15-814 Homework 2 Solutions

October 10, 2017

Task 1 *Prove closure under head expansion.*

Solution: We proceed by structural induction on the type τ.

Case $\tau = \texttt{nat}$: We need to show the 3 conditions for $\mathrm{Red}_{\texttt{nat}}(e)$ to hold.

1. $\cdot \vdash e : \texttt{nat}$ holds by assumption.
2. We know $\mathrm{Red}_{\texttt{nat}}(e')$, i.e. there exists v val such that $e' \mapsto^* v$. Therefore, since we assumed $e \mapsto e'$, we also have $e \mapsto^* v$ by transitivity and definition of $\mapsto^*$.
3. This follows directly from the $\mathrm{Red}_{\texttt{nat}}(e')$ assumption, since we are considering the same v.

Case $\tau = \tau_1 \to \tau_2$: We need to show the 3 conditions for $\mathrm{Red}_{\tau_1 \to \tau_2}(e)$ to hold.

1. $\cdot \vdash e : \tau_1 \to \tau_2$ holds by assumption.
2. We know $\mathrm{Red}_{\tau_1 \to \tau_2}(e')$, i.e. there exists v val such that $e' \mapsto^* v$. Therefore, since we assumed $e \mapsto e'$, we also have $e \mapsto^* v$ by definition of $\mapsto^*$.
3. Consider e'' such that $\mathrm{Red}_{\tau_1}(e'')$. We need to show $\mathrm{Red}_{\tau_2}(e\ e'')$. Since τ_2 is structurally smaller than $\tau_1 \to \tau_2$, we do this by showing the 3 premises of the inductive hypothesis.
   (a) By the third condition of $\mathrm{Red}_{\tau_1 \to \tau_2}(e')$, we have $\mathrm{Red}_{\tau_2}(e'\ e'')$.
   (b) We have $\cdot \vdash e : \tau_1 \to \tau_2$ and $\cdot \vdash e'' : \tau_1$ (by assumptions). Therefore, $\cdot \vdash e\ e'' : \tau_2$ by (App).
   (c) Since $e \mapsto e'$, we have $e\ e'' \mapsto e'\ e''$ by (App-S).

   Therefore, $\mathrm{Red}_{\tau_2}(e\ e'')$ by I.H..

Task 2 *Prove the remaining cases of Theorem 3. You may state (without proof) lemmas about substitution, but be sure to check that they are actually true.*

Solution: In the proof, we will use $\circ$ to denote composition of finite maps. For example, if $\gamma = \{x_1 \hookrightarrow e_1, \ldots, x_n \hookrightarrow e_n\}$, $\gamma' = \{x'_1 \hookrightarrow e'_1, \ldots, x'_n \hookrightarrow e'_n\}$ then $\gamma' \circ \gamma = \{x'_1 \hookrightarrow e'_1, \ldots, x'_n \hookrightarrow e'_n, x_1 \hookrightarrow e_1, \ldots, x_n \hookrightarrow e_n\}$. We implicitly assume that the variable names mapped by γ, γ' do not clash. Furthermore, we will use (without proof) the congruence of $\mapsto^*$ with the congruence rules, and the following lemmas[1]:

Lemma 1 (Simultaneous substitution decomposition) *If $\Vdash \gamma : \Gamma$, then γ only substitutes closed terms for variables, and in particular, composed finite maps containing γ can be decomposed:* $(\gamma' \circ \gamma)(e) = \gamma'(\gamma(e))$

[1] These were my original solutions, but I think that some of my uses of the substitution lemma were unnecessary, see footnote in (Lam) case. The decomposition lemma is probably too much detail here, but observe that you cannot, in general, conclude $(\gamma' \circ \gamma)e = \gamma'(\gamma(e))$ without further assumptions. For example, $\{x \hookrightarrow y, y \hookrightarrow x\}y = x$ but $\{x \hookrightarrow y\}(\{y \hookrightarrow x\}y) = y$.

Lemma 2 (Substitution) *If* $\Vdash$ *, and* ; $'\vdash e$: *, then only substitutes closed terms for variables, and in particular,* $'\vdash$ (e) : .

Proof: The proof proceeds by induction on $\vdash e$: (the case for (App) is omitted). (I write the form of $\vdash e$: followed by the rulename for each relevant case). Note that by de nition of Red , whenever Red (e), then $\cdot \vdash e$: . This also implies that e is a closed term with no free variables.

Case ; x : $\vdash x$: (Hyp): By assumption, we have $\Vdash$; x : , i.e. $= \{:::; x \mapsto e\}$ by de nition of $\Vdash$. In particular, we have Red (e), but $e =$ (x) by de nition of substitution on variable x so we are done.

Case $\vdash \mathtt{z} : \mathtt{nat}$ (Z): We have $(\mathtt{z}) = \mathtt{z}$, so we show $\mathsf{Red}_{\mathtt{nat}}(\mathtt{z})$.

1. $\cdot \vdash \mathtt{z} : \mathtt{nat}$ by (Z).
2. $\mathtt{z}$ val by (Z-V) and $\mathtt{z} \mapsto^* \mathtt{z}$ by re exivity.
3. $\mathtt{z}\downarrow$ by ($\downarrow$-Z).

Case $\vdash \mathtt{s}(e) : \mathtt{nat}$ (S): We need to show $\mathsf{Red}_{\mathtt{nat}}($ $(\mathtt{s}(e)))$, i.e. $\mathsf{Red}_{\mathtt{nat}}(\mathtt{s}($ $(e)))$ by de nition of substitution.

From the premise of the rule, we have $\vdash e : \mathtt{nat}$, and by assumption $\Vdash$. By I.H., we have $\mathsf{Red}_{\mathtt{nat}}($ $(e))$. By de nition of $\mathsf{Red}_{\mathtt{nat}}$, we have $\cdot \vdash$ $(e) : \mathtt{nat}$, $(e) \mapsto^* v$ for some v val and $v\downarrow$.

1. $\cdot \vdash \mathtt{s}($ $(e)) : \mathtt{nat}$ by (S).
2. $\mathtt{s}($ $(e))$ val by (S-V) and $\mathtt{s}($ $(e)) \mapsto^* \mathtt{s}($ $(e))$ by re exivity.
3. $\mathtt{s}($ $(e))\downarrow$ by ($\downarrow$-S) (note that the premises of the rule are satis ed by what we have above).

Case $\vdash \mathtt{natrec}(e; e_0; x{:}y{:}e_1)$: (Rec): We need to show Red ($(\mathtt{natrec}(e; e_0; x{:}y{:}e_1)))$, i.e. Red $(\mathtt{natrec}($ $(e);$ $(e_0); x{:}y{:}$ $(e_1)))$ by de nition of substitution.

From the premises of the rule, we have $\vdash e : \mathtt{nat}$, $\vdash e_0$: and ; $x : \mathtt{nat}; y$: $\vdash e_1$: . By assumption, we also have $\Vdash$.

By the substitution lemma on ; $x : \mathtt{nat}; y$: $\vdash e_1$: , we have $x : \mathtt{nat}; y$: $\vdash$ (e_1) : . By I.H. on the rst two premises, we have $\mathsf{Red}_{\mathtt{nat}}($ $(e))$ and Red ($(e_0))$. From these, we also have $\cdot \vdash$ $(e) : \mathtt{nat}$ and $\cdot \vdash$ (e_0) : . Hence, $\cdot \vdash \mathtt{natrec}($ $(e);$ $(e_0); x{:}y{:}$ $(e_1))$: by (Rec).

From $\mathsf{Red}_{\mathtt{nat}}($ $(e))$, we also have $(e) \mapsto^* v$ for some v where v val and $v\downarrow$. By congruence with (Rec-S), we have $\mathtt{natrec}($ $(e);$ $(e_0); x{:}y{:}$ $(e_1)) \mapsto^* \mathtt{natrec}(v;$ $(e_0); x{:}y{:}$ $(e_1))$.

Using closure under head expansion, it su ces to show Red $(\mathtt{natrec}(v;$ $(e_0); x{:}y{:}$ $(e_1)))$. We do this by nested induction on the $v\downarrow$ judgement. Note that we have $\cdot \vdash (\mathtt{natrec}(v;$ $(e_0); x{:}y{:}$ $(e_1)))$: and $\cdot \vdash v : \mathtt{nat}$ by type preservation.

Sub-case ($\downarrow$-Z): We have $v = \mathtt{z}$, so $\mathtt{natrec}(\mathtt{z};$ $(e_0); x{:}y{:}$ $(e_1)) \mapsto$ (e_0) by (Rec-IZ). Since Red ($(e_0))$, we have Red $(\mathtt{natrec}(\mathtt{z};$ $(e_0); x{:}y{:}$ $(e_1)))$ by closure under head expansion.

Sub-case ($\downarrow$-S): We have $v = \mathtt{s}(e')$ for some $e'; v'$ such that $e' \mapsto^* v'$, v'*val* and $v'\downarrow$. By inversion on the typing judgment for v, we have $\cdot \vdash e' : \mathtt{nat}$. Therefore, $\mathsf{Red}_{\mathtt{nat}}(e')$ by de - nition. By the nested I.H. on $v'\downarrow$, we have Red $(\mathtt{natrec}(v';$ $(e_0); x{:}y{:}$ $(e_1)))$. We also have $\cdot \vdash \mathtt{natrec}(e';$ $(e_0); x{:}y{:}$ $(e_1))$: by (Rec), and therefore, Red $(\mathtt{natrec}(e';$ $(e_0); x{:}y{:}$ $(e_1)))$ by closure under head expansion and congruence with (Rec-S).

Let Red