conv}(F)$. By a reasoning similar to the above, we deduce that $z^j \in F \cap \operatorname{bd}(\mathcal{H}^+)$. We now rewrite

$$x = \sum_{i\in I} \lambda_i x^i + \sum_{j\in J} \frac{\lambda_j}{1-\alpha_j}(z^j - \alpha_j x) \iff \left(1 + \sum_{j\in J} \frac{\alpha_j \lambda_j}{1-\alpha_j}\right) x = \sum_{i\in I} \lambda_i x^i + \sum_{j\in J} \frac{\lambda_j}{1-\alpha_j} z^j.$$

Notice also that

$$\begin{aligned} 1 + \sum_{j\in J} \frac{\alpha_j \lambda_j}{1-\alpha_j} &= \sum_{i\in I} \lambda_i + \sum_{j\in J} \lambda_j + \sum_{j\in J} \frac{\alpha_j \lambda_j}{1-\alpha_j} \\ &= \sum_{i\in I} \lambda_i + \sum_{j\in J} \frac{(1-\alpha_j)\lambda_j}{1-\alpha_j} + \sum_{j\in J} \frac{\alpha_j \lambda_j}{1-\alpha_j} \\ &= \sum_{i\in I} \lambda_i + \sum_{j\in J} \frac{\lambda_j}{1-\alpha_j}. \end{aligned}$$

Then we conclude that x is a convex combination of points $x^i, z^j \in F \cap \mathcal{H}^+$ as desired. □

3.1.1 Additional Hollow Constraints

Here we consider a hollow constraint $y \in \mathcal{H}$ given by $\mathcal{H} = \mathbb{R}^n \setminus \mathcal{P}$ where $\mathcal{P}$ is a given possibly nonconvex set. We impose the following condition on $\mathcal{H} = \mathbb{R}^n \setminus \mathcal{P}$.

Condition 3.7. The set $\mathcal{P} \subseteq \mathbb{R}^n$ satisfies $\mathcal{P} \subseteq \left\{y : \|y\|^2 < 1,\ Ay - b \in \mathcal{K}\right\}$.

If $\mathcal{P} = \bigcup_{i=1}^m E_i$ is a union of open ellipsoids $E_i = \left\{y : y^\top A_i y + 2b_i^\top y + c_i \le 0\right\}$, then Condition 3.7 can be checked by solving

$$v_i = \min_y \left\{1 - \|y\|^2 :\ y^\top A_i y + 2b_i^\top y + c_i \le 0\right\}.$$

In particular, E_i satisfies Condition 3.7 if and only if $v_i > 0$. While this is a nonconvex quadratic program, our developments from the previous section give a tight SOC reformulation for it. In addition, the S-lemma ensures that the associated semidefinite relaxation is tight. Thus, Condition 3.7 can be verified efficiently when $\mathcal{P}$ is a union of ellipsoids.

A number of papers have studied conditions similar to Condition 3.7. Most notably, the *generalized trust region subproblem* corresponds to the case when $\mathcal{H}$ is a single lower bounded, nonconvex quadratic constraint $y^\top Dy \geq l$; this is studied in [5, 8, 32, 36, 39] for example. More recently, Yang et al. [38] study the case when the hollow set $\mathcal{P}$ is the disjoint union of ellipsoids which do not intersect the boundary of the unit ball $\{y : \|y\| \leq 1\}$. A number of these papers [32, 38, 39] show that the natural SDP relaxation is tight. As opposed to these results on tight SDP relaxations, it is shown in [7] that the general quadratic programming problem

$$\min_y \left\{ y^\top Q_0 y + 2g_0^\top y : y^\top Q_i y + 2g_i^\top y + c_i \leq 0,\ i = 1, \ldots, m \right\}$$

is polynomially solvable using a weak feasibility oracle, under the assumption that at least one quadratic constraint $y^\top Q_i y + 2g_i^\top y + c_i \leq 0$ is strictly convex. In a similar vein, [9] also study the TRS with additional ellipsoidal hollow constraints. In [9], instead of giving the convex hull, they explore conditions that allow for polynomial solvability using a combinatorial enumeration technique and thus are able to cover cases where the hollow set $\mathcal{P}$ may not be contained in the unit ball. On a related domain, [8] studies the characterization and separation of valid linear inequalities that convexify the epigraph of a convex, differentiable function whose domain is restricted to the complement of a convex set defined by linear or convex quadratic inequalities.

We note that these papers [5, 7, 8, 9, 32, 36, 38, 39] consider the more general case of minimizing an arbitrary quadratic objective, which can be convex, over a domain given by possibly nonconvex quadratic constraints. On the other hand, our result applies to the special case of minimizing a nonconvex quadratic over the unit ball, a convex quadratic constraint. As a result, we are able to relax the assumptions that the hollow set $\mathcal{P}$ is generated by quadratics and the ellipsoidal hollows are disjoint. Specifically, we show that under Condition 3.7, the original SOC based reformulation of the TRS with additional conic constraints obtained by ignoring $\mathcal{H}$ is tight.

We start with the following lemma on the structure of extreme points of $\text{conv}(X)$. Given a set S, we let $\text{Ext}(S)$ denote the set of extreme points of S and $\text{Rec}(S)$ denote its recession cone.

Lemma 3.7. *Let X be defined as in (9); and suppose that Condition 3.6 holds. Let $x = [y; 1; x_{n+2}]$ be an extreme point of $\text{conv}(X)$. Then x must satisfy $\|y\| = 1$.*

Proof. Suppose for contradiction that $\|y\| < 1$. By Corollary 3.4, $x \in \text{conv}(X)$ must satisfy

$$\begin{aligned} \|y\| &\leq 1 \\ x_{n+2} &\geq y^\top (Q - \lambda_Q I_n) y + 2g^\top y + \lambda_Q \\ Ay - b &\in \mathcal{K}. \end{aligned}$$

Let $d \neq 0$ be the vector given by Condition 3.6. Since d satisfies $Qd = \lambda_Q d$ and $\pm Ad \in \mathcal{K}$, for any $\epsilon \in \mathbb{R}$,

$$\begin{aligned} &(y + \epsilon d)^\top (Q - \lambda_Q I_n)(y + \epsilon d) + 2g^\top (y + \epsilon d) + \lambda_Q \\ &\quad = [y^\top (Q - \lambda_Q I_n) y + 2g^\top y + \lambda_Q] + 2g^\top d\epsilon \\ &\quad \leq x_{n+2} + 2g^\top d\epsilon. \end{aligned}$$

Since $\|y\|^2 < 1$, there exists $\epsilon > 0$ sufficiently small such that $\|y \pm \epsilon d\|^2 \leq 1$. Note that $A(y \pm \epsilon d) - b \in \mathcal{K}$ since $\pm Ad \in \mathcal{K}$. Now define the points

$$x^+ := [y + \epsilon d; 1; x_{n+2} + 2g^\top d\epsilon] \quad \text{and} \quad x^- := [y - \epsilon d; 1; x_{n+2} - 2g^\top d\epsilon].$$

Both x^+, x^- satisfy all constraints of $\text{conv}(X)$, and hence $x^+, x^- \in \text{conv}(X)$. In particular, $x^+ \neq x \neq x^-$, but $x = \frac{1}{2}(x^+ + x^-)$; and thus x cannot be an extreme point of $\text{conv}(X)$. □

Theorem 3.8. *Let X be defined in (9) and $\mathcal{H} = \mathbb{R}^n \setminus \mathcal{P}$ be a set satisfying Condition 3.7. Assume Condition 3.6 also holds. Then*

$$\text{conv}\left(\left\{[y; 1; x_{n+2}] : \begin{array}{rcl} \|y\| & \leq & 1 \\ y & \in & \mathcal{H} \\ Ay - b & \in & \mathcal{K} \\ y^\top Qy + 2g^\top y & \leq & x_{n+2} \end{array}\right\}\right) = \text{conv}(X).$$

Proof. Define $\mathcal{P}^+ = \left\{[y; 1; x_{n+2}] : y \in \mathcal{P},\ y^\top Qy + 2g^\top y \leq x_{n+2}\right\}$. Then

$$\mathcal{H}^+ = \mathbb{R}^{n+2} \setminus \mathcal{P}^+ = \{[y; 1; x_{n+2}] : y \in \mathcal{H}\} \cup \left\{[y; 1; x_{n+2}] : y^\top Qy + 2g^\top y > x_{n+2}\right\},$$

and hence

$$X \cap \mathcal{H}^+ = \left\{[y; 1; x_{n+2}] : \begin{array}{rcl} \|y\| & \leq & 1 \\ y & \in & \mathcal{H} \\ Ay - b & \in & \mathcal{K} \\ y^\top Qy + 2g^\top y & \leq & x_{n+2} \end{array}\right\}.$$

Now by our definition of $\mathcal{P}^+$ and the fact that $\mathcal{P} \subseteq \{y : \|y\| < 1,\ Ay - b \in \mathcal{K}\}$, we have $\text{bd}(\mathcal{P}^+) = \text{bd}(\mathcal{H}^+)$ and $\text{conv}(X) \cap \text{bd}(\mathcal{H}^+) \subseteq X$ because $\text{bd}(\mathcal{P}^+) \subseteq X$. By Lemma 3.6, we have

$$X \cap \mathcal{H}^+ \subseteq \text{conv}(X) \cap \mathcal{H}^+ \subseteq \text{conv}(X \cap \mathcal{H}^+),$$

and taking the convex hull of all sides gives us

$$\text{conv}(\text{conv}(X) \cap \mathcal{H}^+) = \text{conv}(X \cap \mathcal{H}^+).$$

It remains to show that $\text{conv}(\text{conv}(X) \cap \mathcal{H}^+) = \text{conv}(X)$. We can immediately deduce that $\text{conv}(X) \supseteq \text{conv}(\text{conv}(X) \cap \mathcal{H}^+)$; we next prove the reverse direction.

By Condition 3.7, $\mathcal{P} \subseteq \left\{y : \|y\|^2 < 1\right\}$, so by Lemma 3.7, we have $\text{Ext}(\text{conv}(X)) \cap \mathcal{P}^+ = \emptyset$, or equivalently

$$\text{Ext}(\text{conv}(X)) \subseteq \mathcal{H}^+.$$

Intersecting both sides with $\text{conv}(X)$, we obtain $\text{Ext}(\text{conv}(X)) \subseteq \text{conv}(X) \cap \mathcal{H}^+$. Note that $\text{conv}(X) = \text{conv}(\text{Ext}(\text{conv}(X))) + \text{Rec}(\text{conv}(X))$; so

$$\text{conv}(X) = \text{conv}(\text{Ext}(\text{conv}(X))) + \text{Rec}(\text{conv}(X)) \subseteq \text{conv}(\text{conv}(X) \cap \mathcal{H}^+) + \text{Rec}(\text{conv}(X)).$$

Furthermore, from Corollary 3.4, it is easy to see that

$$\text{Rec}(\text{conv}(X) \cap \mathcal{H}^+) = \{[0; 0; x_{n+2}] : x_{n+2} \geq 0\} = \text{Rec}(\text{conv}(X)).$$

Hence, we deduce that

$$\begin{aligned}\operatorname{conv}(X) &\subseteq \operatorname{conv}(\operatorname{conv}(X) \cap \mathcal{H}^+) + \operatorname{Rec}(\operatorname{conv}(X)) \\ &= \operatorname{conv}(\operatorname{conv}(X) \cap \mathcal{H}^+) + \operatorname{Rec}(\operatorname{conv}(X) \cap \mathcal{H}^+) \\ &= \operatorname{conv}(\operatorname{conv}(X) \cap \mathcal{H}^+).\end{aligned}$$

The last equality holds because $\operatorname{conv}(S) = \operatorname{conv}(S) + \operatorname{Rec}(S)$ for any set S. This gives us the result. □

3.1.2 Additional Convex Constraints

Lemma 3.6 allows us to add further convex constraints $x \in \mathcal{H}$ as long as these new constraints satisfy the following boundary condition.

Condition 3.8. The new set $\mathcal{H}$ satisfies $\mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1 \cap \operatorname{bd}(\mathcal{H}) \subseteq \mathcal{F}_1$.

Corollary 3.9. *Suppose $\mathcal{H}$ is convex and Conditions 3.6 and 3.8 are satisfied. Then we have*

$$\operatorname{conv}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1 \cap \mathcal{H}) = \mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1 \cap \mathcal{H}.$$

Proof. From Theorem 3.3, we have $\operatorname{conv}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1) = \mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1$. When $\mathcal{H}$ is convex, this immediately implies $\operatorname{conv}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1 \cap \mathcal{H}) \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1 \cap \mathcal{H}$. Condition 3.8 gives us $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1 \cap \operatorname{bd}(\mathcal{H}) \subseteq \mathcal{F}_1$. Then by selecting $F = \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1$, we satisfy the premise $\operatorname{conv}(F) \cap \operatorname{bd}(\mathcal{H}) \subseteq F \cap \mathcal{H}$ of Lemma 3.6. From Lemma 3.6, we deduce

$$\operatorname{conv}\left(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap \mathcal{K}^+ \cap H^1 \cap \mathcal{H}\right) \supseteq \mathcal{F}_0^+ \cap \mathcal{F}_s \cap \mathcal{K}^+ \cap H^1 \cap \mathcal{H}.$$

□

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