side of inequality (11) evaluated at o_{ν_1} and o_i is

$$\begin{aligned} w_{j^*_{\nu_1}} - w_{j^*_i} + v^-_{\bar{j}_{\nu_1}} - v^-_{\bar{j}_i} - \sum_{j \in V_{\nu_1} \setminus V_i} \theta_j &= w_{j^*_{\nu_1}} - w_{j^*_i} + v^-_{\bar{j}_{\nu_1}} - v^-_{\bar{j}_i} - \sum_{j \in V_{\nu_1+1} \setminus V_{\nu_2}} \theta_j \\ &= w_{j^*_{\nu_1}} - w_{j^*_i} + v^-_{\bar{j}_{\nu_1}} - v^-_{\bar{j}_i} - 2 \sum_{j \in V_{\nu_1+1} \setminus V_{\nu_2}} f_j \\ &= w_{j^*_{\nu_1}} - w_{j^*_i} + v^-_{\bar{j}_{\nu_1}} - v^-_{\bar{j}_i} - 2 \sum_{j \in V_{\nu_1} \setminus V_i} f_j \\ &= 2(F(o_{\nu_1}) - F(o_i)) = 0, \end{aligned}$$

where the first and third equations follow from $V_{\nu_1} \setminus V_i = V_{\nu_1+1} \setminus V_{\nu_2}$, and the second equation from the relation between the coefficients θ and f. As a result, the left-hand side of inequality (11) evaluated at o_{ν_1} is equal to that evaluated at o_i. Because we have shown that inequality (11) is tight at o_{ν_1}, it must be tight at o_i as well.

□

We are now ready to give the proof of Theorem 3.1.

Proof. [Proof of Theorem 3.1] From Proposition 3.1, we observe that $\mathcal{O}$ is composed of points of form $A(\emptyset) = C(\emptyset)$, D, and $A(V)$, $B(V)$, $C(V)$ for some $\emptyset \neq V \subset \Omega$.

First, we show that if $p_1 = 1$, then we can find an inequality (9) or (10) that is tight for this single point. If $o_1 = A(\emptyset)$ (or $o_1 = D$), then clearly every mixing inequality (9) ((10)) for any nonempty sequence S (T) is satisfied at equality by this point. Otherwise, if $o_1 = A(V)$ for some $\emptyset \neq V \subseteq \Omega$, then we consider two cases. If $w_{j^*_1} = w_{\alpha_1}$, then we let $S = \{j^*_1\}$. In this case, because $z_{j^*_1} = 0$, the left-hand side of inequality (9) generated by this choice of S is $w_{j^*_1} = w_{\alpha_1}$, implying that inequality (9) is tight at the point o_1. Otherwise, if $w_{j^*_1} < w_{\alpha_1}$, then we have $S = \{\alpha_1, j^*_1\}$ and $z^1_{\alpha_1} = 1$. In this case, the left-hand side of inequality (9) for this choice of S is $w_{j^*_1} + (w_{\alpha_1} - w_{j^*_1}) z_{\alpha_1} = w_{\alpha_1}$, which also implies that inequality (9) is tight at the point o_1. Otherwise, if $o_1 = B(V)$ or $o_1 = C(V)$, for some $\emptyset \neq V \subseteq \Omega$, then we can find a mixing inequality (10) using a procedure similar to the preceding discussion.

Next, if $p_1 \geq 2$, then we break the proof into several cases based on the objective coefficient c_d:

(i) Suppose $c_d > 0$. In this case, for any two points $o_i, o_j \in \mathcal{O}$ for $i, j \in [p_1]$, we first claim that both o_i, o_j are of the form $A(V)$ or $C(V)$ for some $V \subseteq \Omega$, such that we have both $\psi(\mathbf{w}, V_i, V_j) = 0$ and there exists $o_k \in \mathcal{O}$ satisfying $o_k \in \{A(V_i \cap V_j), C(V_i \cap V_j)\}$.

- Suppose $c_d > 1$. From Lemma 3.5(i), we deduce that only points of the form $A(V)$ for some $V \subseteq \Omega$ can be optimal for this type of a cost vector. For any two points $o_i = A(V_i)$, $o_j = A(V_j)$ such that $o_i, o_j \in \mathcal{O}$, Lemma 3.2(iii) implies that both $A(V_i \cap V_j) \in \mathcal{O}$ and $\psi(\mathbf{w}, V_i, V_j) = 0$.
- Suppose that $c_d = 1$. From Lemma 3.5(ii), we deduce that only points of the form $A(V)$ or $C(V)$ for some $V \subseteq \Omega$ can be optimal for this type of a cost vector. Given that $c_d = 1$, for any $o_i \in \mathcal{O}$, $i \in [p_1]$, $A(V_i)$ is an optimal solution if and only if $C(V_i)$ is an optimal solution because the sum $y_p + y_d$ is the same for the solutions $C(V)$ and $A(V)$ corresponding to the same set V. Then for any two points $o_1 \in \{A(V_1), C(V_1)\}$, $o_2 \in \{A(V_2), C(V_2)\}$ such that $o_1, o_2 \in \mathcal{O}$, Lemma 3.2(iii) implies that $\psi(\mathbf{w}, V_1, V_2) = 0$ and $A(V_i \cap V_j) \in \mathcal{O}$ and $C(V_i \cap V_j) \in \mathcal{O}$.
- If $0 < c_d < 1$, then $F\Big(D\Big) > F\Big(C(\emptyset)\Big)$. Hence, from Lemma 3.5(v), the points of the form

$C(V)$ for some $V \subseteq \Omega$ can be optimal, but $D \notin \mathcal{O}$. Moreover, for any $C(V_i), C(V_j) \in \mathcal{O}$ for $V_i, V_j \subseteq \Omega, V_i \neq V_j$, from Lemma 3.4 we have $\psi(\mathbf{w}, V_i, V_j) = 0$ and $C(V_i \cap V_j) \in \mathcal{O}$.

Then Lemma 3.6 shows that we can always find a sequence S such that the mixing inequality (9) corresponding to S is tight for all solutions in $\mathcal{O}$.

(ii) Suppose $c_d = 0$. From Lemma 3.5(v), we deduce that only points of the form $C(V)$ for some $V \subseteq \Omega$ and D can be optimal for this type of a cost vector. Recall from Lemma 3.4, we have for any $C(V_i), C(V_j) \in \mathcal{O}$ for $V_i, V_j \subseteq \Omega, V_i \neq V_j$ that $\psi(\mathbf{w}, V_i, V_j) = 0$ and $C(V_i \cap V_j) \in \mathcal{O}$. Consequently, observe that if $D \notin \mathcal{O}$, then all points in $\mathcal{O}$ are of the form $C(V)$ for some $V \subseteq \Omega$. In such a case, Lemma 3.6 shows that we can always find a sequence S such that the mixing inequality (9) corresponding to S is tight for all solutions in $\mathcal{O}$. Therefore, without loss of generality we can assume that $D \in \mathcal{O}$.

Now note that $F\Big(D\Big) = F\Big(C(\emptyset)\Big)$ because $c_d = 0$. Therefore, we have both $D, C(\emptyset) \in \mathcal{O}$.

- When $p_1 = 2$, since $D, C(\emptyset) \in \mathcal{O}$, we have $\mathcal{O} = \{D, C(\emptyset)\}$. In this case, the generalized mixing inequality (11) corresponding to $\Pi = \{\alpha_1 \to \beta_1\}$ is tight for all solutions in $\mathcal{O}$.
- When $p_1 > 2$, we have $\mathcal{O} \supset \{D, C(\emptyset)\}$. Then there exists $o_i = C(V_i) \in \mathcal{O}$, $i \in [p_1]$ for some $\emptyset \neq V_i \subseteq \Omega$. In this case, Lemma 3.8 shows that we can always find a sequence $\Pi \subseteq \Omega$ such that the generalized mixing inequality (11) corresponding to Π is tight for all solutions in $\mathcal{O}$.

(iii) Suppose $c_d < 0$. In this case, for any two points $o_i, o_j \in \mathcal{O}$, we first claim that both o_i, o_j are of the form D, $B(V)$ or $C(V)$ for some $\emptyset \neq V \subseteq \Omega$. Moreover, we claim that for any $o_i \in \{B(V_i), C(V_i)\}$ and $o_j \in \{B(V_j), C(V_j)\}$ with $\emptyset \neq V_i, V_j \subseteq \Omega$, we have both $\psi(\mathbf{v}, V_i, V_j) = 0$ and there exists $o_k \in \mathcal{O}$ satisfying $o_k \in \{B(V_i \cap V_j), C(V_i \cap V_j)\}$.

- Suppose $c_d < -1$. From Lemma 3.5(iii), we deduce that only points of form D or $B(V)$ for some $\emptyset \neq V \subseteq \Omega$ can be optimal for this type of a cost vector. For any two points $o_i = B(V_i)$, $o_j = B(V_j)$ such that $o_i, o_j \in \mathcal{O}$, Lemma 3.3 implies that $\psi(\mathbf{v}, V_i, V_j) = 0$ and $B(V_i \cap V_j) \in \mathcal{O}$. Because $B(V_i \cap V_j) \in \mathcal{O}$ and $B(\emptyset) \notin \mathcal{O}$, we deduce $V_i \cap V_j \neq \emptyset$.
- Suppose that $c_d = -1$. From Lemma 3.5(iv), we deduce that only points of the form D, $B(V)$ or $C(V)$ for some $\emptyset \neq V \subseteq \Omega$ can be optimal for this type of a cost vector. Given that $c_d = -1$, for any $o_i \in \mathcal{O}$, $i \in [p_1]$, $B(V_i) \in \mathcal{O}$ if and only if $C(V_i)) \in \mathcal{O}$ because the term $y_p - y_d$ is the same for the solutions $C(V)$ and $B(V)$ corresponding to the same set V. Then for any two points $o_i \in \{B(V_i), C(V_i)\}$, $o_j \in \{B(V_j), C(V_j)\}$ such that $o_i, o_j \in \mathcal{O}$, we deduce $C(V_i), C(V_j) \in \mathcal{O}$ and then Lemma 3.4 implies that $\psi(\mathbf{v}, V_i, V_j) = 0$ and $C(V_i \cap V_j) \in \mathcal{O}$. Because $C(V_i \cap V_j) \in \mathcal{O}$ and $C(\emptyset) \notin \mathcal{O}$, we conclude that $V_i \cap V_j \neq \emptyset$, and $B(V_i \cap V_j) \in \mathcal{O}$.
- If $-1 < c_d < 0$, then $F\Big(D\Big) < F\Big(C(\emptyset)\Big)$. Hence, from Lemma 3.5(v), only the points of the form D or $C(V)$ for some $\emptyset \neq V \subseteq \Omega$ can be optimal. Moreover, for any $C(V_i), C(V_j) \in \mathcal{O}$ for $\emptyset \neq V_i, V_j \subseteq \Omega$, Lemma 3.4 implies $\psi(\mathbf{v}, V_i, V_j) = 0$ and $C(V_i \cap V_j) \in \mathcal{O}$. Once again, because $C(V_i \cap V_j) \in \mathcal{O}$ and $C(\emptyset) \notin \mathcal{O}$, we deduce that $V_i \cap V_j \neq \emptyset$.

Then Lemma 3.7 shows that we can always find a sequence T such that the mixing inequality (10) corresponding to T is tight for all solutions in $\mathcal{O}$.

□

4. Separation of Inequalities (11) In this section, we discuss exact and heuristic separation approaches for inequality (11). Let $(\hat{y}_p, \hat{y}_d, \hat{\mathbf{z}})$ be a fractional solution. In order to find the most violated

inequality (11), we need to find a sequence $\Pi = \{\pi_1 \to \pi_2 \to \cdots \to \pi_\tau\}$ that minimizes the value of the term $\sum_{j=1}^{\tau} \Big((w_{\pi_j} - \bar{w}_{\Pi,j})_+ + (v_{\pi_j} - \bar{v}_{\Pi,j})_+\Big)\hat{z}_{\pi_j}$. Throughout this section, this value is interpreted as *cost*.

4.1 An Exact Separation Approach In this section, we give a polynomial-time dynamic programming algorithm to separate inequality (11) exactly. Without loss of generality, we assume that the sequence Π has length m. Here, we only consider the case where $\alpha_1 \in \Pi$ and $\beta_1 \in \Pi$, because otherwise the resulting inequality can be strengthened by including α_1 and β_1.

In our dynamic programming algorithm, the states are given by $(i, j, \bar{W}_{i-1}, \bar{V}_{i-1})$ for $i, j \in \Omega = [m]$, $\bar{W}_{i-1} \geq w_j$ and $\bar{V}_{i-1} \geq v_j$, where $\bar{W}_{i-1}$ and $\bar{V}_{i-1}$ represent the values of $\bar{w}_{\Pi,i-1}$ and $\bar{v}_{\Pi,i-1}$ for the constructed sequence Π, respectively. Note that for any $k \in [m]$, the eligible values of $\bar{W}_k$ and $\bar{V}_k$ are from the entries of the vectors $\mathbf{w}$ and $\mathbf{v}$, respectively. The state function is $\bar{G}_i(j, \bar{W}_{i-1}, \bar{V}_{i-1})$, which is defined as the *minimum* cost of the subsequence $\pi_i \to \pi_{i+1} \to \cdots \to \pi_m$, where item j is the first item in this subsequence (i.e., $\pi_i = j$), $\max\{w_{\pi_j} : i \leq j \leq m\} = \bar{w}_{\Pi,i-1} = \bar{W}_{i-1}$, and $\max\{v_{\pi_j} : i \leq j \leq m\} = \bar{v}_{\Pi,i-1} = \bar{V}_{i-1}$. Note that there are $\mathrm{O}(m^4)$ many possible states $(i, j, \bar{W}_{i-1}, \bar{V}_{i-1})$, because $i,\ j \in [m]$ and $\bar{W}_{i-1} = w_k$ for some $k \in \Omega$ and $\bar{V}_{i-1} = v_{k'}$ for some $k' \in \Omega$.

Next, the boundary condition is defined as:

$$\bar{G}_m(j, \bar{W}_{m-1}, \bar{V}_{m-1}) = \begin{cases} (w_j + v_j)\hat{z}_j, & \text{if } \bar{W}_{m-1} = w_j, \text{ and } \bar{V}_{m-1} = v_j, \\ +\infty, & \text{if } \bar{W}_{m-1} > w_j, \text{ or } \bar{V}_{m-1} > v_j, \end{cases}$$

where the state $\bar{G}_m(j, \bar{W}_{m-1}, \bar{V}_{m-1})$, in which $\bar{W}_{m-1} > w_j$ or $\bar{V}_{m-1} > v_j$ is infeasible, because if item $j = \pi_m$, then we must have $\bar{W}_{m-1} = w_j$ and $\bar{V}_{m-1} = v_j$. The optimal solution is then given by $\min\Big\{\bar{G}_1(\alpha_1, w_{\alpha_1}, v_{\beta_1}),\ \bar{G}_1(\beta_1, w_{\alpha_1}, v_{\beta_1})\Big\}$, because α_1 and β_1 are in Π, and without loss of generality, we have $w_{\pi_1} = w_{\alpha_1}$ or $v_{\pi_1} = v_{\beta_1}$.

Finally, we give the backward transition function

$$\bar{G}_i(j, \bar{W}_{i-1}, \bar{V}_{i-1}) =$$
$$\begin{cases} \min\limits_{j' \in \Omega} \Big\{\bar{G}_{i+1}(j', \bar{W}_{i-1}, \bar{V}_{i-1})\Big\}, & \text{if } \bar{W}_{i-1} > w_j, \text{ and } \bar{V}_{i-1} > v_j, \\ \min\limits_{j' \in \Omega, \bar{W}_i \leq w_j, \bar{V}_i \leq v_j} \Big\{\bar{G}_{i+1}(j', \bar{W}_i, \bar{V}_i) + (w_j + v_j - \bar{W}_i - \bar{V}_i)\hat{z}_j\Big\}, & \text{if } \bar{W}_{i-1} = w_j, \text{ and } \bar{V}_{i-1} = v_j, \\ \min\limits_{j' \in \Omega, \bar{W}_i \leq w_j} \Big\{\bar{G}_{i+1}(j', \bar{W}_i, \bar{V}_{i-1}) + (w_j - \bar{W}_i)\hat{z}_j\Big\}, & \text{if } \bar{W}_{i-1} = w_j, \text{ and } \bar{V}_{i-1} > v_j, \\ \min\limits_{j' \in \Omega, \bar{V}_i \leq v_j} \Big\{\bar{G}_{i+1}(j', \bar{W}_{i-1}, \bar{V}_i) + (v_j - \bar{V}_i)\hat{z}_j\Big\}, & \text{if } \bar{W}_{i-1} > w_j, \text{ and } \bar{V}_{i-1} = v_j. \end{cases}$$

The running time of executing the transition function is $\mathrm{O}(m^3)$, so the total running time of this dynamic programming algorithm is $O(m^7)$.

4.2 Heuristic Separation Approaches Given the inefficiency of the exact separation algorithm in Section 4.1, in this section we discuss simple heuristic separation approaches for (11) with complexity $O(m \log m)$. The underlying idea for these heuristics is that instead of trying to minimize the overall term $\sum_{j=1}^{\tau} \Big((w_{\pi_j} - \bar{w}_{\Pi,j})_+ + (v_{\pi_j} - \bar{v}_{\Pi,j})_+\Big)\hat{z}_{\pi_j}$, we can aim to minimize the partial sum terms for involving only w_j's or only v_j's.

Suppose that the polynomial-time separation algorithm in [5] is applied to find the mixing sequences

$S^* = \{s_1 = \alpha_1 \to s_2 \to \cdots \to s_\eta\}$ and $T^* = \{t_1 = \beta_1 \to t-2 \to \cdots \to t_\rho\}$ that maximize the violation of inequalities (9) and (10), respectively. Next, based on the sequences S^* and T^*, as a heuristic separation procedure for (11), we generate the sequence $\Pi := t_1 \to t_2 \to \cdots \to t_\rho \to s_1 \to s_2 \to \cdots \to s_\eta$, where we append sequence S^* after the sequence T^*. Then clearly $R[\Pi] \supseteq S^*$ because $s_1 = \alpha_1$ and by definition of the sequences α and S^*, we have $w_{s_1} = w_{\alpha_1} \geq w_{t_i}$ for all $i \in [\rho]$ as well as $w_{s_1} \geq w_{s_i}$ for all $i \in [\eta]$ which implies $\bar{w}_{\Pi,1} = w_{\alpha_1}$. In fact, we have $R[\Pi] = S^*$ whenever the values of w_j are distinct because then we would have $w_{s_1} = w_{\alpha_1} > w_{t_i}$ for all $i \in [\rho]$. Therefore, the sequence Π generated by this procedure minimizes the partial summation term $\sum_{j=1}^{\tau}(w_{\pi_j} - \bar{w}_{\Pi,j})_+\hat{z}_{\pi_j}$ in (11). Besides, the inequality (11) generated by this choice of Π is different from inequality (9) for S^* because $G[\Pi] \neq \emptyset$ (note $\beta_1 = t_1 \in G[\Pi]$ and $s_\eta \in G[\Pi]$) and the right hand side values of the two inequalities will differ whenever $\mathbf{v} > \mathbf{0}$.

Similarly, an alternative heuristic separation for (11) is given by generating a sequence Π where we append T^* *after* S^*, then the resulting sequence Π minimizes the partial summation term $\sum_{j=1}^{\tau}(v_{\pi_j} - \bar{v}_{\Pi,j})_+\hat{z}_{\pi_j}$, because $G[\Pi] \supseteq T^*$ (and whenever the values of v_j are distinct, $G[\Pi] = T^*$).

5. Preliminary Numerical Study and Future Directions In this section, we study the computational performance of the proposed inequality (11) against adding only mixing inequalities (9) and (10) on randomly generated test instances. Our test instances are deterministic equivalents of chance-constrained programs taking the form $\min\{\boldsymbol{\xi}^\top \mathbf{x} : (8a) - (8b), (6b) - (6c),\ \mathbf{x} \geq \mathbf{0},\ y_p = \sum_{i=1}^{5} p_i x_i,\ y_d = \sum_{i=1}^{5} d_i x_i,\ 0 \leq y_d \leq u_d\}$. We generate different classes of problems with varying scenario sizes by selecting $m = k \cdot 1000$ where $k \in [4]$, and $\epsilon \in \{0.1, 0.15, 0.2\}$. For each problem class, we generate three instances and report the averages. We assume that each scenario is equally likely. For $i \in [5]$, we generate p_i and d_i from uniform distribution $U[0,1]$, and ξ_i from $U[1,2]$. In addition, for all $j \in \Omega$, q_j is generated as $\max\{q'_j, 0\}$, where q'_j follows the normal distribution $N(40,10)$, and $h_j = \min\{q_j, h'_j\}$, where h'_j follows $N(20,10)$. Furthermore, u_d is taken as $\max_{j\in\Omega}\{h_j + q_j\}$. This data generation scheme ensures that **Assumptions A1**, **A2**, and **A3** are satisfied. Our test instances are available in an Online Supplement at http://faculty.washington.edu/simge/IntMixOS.zip.

All runs are executed on a Windows 7 with 2.27GHZ Intel(R) Core(TM) i3 CPU and 2.0 GB RAM. We implemented our algorithms using C programming language, with Microsoft Visual Studio 2008 and CPLEX 12.4 in its default setting. A time limit of one hour is used for each problem instance.

In our computational study, we separate and add inequalities (9)–(11) only at the root node using the user cut callback function of CPLEX. In particular, at each fractional solution $(\hat{\mathbf{x}}, \hat{y}_p, \hat{y}_d, \hat{\mathbf{z}})$ at the root node, we add at most one violated inequality of each type (9), (10) and (11), in this order, and re-solve the linear programming relaxation until either there are no further violated cuts from the given cut class or a predetermined cut limit is reached. We apply the polynomial-time separation algorithm from [2] to find the mixing sequences $S^* = \{s_1 = \alpha_1 \to s_2 \to \cdots \to s_\eta\}$ and $T^* = \{t_1 = \beta_1 \to t-2 \to \cdots \to t_\rho\}$ that maximize the violation of inequalities (9) and (10), respectively. The exact separation algorithm presented in Section 4.1 is inefficient. Hence, we use some of the heuristic separation ideas presented in Section 4.2. In particular, we use the heuristic that constructs Π by appending sequence S^* after the sequence T^*. In our preliminary tests, the alternative heuristic of appending T^* *after* S^* did not provide significant improvements in the computational performance. While our focus in this paper is not on finding the most effective heuristic to separate inequality (11), we observe that our heuristic strikes a good balance between efficiency of cut generation and effectiveness in improving the integrality gaps and computational times.

In our preliminary numerical tests, we observe that a significant amount of time was being spent at the root node without too much gap improvement after a certain number of cuts are added. Thus, in our computational study, we imposed a limit on the number of cuts added at the root node. In particular, the number of mixing inequalities (9), (10) and new inequalities (11) that can be added is limited to $m \times \gamma$ for each class of inequalities, where we take $\gamma \in \{0.1, 0.2\}$ e.g., for instances with $m = 4000$ and $\gamma = 0.1$, the cut limit is set to 400 for each class of inequalities. In our experiments, the cut limits for mixing inequalities (9) and (10) were hit in every instance for both settings for γ. This was not the case for inequalities (11).

Table 2: Effectiveness of inequalities (11) on random two-sided chance-constrained problem instances.

Instances		DEP & Mix. Ineq. ($\gamma = 0.1$)			DEP & Mix. Ineq. ($\gamma = 0.2$)			DEP & Mix. Ineq. & New Ineq. ($\gamma = 0.1$)			
ϵ	m (10^3)	Time (#, %)	R.Gap (%)	Nodes	Time (#, %)	R.Gap (%)	Nodes	Time (#, %)	R.Gap (%)	Nodes	Cuts
0.10	1	45	12.0	2240	45	11.5	2328	30	11.2	262	88
	2	259	14.0	4926	220	13.2	4025	107	12.4	2615	200
	3	503	18.0	32472	480	16.2	280162	225	13.2	8327	273
	4	1240	22.4	501547	2876 (2, 2.9)	24.0	612957	826	19.0	242918	300
0.15	1	274	17.2	9325	300	18.0	10042	200	13.0	75208	100
	2	1452	19.3	125726	1328	18.5	102284	925	18.2	86122	180
	3	2507	23.0	190742	2775	23.2	204182	1550	20.5	178252	290
	4	3351 (1, 5.2)	24.2	228063	3600 (0, 5.7)	24.3	259175	2539	23.0	115254	377
0.20	1	841	19.7	54028	886	19.7	53325	623	19.0	49128	95
	2	3072	20.4	90426	3286(2, 0.8)	21.0	11050	2528	19.5	77296	185
	3	3600(0, 4.7)	22.0	105134	3600 (0, 4.5)	21.5	122057	2755(2, 1.2)	20.9	140231	282
	4	3600 (0, 6.5)	22.6	338036	3600 (0, 7.2)	22.3	289625	3600(0, 2.6)	21.7	286855	400

Table 2 summarizes our computational results. In Table 2, the section "DEP & Mix. Ineq." reports the results of using mixing inequalities (9) and (10) only, where the cuts limits of both type of mixing inequalities are set to $m \times \gamma$ for $\gamma = 0.1$ and 0.2, and the section "DEP & Mix. Ineq. (9), (10) & New Ineq." reports where both mixing inequalities and the new inequalities (11) are utilized. The column "Time" reports the average solution time in seconds for the instances that are solved to optimality within the time limit. Whenever all instances are not solved to optimality within the time limit, we report two additional statistics in parentheses (#, %). The first number in the parentheses is the number of instances that are solved to optimality within the time limit, and the second one is the average percentage final gap for the instances that terminate due to the time limit. The column "R.Gap" reports the root node gap for the instances after adding the violated inequalities. Furthermore, the column "Nodes" displays the average number of branch-and-bound nodes explored during the process. Finally, the column "Cut" in the section "DEP & Mix. Ineq. & New Ineq." reports the average number of inequalities (11) that are added in addition to the $m \times \gamma$ mixing inequalities (9) and (10).

First, comparing the results of the settings "DEP & Mix. Ineq." with $\gamma = 0.1$ and $\gamma = 0.2$, we see the diminishing returns of adding more mixing inequalities. This observation forms the basis for establishing cut limits. We observe that the setting with $\gamma = 0.1$ provides better results in general. Hence, we set the cut limit as $m \times 0.1$ for the setting "DEP & Mix. Ineq. & New Ineq.". Table 2 indicates the new inequality class (11) is computationally effective: the solution time, ending gap, root node gap and number of branch-and-bound (B&B) nodes generated are generally better for the option with the new inequality (11) than the option without the inequality (11). In particular, the improvements in the overall solution times and the number of B&B nodes are quite significant. Besides, the column "Cut" shows that the proposed inequalities are useful in terms of cutting off the fractional solution. On the contrary, if we do not add the proposed inequalities (11) and add more mixing inequalities (9) and (10) instead, i.e., compare the sections "DEP &

Mix. Ineq." for $\gamma = 0.2$ and "DEP & Mix. Ineq. & New Ineq.", then the computational performance for $\gamma = 0.2$ setting is still worse, both in terms of solution time (always) and average root gap (mostly).

In conclusion, not only do the proposed inequalities have desirable theoretical properties, namely that they are convex-hull defining for the set $\mathcal{P}$ for which they are derived, but they are also effective in practice where $\mathcal{P}$ appears as a substructure. Several interesting questions are left for future studies: study of an intersection of mixing sets with general coefficients on multiple shared continuous variables, strengthening of the proposed inequalities in the presence of cardinality/knapsack constraints for chance-constrained applications, and lifting the assumption on the nonnegativity i.e., Assumption **A3**, of the data defining the particular generalization of the intersection of mixing set $\mathcal{P}$.

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Appendix A. Proof of Proposition 3.4 Proof. We first establish the necessity of the condition $w_{s_1} = w_{\alpha_1}$ for inequality (9) to be a facet. Suppose $w_{s_1} < w_{\alpha_1}$. Note that inequality (9) given for $S' = \{\alpha_1 \to s_1 \to s_2 \to \cdots \to s_\eta\}$ is simply

$$y_p + y_d + \sum_{j=1}^{\eta} (w_{s_j} - w_{s_{j+1}}) z_{s_j} \geq w_{s_1} + (w_{\alpha_1} - w_{s_1})(1 - z_{\alpha_1}).$$

This inequality is stronger than the original inequality (9) given for $S = \{s_1 \to s_2 \to \cdots \to s_\eta\}$ because $(w_{\alpha_1} - w_{s_1})(1 - z_{\alpha_1}) \geq 0$. Hence, this establishes the necessity of condition $w_{s_1} = w_{\alpha_1}$. The argument for the necessity of condition $v_{t_1} = v_{\beta_1}$ for inequality (10) to be a facet is identical.

To see that inequality (9) is facet defining if $w_{s_1} = w_{\alpha_1}$, first, for all $j \in \Omega \setminus S$, we consider the points $(w_{\alpha_1}, 0, \mathbf{e}_j)$. These points are feasible (see the proof of Proposition 3.3). In addition, these points satisfy inequality (9) at equality and are affinely independent. Next, for all $j \in [\eta]$, we consider the points $A\Big(\cup_{i=j}^{\eta} s_i\Big) = (w_{s_j}, 0, \sum_{i \in \Omega \setminus (\cup_{i=j}^{\eta} s_i)} \mathbf{e}_{s_i})$. The feasibility of these points follow from Lemma 3.1. In addition, these points satisfy inequality (9) at equality and are affinely independent. Finally, we consider the feasible points $A(\emptyset)$ and $C(\Omega)$, which are affinely independent from all other points. In addition, $A(\emptyset)$ and $C(\Omega)$ satisfy inequality (9) at equality. Hence, we obtain $m + 2$ affinely independent points that are feasible and satisfy inequality (9) at equality. This proves that inequality (9) is facet-defining for $\text{conv}(\mathcal{P})$.

The proof for inequality (10) to be facet defining when $v_{t_1} = v_{\beta_1}$ is similar. In this case, we consider the points D, $C(\Omega)$, $C(\Omega \setminus \{j\})$, for all $j \in \Omega \setminus T$, and $B\Big(\cup_{i=j}^{\rho} t_i\Big)$, for all $j \in [\rho]$ These points are feasible from Lemma 3.1 and are also affinely independent. □

Appendix B. Proof of Proposition 3.5 Proof. If $w_{r_1} < w_{\alpha_1}$, then we can attach α_1 at the *beginning* of the sequence Π to obtain another valid inequality of form (11) (or equivalently (12))

$$2y_p + \sum_{j=1}^{\tau_R} (w_{r_j} - w_{r_{j+1}}) z_{r_j} + \sum_{j=1}^{\tau_G} (v_{g_j} - v_{g_{j+1}}) z_{g_j} \geq w_{r_1} + v_{g_1} + (w_{\alpha_1} - w_{r_1})(1 - z_{\alpha_1}).$$

The resulting inequality is at least as strong as the original inequality because $w_{\alpha_1} > w_{r_1}$ and $1 - z_{\alpha_1} \geq 0$. Similarly, if $v_{g_1} < v_{\beta_1}$, then we can attach β_1 at the beginning of the sequence Π to obtain another inequality that is at least as strong as the original inequality. This shows the necessity of the facet conditions.

To see the sufficiency, first consider the feasible points $C(\emptyset)$ and D (see Lemma 3.1 for their feasibility). These points satisfy inequality (11) at equality. Next, we consider the feasible point $C(\Omega)$, which satisfies inequality (11) at equality. Now, consider the points $(\frac{w_{\alpha_1}+v_{\beta_1}}{2}, \frac{w_{\alpha_1}-v_{\beta_1}}{2}, \mathbf{e}_j)$, for all $j \in \Omega \setminus \Pi$. For each $j \in \Omega \setminus \Pi$, using Observation 3.1(i) and the feasibility of the point $C(\Omega) = (\frac{w_{\alpha_1}+v_{\beta_1}}{2}, \frac{w_{\alpha_1}-v_{\beta_1}}{2}, \mathbf{0})$, we conclude that these points are also feasible. Since $j \notin \Pi$, these points satisfy (11) at equality as well. Note that the points considered thus far are affinely independent.

Next, for all $j \in [\tau] \setminus \{1\}$ such that $\pi_j \in \Pi$, if $w_{\pi_j} < \bar{w}_{\Pi,j}$ and $v_{\pi_j} < \bar{v}_{\Pi,j}$, then we consider the point $(\frac{w_{\alpha_1}+v_{\beta_1}}{2}, \frac{w_{\alpha_1}-v_{\beta_1}}{2}, \mathbf{e}_{\pi_j})$. For each such j, the feasibility of the associated point follows from the feasibility of $C(\Omega)$ and Observation 3.1(i). In addition, this point also satisfies inequality (11) at equality, because $(w_{\pi_j} - \bar{w}_{\Pi,j})_+ = (v_{\pi_j} - \bar{v}_{\Pi,j})_+ = 0$, so the left-hand side of inequality (11), after substituting this point, becomes $w_{\alpha_1} + v_{\beta_1}$. Otherwise, if $w_{\pi_j} \geq \bar{w}_{\Pi,j}$ or $v_{\pi_j} \geq \bar{v}_{\Pi,j}$ for some $j \in [\tau] \setminus \{1\}$, then we consider the following feasible point $C\Big(\Pi \setminus (\cup_{i=1}^{j-1}\{\pi_i\})\Big) = (\frac{\bar{w}_{\Pi,j-1}+\bar{v}_{\Pi,j-1}}{2}, \frac{\bar{w}_{\Pi,j-1}-\bar{v}_{\Pi,j-1}}{2}, \sum_{i=1}^{j-1} \mathbf{e}_{\pi_i} + \sum_{i\in(\Omega\setminus\Pi)} \mathbf{e}_i)$. Note also

$$\sum_{i=1}^{j-1}\left(\left(w_{\pi_i} - \bar{w}_{\Pi,i}\right)_+\right) + \bar{w}_{\Pi,j-1} = \max_{\ell\in[\tau]} w_{\pi_\ell} = w_{\alpha_1},$$

and

$$\sum_{i=1}^{j-1}\left(\left(v_{\pi_i} - \bar{v}_{\Pi,i}\right)_+\right) + \bar{v}_{\Pi,j-1} = \max_{\ell\in[\tau]} v_{\pi_\ell} = v_{\beta_1},$$

because $\alpha_1 \in \Pi$ and $\beta_1 \in \Pi$. Thus, the point $C\Big(\Pi \setminus (\cup_{i=1}^{j-1}\{\pi_i\})\Big)$ satisfies inequality (11) at equality as well. Also, these points are affinely independent from the points listed earlier. Hence, in total, we obtain $m+2$ affinely independent feasible points that satisfy inequality (11) at equality. This completes the proof. □

Appendix C. Proof of Lemma 3.7 Proof. To prove our claim, first, observe that if $D \in \widehat{\mathcal{O}}$, then substituting the point D into inequality (10) defined by the sequence T as defined in the premise of the lemma, the left-hand side becomes $-u_d + \sum_{j=1}^{\rho}(v_{t_j} - v_{t_{j+1}}) = v_{t_1} = v_{\beta_1}$ (recall $v_{t_{\rho+1}}$ in (10)), which proves that inequality (10) defined by T is tight at $D \in \widehat{\mathcal{O}}$.

Next consider any solution $\hat{o}_i = B(V_i) \in \widehat{\mathcal{O}}$ or $\hat{o}_i = C(V_i) \in \widehat{\mathcal{O}}$ with $V_i \neq \emptyset$ and $i \in [\hat{p}]$. From the definitions of $B(V_i)$ and $C(V_i)$, we have $y_p^i - y_d^i = v_{\bar{j}_i}$ (recall the definition of $\bar{j}_i$) and $z_k^i = 1$ for all $k \in \Omega$ such that $v_k > v_{\bar{j}_i}$ (from inequality (1b) in the original constraint set). Also, by definition of T, $\bar{j}_i = t_{k_i}$ for some $k_i \in [\rho]$, and we have $v_{t_j} \geq v_{t_{k_i}}$ for $j \in [k_i - 1]$; hence, $z_{t_j}^i = 1$ for all $j \in [k_i - 1]$ such that $v_{t_j} > v_{t_{k_i}}$. Then $\sum_{j=1}^{k_i}(v_{t_j} - v_{t_{j+1}})z_{t_j}^i = v_{\beta_1} - v_{t_{k_i}}$ where the equality holds because $z_{t_{k_i}}^i = 0$, $t_1 = \beta_1$ and for $j \in [k_i - 1]$ we have $z_{t_j}^i = 1$ if $v_{t_j} > v_{t_{k_i}}$. Substituting this term and the relation $y_p^i - y_d^i = v_{\bar{j}_i} = v_{t_{k_i}}$ in inequality (10) leads to the equivalent inequality given by

$$v_{t_{k_i}} + v_{\beta_1} - v_{t_{k_i}} + \sum_{j=k_i+1}^{\rho}(v_{t_j} - v_{t_{j+1}})z_{t_j}^i \geq v_{\beta_1}. \tag{27}$$

Suppose, for contradiction, that $\hat{o}_i$ does not satisfy inequality (10) at equality for this choice of T. Then, from (27), we see that we must have $\sum_{j=k_i+1}^{\rho}(v_{t_j} - v_{t_{j+1}})z_{t_j}^i > 0$. In other words, there exists $t_{j'} \in T$ for some $j' \in [\rho] \setminus [k_i]$ with both $z_{t_{j'}}^i = 1$ (i.e., $t_{j'} \notin V_i$) and $v_{t_{j'}} - v_{t_{j'+1}} > 0$. This along with Assumption **A3**, implies that $v_{t_{j'}} > 0$. Moreover, from $j' \in [\rho] \setminus [k_i]$, $t_{k_i} = \bar{j}_i$ and the definition of the sequence T, we deduce $v_{t'_j} \leq v_{t_{k_i}} = v_{\bar{j}_i}$.

Because $t_{j'} \in T \setminus V_i$, there exists another point, say $\hat{o}_\ell = B(V_\ell) \in \widehat{\mathcal{O}}$ or $\hat{o}_\ell = C(V_\ell) \in \widehat{\mathcal{O}}$, such that

$t_{j'} = \arg\max\left\{ v_j \mid z_j^\ell = 0, j \in \Omega \right\} = \bar{j}_\ell$. Hence, $t_{j'} \in V_\ell \setminus V_i$. We have $\min\left\{ \max_{j\in V_i} v_j,\ \max_{j\in V_\ell} v_j \right\} = \min\{v_{\bar{j}_i}, v_{t_{j'}}\} = v_{t_{j'}} = \max_{j\in V_\ell} v_j > \max_{j\in(V_i\cap V_\ell)} v_j$, where in the equations we have used respectively the definitions of $\bar{j}_i$ and $\bar{j}_\ell$ along with $t_{j'} = \bar{j}_\ell$, the fact that $v_{t_{j'}} \le v_{\bar{j}_i}$. Whenever $V_i \cap V_\ell = \emptyset$, the strict inequality follows from $v_{t_{j'}} > 0$ and our convention that $\max_{j\in V} v_j = 0$ for $V = \emptyset$. Whenever $V_i \cap V_\ell \neq \emptyset$, recall that if $\hat{o}_i \in \{B(V_i), C(V_i)\}$ is in $\widehat{\mathcal{O}}$ and $\hat{o}_\ell \in \{B(V_\ell), C(V_\ell)\}$ is in $\widehat{\mathcal{O}}$, then from the premise of the lemma, we have $V_\kappa := V_i \cap V_\ell$ is such that $\hat{o}_\kappa \in \{B(V_\kappa), C(V_\kappa)\}$ is also in $\widehat{\mathcal{O}}$ which implies that the strict inequality above follows from $\bar{j}_\ell = t_{j'} \notin V_i \cap V_\ell$, hence $\bar{j}_\kappa = t_{k_\kappa}$ for some $\rho \ge k_\kappa \ge j' + 1$ and that $v_{t_{j'}} > v_{t_{j'+1}} \ge v_{\bar{j}_\kappa}$. Consequently, we reach a contradiction because this inequality implies $\psi(\mathbf{v}, V_i, V_\ell) > 0$, which contradicts the premise of the lemma. As a result, $t_{j'}$ cannot exist, i.e., $z_{t_j}^i = 0$ for all $j = k_i + 1, \ldots, \rho$ in inequality (27). Hence, inequality (10) for this choice of T must be tight at any solution $\hat{o}_i \in \widehat{\mathcal{O}}$ satisfying the premise of the lemma. □

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