

Dedicated to the memory of Raphael Siev

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Instantons on the Taub-NUT Space

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Abstract

We present a construction for self-dual Yang-Mills connections on the Taub-NUT space. We illustrate it by finding explicit expressions for all $SU(2)$ instantons of instanton number one and generic monodromy at infinity.

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1 Introduction

We are looking for a Hermitian connection, given by a $u(n)$ -valued one-form iA , on an n -dimensional Hermitian bundle \mathcal{E} over the Taub-NUT space with the curvature two-form $F = dA - iA \wedge A$ satisfying the self-duality condition

$$F = *F. \quad (1)$$

Here $*$ denotes the Hodge star operator taking a two-form to its dual. We require the connection A to have finite action $S = \int \text{tr} F \wedge *F$.

Everywhere outside one point 0 the Taub-NUT space itself can be thought of as being fibered by a circle S^1 over a base $\mathbb{R}^3 \setminus 0$. Choosing $\tau \sim \tau + 4\pi$ to be the periodic coordinate on the S^1 fiber and $\vec{x} = (x_1, x_2, x_3)$ to be coordinates on \mathbb{R}^3 the Taub-NUT metric¹ is

$$ds^2 = \frac{1}{4} \left(\left(l + \frac{1}{|\vec{x}|} \right) d\vec{x}^2 + \frac{1}{\left(l + \frac{1}{|\vec{x}|} \right)} (d\tau + \vec{\omega} \cdot d\vec{x})^2 \right), \quad (2)$$

where l is some fixed parameter determining the asymptotic size of the S^1 and $\frac{\partial}{\partial x_i} \frac{1}{|\vec{x}|} = \epsilon_{ijk} \frac{\partial}{\partial x_j} \omega_k$. This metric degenerates to a flat metric \mathbb{R}^4 as $l \rightarrow 0$. Its noncompact cycle $\mathcal{C} : \{(\tau, \vec{x}) | x_1 = x_2 = 0, x_3 \geq 0\}$ degenerates to a plane in this limit.

The Taub-NUT space is equipped with a natural line bundle with a connection $a = \frac{1}{2V}(d\tau + \omega)$. This connection has a self-dual curvature. As a matter of fact, it has a one parameter family of line bundles with the following Abelian connections:

$$a_s = sa = \frac{s}{2} \frac{d\tau + \omega}{V}, \quad (3)$$

parameterized by $s \in [-l/2, l/2]$. These connections are Abelian instantons, as their curvature is self-dual in the orientation (τ, x_1, x_2, x_3) . Note that the relation between the left and right ends of this interval is given by tensoring with a line bundle \mathcal{L}_l , which is trivial since

$$\int_{\mathcal{C}} d(a_{l/2} - a_{-l/2}) = 2\pi. \quad (4)$$

¹The factor of $\frac{1}{4}$ in the metric is chosen for future convenience and the apparent singularity at the origin of \mathbb{R}^3 is merely a coordinate singularity.

There has been a lot of work exploring instantons in various backgrounds. The ADHM original construction [1] provides all instantons on \mathbb{R}^4 . Nahm modified this construction in [2, 3] to provide calorons, i.e. instantons on $\mathbb{R}^3 \times S^1$. Orbifolding the ADHM construction Kronheimer and Nakajima [4] obtained instantons on ALE spaces. In [5] Nekrasov and Schwarz modified the ADHM construction to construct instantons on noncommutative \mathbb{R}^4 . All of these constructions have a string theory interpretation [6, 7, 8] and emerge from the sigma model analysis of appropriate D-brane configurations.

Based on these general constructions some explicit solutions at a general position were obtained in [9, 10] for a caloron and in [11] for instantons on certain ALE spaces.

We would like to point out that in all these cases the underlying space is flat, or it has a useful flat limit. Here, we aim to find a general construction for generic² instantons on an essentially curved space. In particular, building on the bow formalism introduced in [16] to study the moduli spaces of instantons on the Taub-NUT space, we find expressions for the instanton connection. As an illustration of our construction we find explicit solution for a single instanton on a Taub-NUT space.

1.1 Instanton Number and Monopole Charges

A generic self-dual $U(n)$ configuration on the Taub-NUT space possesses two types of topological charges: an instanton number and n monopole charges. The instanton number as well as the monopole charges are given by integers. A detailed discussion of various charges of instantons on multi-Taub-NUT and their relation with the corresponding brane configurations appeared recently in [17]. Here we define these numbers in a somewhat different fashion.

For any given $\vec{x} \in \mathbb{R}$ consider $W(\vec{x}, \tau) \in U(n)$ satisfying $(\partial_\tau - iA_\tau)W(\vec{x}, \tau) = 0$ and $W(\vec{x}, 0) = 1$, so that the monodromy around the circle S_x^1 is $W(\vec{x}, 4\pi)$. The finite action condition implies that the conjugacy class of $\lim_{|\vec{x}| \rightarrow \infty} W(\vec{x}, 4\pi)$ is well defined and does not depend on the direction in which we approach the infinity. Write the eigenvalues of $\lim_{|\vec{x}| \rightarrow \infty} W(\vec{x}, 4\pi)$ as

$$\exp\left(\frac{2\pi i \lambda_1}{l}\right), \exp\left(\frac{2\pi i \lambda_2}{l}\right), \dots, \exp\left(\frac{2\pi i \lambda_n}{l}\right).$$

Here we restrict our attention to the so called ‘maximal symmetry breaking’ case presuming all λ_j distinct and ordered: $-\frac{l}{2} < \lambda_1 < \lambda_2 < \dots < \lambda_n < \frac{l}{2}$.

²Some special instantons on the Taub-NUT space were obtained in [12, 13, 14, 15].

Consider a sphere $S_R^2 \in \mathbb{R}^3$ of large radius $|\vec{x}| = R$. Any point in this sphere determines a τ -circle in the Taub-NUT space, so that the union of all these circles is a squashed three sphere S_R^3 . Thus S_R^3 is fibered by circles over the S_R^2 and for a Taub-NUT space this fibration is the Hopf fibration $S^1 \rightarrow S_R^3 \rightarrow S_R^2$. Since the total action is finite there is a gauge transformation on S_R^3 such that for large radius R the connection A restricted to S_R^3 has τ -independent components. Let us write this connection with τ -independent components in the form $A = \hat{A} - \hat{\Phi} \frac{d\tau + \omega}{V}$. Then, the self-duality condition for A is equivalent [18] to the Bogomolny equation

$$\hat{F} = *_3 D_{\hat{A}} \hat{\Phi}, \quad (5)$$

for $(\hat{A}, \hat{\Phi})$. Here $*$ is the three-dimensional Hodge star operation for the flat metric $dx_1^2 + dx_2^2 + dx_3^2$ and \hat{F} is the curvature form of \hat{A} . The asymptotic eigenvalues of $\hat{\Phi}$ are given by the eigenvalues of the monodromy operator $W(\vec{x}, 4\pi)$ moreover, the asymptotic behavior of $\hat{\Phi}$ eigenvalues is the same as for a BPS monopole, namely the eigenvalues are

$$\lambda_1 + \frac{j_1}{x} + O(x^{-2}), \quad \lambda_2 + \frac{j_2}{x} + O(x^{-2}), \quad \dots, \quad \lambda_n + \frac{j_n}{x} + O(x^{-2}), \quad (6)$$

with j_1, j_2, \dots, j_n integers.

Let us describe this construction in different terms, making clear that j 's are indeed integers. Considering the eigen-spaces of the monodromy operator $W(\vec{x}, \tau + 4\pi)W^{-1}(\vec{x}, \tau)$ splits the bundle $\mathcal{E}|_{S_R^3}$ into k line bundles $\mathcal{E}|_{S_R^3} = \mathcal{L}_{\lambda_1} \oplus \mathcal{L}_{\lambda_2} \oplus \dots \mathcal{L}_{\lambda_n}$. Moreover, since the eigenvalue λ is independent of the base, each of these line bundles can be trivialized on all S^1 Hopf fibers simultaneously. Thus we have a well defined pushdown line bundles over the base of the Hopf fibration S_R^2 . Chern classes of these are j_1, j_2, \dots, j_n . We now use these integers to define the monopole charges of the configuration.

Let $M = \min(j_1, j_1 + j_2, \dots, j_1 + j_2 + \dots + j_n)$. The monopole charges of an instanton on a Taub-NUT are defined as

$$(m_1, m_2, \dots, m_n) = (j_1 - M, j_1 + j_2 - M, \dots, j_1 + j_2 + \dots + j_n - M).$$

Note, that from the way they are defined, one of these charges, say m_p , has to vanish. Nevertheless, we keep it among the charges and its position p is significant as will be clear momentarily.

Intuitively, since the total action is finite, the asymptotic connection can be put into a form independent of the τ coordinate. Then, asymptotically, it

can be reduced [18] to a monopole on the base \mathbb{R}^3 . It is the charges of these monopoles that we defined above.

The instanton number is less straightforward to define. One can write an explicit expression given by the Chern number minus the contributions of the monopole charges. To make clear that it is an integer, we define it here as an index of the Weyl operator for the connection $A + \frac{1}{2}(\lambda_p + \lambda_{p+1})a$:

$$k_0 = \text{Ind } \mathcal{D}_{A + \frac{1}{2}(\lambda_p + \lambda_{p+1})a}. \quad (7)$$

Thus a general $U(n)$ instanton on a Taub-NUT has an instanton number n_0 and monopole charge (m_1, m_2, \dots, m_n) .

Kronheimer [18] demonstrated equivalence of the ‘pure monopole’ case of $k_0 = 0$ to singular monopoles studied in [19, 20]. In particular, explicit solutions for $k_0 = 0$ and $m = 1$ are equivalent to singular monopole solutions presented in [21, 22]. In this paper we focus our attention on the pure instanton case of $m = 0$, and obtain the explicit solution with $k_0 = 1$, i.e. a single instanton on the Taub-NUT space with no monopole charge. The explicit metric on the moduli space of such solutions was found in [16].

2 Ingredients

The data specifying an instanton on a Taub-NUT space will be encoded in terms of a *bow diagram*. There are two basic ingredients in our construction:

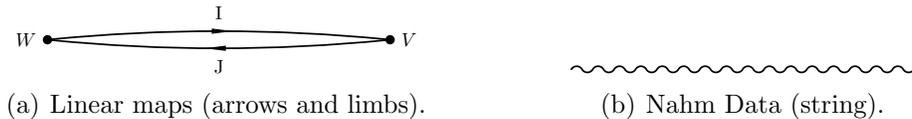


Figure 1: Components of bow diagrams.

2.1 Arrows and Limbs

Figure 1a represents a pair of complex vector spaces $V = \mathbb{C}^v$ and $W = \mathbb{C}^w$ with maps $J : V \rightarrow W$ and $I : W \rightarrow V$. The linear space formed by the pair

of maps (I, J) has a natural hyperkähler structure, which is respected by the action of $U(v)$ and $U(w)$. The hyperkähler moment map of the $U(v)$ action $g_v : (I, J) \mapsto (g_v^{-1}I, Jg_v)$ is

$$\mu_V^{\mathbb{C}} = \mu_V^1 + i\mu_V^2 = IJ, \quad \mu_V^{\mathbb{R}} = \mu_V^3 = \frac{1}{2}(J^\dagger J - II^\dagger), \quad (8)$$

while for the $U(w)$ action $g_w : (I, J) \mapsto (Ig_w, g_w^{-1}J)$ the moment map is

$$\mu_W^{\mathbb{C}} = \mu_W^1 + i\mu_W^2 = -JI, \quad \mu_W^{\mathbb{R}} = \mu_W^3 = \frac{1}{2}(I^\dagger I - JJ^\dagger). \quad (9)$$

It is convenient to assemble the pair (I, J) into

$$Q = \begin{pmatrix} J^\dagger \\ I \end{pmatrix} \text{ and } \hat{\mathcal{O}} = \begin{pmatrix} I^\dagger \\ -J \end{pmatrix}, \quad (10)$$

(pronounced “kyu” and “yuk”) so that $Q : W \rightarrow V \otimes S$ and $\hat{\mathcal{O}} : V \rightarrow W \otimes S$ with the three complex structures $e_j = -i\sigma_j$ acting on Q 's. $S \approx \mathbb{C}^2$ is a two dimensional space of spinors providing the representation of quaternions, with the quaternionic units $e_j = i\sigma_j$, i.e.

$$e_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The natural metric on the linear space of all pairs of maps is $ds^2 = \text{tr}_W dQ^\dagger dQ = \text{tr}_W (dJdJ^\dagger + dI^\dagger dI)$. The three symplectic forms $\omega_j \equiv g(\cdot, e_j \cdot) = \text{tr}_W (dQ^\dagger \wedge e_j dQ)$ can be combined into $\lambda \equiv \omega_j \sigma_j = 2i \text{Vec tr}_V dQ \wedge dQ^\dagger$. Here we introduce a ‘vector operation’ Vec defined by

$$\text{Vec}(M^0 \otimes 1_{2 \times 2} + M^j \otimes \sigma_j) = M^j \otimes \sigma_j. \quad (11)$$

Since $-i\sigma_j$ represent the quaternionic imaginary units, this operation basically amounts to taking the imaginary part of a quaternion.

With this notation the moment maps are

$$\lambda_V = \mu_V^i \sigma_i = \text{Vec}(QQ^\dagger) \text{ and } \lambda_W = \mu_W^i \sigma_i = \text{Vec}(\hat{\mathcal{O}}\hat{\mathcal{O}}^\dagger). \quad (12)$$

2.2 String

Figure 1b represents an interval \mathcal{I} parameterised by s with a bundle $E \rightarrow \mathcal{I}$ endowed with a Hermitian structure, a connection $D_s = d/ds - iT_0$, and a triplet $\vec{T} = (T_1, T_2, T_3)$ of endomorphisms of E . In other words for a given trivialization of E we have a quadruplet of hermitian matrix valued functions $(T_0(s), T_1(s), T_2(s), T_3(s))$. These also form a linear space with a natural metric $ds^2 = \int \text{tr}_E (dT_0^2 + dT_1^2 + dT_2^2 + dT_3^2)$ and hyperkähler structure invariant with respect to the following gauge group action

$$g(s) : \begin{pmatrix} T_0(s) \\ T_1(s) \\ T_2(s) \\ T_3(s) \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}T_0g + ig^{-1}\frac{d}{ds}g \\ g^{-1}T_1g \\ g^{-1}T_2g \\ g^{-1}T_3g \end{pmatrix}. \quad (13)$$

The corresponding moment maps are

$$\mu^1 = -\frac{d}{ds}T_1 + i[T_0, T_1] + i[T_2, T_3], \quad (14)$$

$$\mu^2 = -\frac{d}{ds}T_2 + i[T_0, T_2] + i[T_3, T_1], \quad (15)$$

$$\mu^3 = -\frac{d}{ds}T_3 + i[T_0, T_3] + i[T_1, T_2]. \quad (16)$$

It is convenient to introduce $\Upsilon = \sigma_1 \otimes T_1 + \sigma_2 \otimes T_2 + \sigma_3 \otimes T_3$ so that the moment map $\lambda = -[\frac{d}{ds} - iT_0, \Upsilon] + \text{Vec } \Upsilon \Upsilon$.

Assembling the Nahm data into a quaternion $T = T_0 + T_j \otimes e_j = T_0 - i\Upsilon$ we write the above metric on the linear space of all the Nahm data in the form

$$ds^2 = \frac{1}{2} \int \text{tr}_S \text{tr}_E \delta T^\dagger \delta T ds. \quad (17)$$

The three symplectic forms $\omega_j = g(\cdot, e_j \cdot)$ are encoded in $\lambda = \frac{i}{2} \int \text{tr}_E \delta T \wedge \delta T^\dagger ds$. Letting $D_s = \frac{d}{ds} - iT_0$ we note that the moment maps can be written in terms of the Weyl operator $D_s + \Upsilon$ and its conjugate as

$$\lambda = \text{Vec}(D_s - \Upsilon)(-D_s - \Upsilon). \quad (18)$$

3 Taub-NUT as Hyperkähler Quotient

This section contains a description of the Taub-NUT space using the ingredients we have defined in the previous section. This description naturally

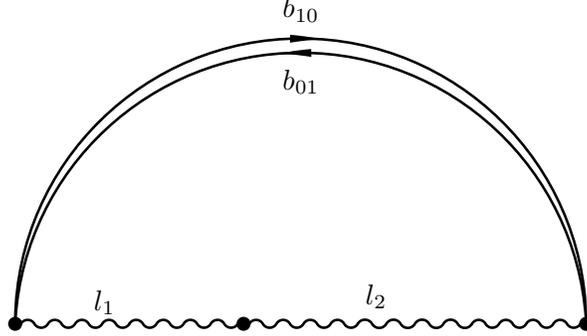


Figure 2: Taub-NUT Bow Diagram.

leads us to a family of self-dual harmonic forms³ which are essential for the instanton construction that follows. Our exposition in this section is close to that of Gibbons and Rychenkova [23], with some change of notation. Just as for the construction [4] of instantons on ALE spaces it was essential to know the realization of the underlying ALE space as a hyperkähler quotient of linear spaces [24], this section is the groundwork for the construction of instantons on the Taub-NUT.

The bow diagram on Figure 2 represents Nahm data of rank 1 associated with a line bundle $e \rightarrow \mathcal{I}$ on an interval $[-l/2, l/2]$ of length l , as well as the maps b_{10} and b_{01} between the one-dimensional complex vector spaces $e_0 = e|_{s=-l/2}$ and $e_1 = e|_{s=l/2}$ at the ends of the interval. The gauge transformation $h(s)$ acts on these data as follows:

$$\begin{pmatrix} t_0 \\ t_j \\ b_{01} \\ b_{10} \end{pmatrix} \mapsto \begin{pmatrix} h^{-1}t_0h + ih^{-1}\frac{d}{ds}h \\ h^{-1}t_jh \\ h^{-1}(-\frac{l}{2})b_{01}h(\frac{l}{2}) \\ h^{-1}(\frac{l}{2})b_{10}h(-\frac{l}{2}) \end{pmatrix} \quad (19)$$

Introducing $\mathbf{t} = t_1 + it_2$ and $\mathbf{D} = d/ds - it_0 - t_3$ the vanishing of moment maps can be written in complex notation as

$$\begin{aligned} [\mathbf{D}, \mathbf{t}] - \delta_{(s+\frac{l}{2})}b_{01}b_{10} + \delta_{(s-\frac{l}{2})}b_{10}b_{01} &= 0, \\ [\mathbf{D}^\dagger, \mathbf{D}] + [\mathbf{t}^\dagger, \mathbf{t}] + \delta_{(s+\frac{l}{2})}(b_{10}^\dagger b_{10} - b_{01}b_{01}^\dagger) + \delta_{(s-\frac{l}{2})}(b_{01}^\dagger b_{01} - b_{10}b_{10}^\dagger) &= 0. \end{aligned} \quad (20)$$

³A description of these in terms of the hyperkähler reduction recently appeared in [17].

Let us distinguish some point s_0 on the Nahm interval. Say this point divides this interval into two intervals of lengths l_L and l_R , i.e. $l_L + l_R = l$ and at this distinguished point $s = s_0 = l_L - l/2 = l/2 - l_R$. Let us assume $s_0 > 0$. We shall perform the hyperkähler quotient step-by-step, so that the last step is the quotient with respect to the $U(1)$ at the distinguished point. This will allow us to identify the natural line bundle over the Taub-NUT space and its natural connection corresponding to this $U(1)$.

First we perform HKQ on each open interval separately. The intervals are of lengths l_L and l_R . Since the computations are identical we focus on the interval of length l_R to the right of s_0 . Since the Nahm data is Abelian, the vanishing of the moment maps implies $dt_j/ds = 0$, thus the vector $\vec{t} = (t_1, t_2, t_3)$ is constant. t_0 can be made constant using gauge transformations that are trivial at the ends of the interval. There is a large gauge transformation $G = \exp(2\pi i(s - s_0)/l_R)$ satisfying $g(s_0) = g(l/2) = 1$. This gauge transformation takes t_{0R} to $t_{0R} + 2\pi/l_R$. Thus the result of this HKQ is $\mathbb{R}^3 \times S^1$ with coordinates t_{1R}, t_{2R}, t_{3R} and $t_{0R} \sim t_{0R} + 2\pi/l_R$ and the metric

$$ds^2 = \int_{s_0}^l (dt_{0R}^2 + d\vec{t}_R^2) ds = l_R (dt_{0R}^2 + d\vec{t}_R^2). \quad (21)$$

The resulting metric on the Nahm data on the left interval is given by the same expression with l_R replaces by l_L .

Now we perform HKQ with respect to the $U(1)$ at $s = l/2$, which can be realized by $h = \exp(i\phi \frac{s-s_0}{l_R})$. Exploiting the fact that all the data is Abelian, we assemble the linear data into a quaternion

$$q = q^0 + q^i e_i = (b_-, b_+) = \begin{pmatrix} b_{01}^\dagger & b_{10}^\dagger \\ -b_{10} & b_{01} \end{pmatrix}. \quad (22)$$

b_- and b_+ play the role of Q and $\hat{\mathcal{O}}$. The natural metric is $ds^2 = \frac{1}{2} \text{tr}_S dq^\dagger dq = db_-^\dagger db_- = db_+^\dagger db_+$, and the resulting symplectic forms are given by $\lambda = i \text{Vec} dq \wedge dq^\dagger$. A gauge transformation $h(s)$ with $h(-l/2) = \exp(i\phi_L)$ and $h(l/2) = \exp(i\phi_R)$ sends q to $q \exp(e_3(\phi_R - \phi_L))$. The resulting moment maps are $\lambda_R = \frac{1}{2} q \sigma_3 q^\dagger$ and $\lambda = -\frac{1}{2} q \sigma_3 q^\dagger$.

The rightmost $U(1)$ acts as $\exp(i\phi \frac{s-s_0}{l_R}) : (q, t_{0R}, \vec{t}_R) \mapsto (qe^{e_3\phi}, t_{0R} - \phi/l_R, \vec{t}_R)$, with the moment map $\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 = \frac{1}{2} q e_3 \bar{q} + t_1 e_1 + t_2 e_2 + t_3 e_3$. Let $q = a e^{e_3 \psi/2}$ where a is a pure imaginary quaternion, and let \vec{x} be such that $x_1 e_1 + x_2 e_2 + x_3 e_3 = q e_3 \bar{q}$. The periodic coordinate $\psi \sim \psi + 4\pi$ and the

components of \vec{x} provide new coordinates on \mathbb{R}^4 . Thus the metric is

$$ds^2 = \frac{1}{2} \text{tr}_S dq^\dagger dq + l_R (dt_{0R}^2 + d\vec{t}_R^2) \quad (23)$$

$$= \frac{1}{4} \left(\frac{1}{|\vec{x}|} d\vec{x}^2 + |\vec{x}| (d\psi + \omega)^2 \right) + l_R (dt_{0R}^2 + d\vec{t}_R^2), \quad (24)$$

where

$$i|\vec{x}|(\omega + d\psi) = \frac{i}{2} \text{tr}(qe_3 dq^\dagger - dqe_3 q^\dagger) = db_-^\dagger b_- - b_-^\dagger db_- = -db_+^\dagger b_+ + b_+^\dagger db_+. \quad (25)$$

The $U(1)$ is acting by $e^{i\phi} : (\psi, t_{0R}) \mapsto (\psi + 2\phi, t_{0R} - \phi/l_R)$. The invariant of this action is $\sigma = \psi + 2l_R t_{0R}$ and the vanishing of the moment maps implies $\vec{t}_R = -\frac{1}{2}\vec{x}$. One can readily verify that the above metric becomes

$$ds^2 = \frac{1}{4} \left(\left(l_R + \frac{1}{|\vec{x}|} \right) d\vec{x}^2 + \frac{(d\sigma + \omega)^2}{l_R + 1/|\vec{x}|} \right) + l_R r \left(l_R + \frac{1}{|\vec{x}|} \right) \left(dt_{0R} + \frac{1}{2} \frac{d\sigma + \omega}{l_R + 1/|\vec{x}|} \right)^2. \quad (26)$$

The last step in the hyperkähler reduction procedure is reduction with respect to the $U(1)$ at the distinguished point $s = s_0$. We use the gauge transformation

$$h(s) = \begin{cases} \exp(i \frac{s}{s_0} \varepsilon), & \text{for } s \leq s_0 \\ \exp(i \frac{l/2 - s}{l/2 - s_0} \varepsilon), & \text{for } s > s_0 \end{cases}, \quad (27)$$

that is continuous and equals to identity at $s = 0$ and $l/2$. At $s = s_0$ this gauge transformation is $h(s_0) = e^{i\varepsilon}$. It has the following action

$$h(s) : \begin{pmatrix} t_{0L} \\ \vec{t}_L \\ t_{0R} \\ \vec{t}_R \\ q \end{pmatrix} \mapsto \begin{pmatrix} t_{0L} - \varepsilon/s_0 \\ \vec{t}_L \\ t_{0R} + \varepsilon/l_R \\ \vec{t}_R \\ q \exp\left(e_3 \frac{l}{2s_0} \varepsilon\right) \end{pmatrix}. \quad (28)$$

The corresponding moment map is $\lambda = \frac{l_L}{s_0} \lambda_L - \lambda_R + \frac{l}{2s_0} \frac{1}{2} \lambda$. Since the vanishing of the moment maps of the first stage of the reduction implies $\lambda_R = -\frac{1}{2}\lambda$, we have $\lambda = \frac{l_L}{s_0} (\lambda_L + \frac{1}{2}\lambda)$. Putting λ equal to zero we have $\lambda_L = -\frac{1}{2}\lambda$ as well.

The metric is

$$ds^2 = \frac{1}{4} \left(\left(l_R + \frac{1}{|\vec{x}|} \right) d\vec{x}^2 + \frac{(d\sigma + \omega)^2}{l_R + 1/|\vec{x}|} \right) + l_L (dt_{0L}^2 + d\vec{t}_L^2). \quad (29)$$

Under the above gauge transformation the angle $\sigma = \psi + 2l_R t_{0R} \mapsto \sigma + 2\frac{l_L}{s_0}\varepsilon$. The invariant is $\tau = \sigma - 2l_L t_{0L} = \psi + 2l_R t_{0R} - 2l_L t_{0L}$, and we choose $\varepsilon \sim \varepsilon + 2\pi$ instead of σ as a coordinate along the circle of the gauge transformation. In these coordinates the above metric can be rewritten as

$$ds^2 = \frac{1}{4} \left[\left(l + \frac{1}{|\vec{x}|} \right) d\vec{x}^2 + \frac{1}{l + 1/|\vec{x}|} (d\tau + \omega)^2 \right] + \frac{l_L(l + \frac{1}{|\vec{x}|})}{s_0^2(l_R + 1/|\vec{x}|)} \left(d\varepsilon + \frac{s_0}{2} \frac{d\tau + \omega}{(l + 1/|\vec{x}|)} \right)^2. \quad (30)$$

The first part of the expression (30) is the resulting hyperkahler metric of the Taub-NUT space

$$4 ds_{TN}^2 = \left(l + \frac{1}{|\vec{x}|} \right) d\vec{x}^2 + \frac{1}{l + 1/|\vec{x}|} (d\tau + \omega)^2, \quad (31)$$

here the one-form ω satisfies $d\omega = *_3 d\frac{1}{|\vec{x}|}$. The second part of the expression (30) provides the natural connection $D = d + is_0 a$ with the one form $s_0 a$, where

$$a = \frac{1}{2} \frac{d\tau + \omega}{l + \frac{1}{|\vec{x}|}}. \quad (32)$$

3.1 Basis of self-dual two-forms

Let $V = l + 1/|\vec{x}|$ and $a = \frac{d\tau + \omega}{2V}$. Here we observe the following relation

$$\left(\frac{1}{2} d\vec{x} - ia \right)^\dagger \wedge \left(\frac{1}{2} d\vec{x} - ia \right) = \frac{i}{2} \sigma_k \left(\frac{d\tau + \omega}{V} \wedge dx^k + \frac{1}{2} \epsilon_{ijk} dx^i dx^j \right). \quad (33)$$

Note that the components of the right-hand-side are self-dual two-forms in the orientation (τ, x^1, x^2, x^3) providing a basis of self-dual two-forms on the Taub-NUT. Let us note for future use that since $\frac{1}{2} d\vec{x} - ia = -(d\vec{\lambda} + ia)$, in terms of the τ and \vec{t} coordinates the combination $(d\vec{\lambda} + ia)^\dagger \wedge (d\vec{t} + ia)$ is self-dual.

4 Instanton Data

Instanton data for an $SU(2)$ instanton with no monopole charges is represented by the bow diagram in Figure 3. It consists of

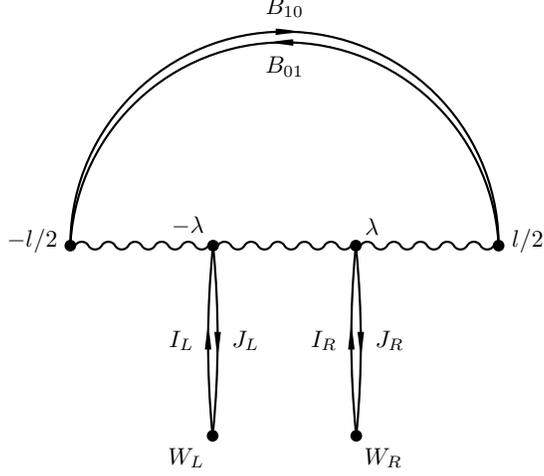


Figure 3: Bow diagram for Instanton on Taub-NUT.

- a rank k_0 vector bundle $E \rightarrow [-l/2, l/2]$ with the Nahm data (T_0, \vec{T}) on the intervals $[-l/2, -\lambda]$, $[-\lambda, \lambda]$, and $[\lambda, l/2]$ (we do not presume continuity at $s = \pm\lambda$),
- linear maps $B_{10} : E_{-l/2} \rightarrow E_{l/2}$ and $B_{01} : E_{l/2} \rightarrow E_{-l/2}$,
- linear maps $I_L : W_L \rightarrow E_{-\lambda}$, $J_L : E_{-\lambda} \rightarrow W_L$, $I_R : W_R \rightarrow E_{\lambda}$, and $J_R : E_{\lambda} \rightarrow W_R$.

The group of gauge transformations acts on this data as follows

$$\begin{pmatrix} T_0 \\ T_j \\ B_{01} \\ B_{10} \\ I_\alpha \\ J_\alpha \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}(s)T_0g(s) + ig^{-1}(s)\frac{d}{ds}g(s) \\ g^{-1}(s)T_jg(s) \\ g^{-1}(-\frac{l}{2})B_{01}g(\frac{l}{2}) \\ g^{-1}(\frac{l}{2})B_{10}g(-\frac{l}{2}) \\ g^{-1}(\lambda_\alpha)I_\alpha \\ J_\alpha g(\lambda_\alpha) \end{pmatrix}, \quad (34)$$

where the index α takes the values L and R and we introduced $\lambda_L = -\lambda$ and $\lambda_R = \lambda$.

Introducing the complex notation $D = \frac{d}{ds} - iT_0 - T_3$ and $T = T_1 + iT_2$, the moment maps are written as

$$[D, T] - \delta_{(s+\frac{l}{2})} B_{01} B_{10} + \delta_{(s-\frac{l}{2})} B_{10} B_{01} + \sum_{\alpha \in \{L, R\}} \delta_{(s-\lambda_\alpha)} I_\alpha J_\alpha = 0 \quad (35)$$

$$\begin{aligned} [D^\dagger, D] + [T^\dagger, T] + \delta_{(s+\frac{l}{2})} (B_{10}^\dagger B_{10} - B_{01} B_{01}^\dagger) + \delta_{(s-\frac{l}{2})} (B_{01}^\dagger B_{01} - B_{10} B_{10}^\dagger) + \\ + \sum_{\alpha \in \{L, R\}} \delta_{(s-\lambda_\alpha)} (J_\alpha^\dagger J_\alpha - I_\alpha I_\alpha^\dagger) = 0. \end{aligned}$$

These conditions can be written compactly if we introduce

$$B_- = \begin{pmatrix} B_{10}^\dagger \\ -B_{01} \end{pmatrix}, \quad B_+ = \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix}, \quad (36)$$

and $\mathcal{D} = -\frac{d}{ds} + iT_0 + \mathbb{Y} = \begin{pmatrix} -D & T^\dagger \\ T & D^\dagger \end{pmatrix}$. Then the moment maps are given by

$$\mathcal{M} = \text{Vec} \left(\mathcal{D}^\dagger \mathcal{D} + \sum_{\alpha} \delta(s - \lambda_\alpha) Q_\alpha Q_\alpha^\dagger + \delta(s + \frac{l}{2}) B_- B_-^\dagger + \delta(s - \frac{l}{2}) B_+ B_+^\dagger \right). \quad (37)$$

5 The Nahm Transform

5.1 Weyl Operator

A central role in the ADHM-Nahm transform [1, 2] is played by a certain linear operator. In the case at hand it is a modification of the Weyl operator. The details of similar construction can be found in [3, 25] for the case of calorons.

Let H be the space of L^2 sections of $E \otimes S$ that are continuous at λ_L and λ_R and have L^2 derivatives on $\mathcal{I} \setminus \{\alpha_L, \alpha_R\}$. Let $\tilde{\mathcal{H}}$ be the direct sum of the space of L^2 sections of $E \otimes S$ with spaces $E_{\lambda_L}, E_{\lambda_R}, E_{-l/2}$, and $E_{l/2}$. Given the instanton data of the bow diagram in Figure 3 we introduce the operator $\mathfrak{D} : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ by

$$\mathfrak{D} : f \mapsto \begin{pmatrix} (-\frac{d}{ds} + iT_0 + \mathbb{Y})f \\ (J_L, I_L^\dagger)f(-\lambda) \\ (J_R, I_R^\dagger)f(\lambda) \\ (B_{01}, B_{10}^\dagger)f(l/2) \\ (-B_{10}, B_{01}^\dagger)f(-l/2) \end{pmatrix}. \quad (38)$$

Let us denote by ψ an L^2 section of the restriction of $E \otimes S$ to $\mathcal{I} \setminus \{\alpha_L, \alpha_R\}$, $\chi_\alpha \in E_{\lambda_\alpha}$, $v_- \in E_{-l/2}$ and $v_+ \in E_{l/2}$. Integrating by parts we find that the cokernel of \mathfrak{D} is given by $(\psi(s), \chi_L, \chi_R, v_-, v_+) \in \tilde{\mathcal{H}}$ satisfying

$$\left(\frac{d}{ds} - iT_0 + \Upsilon \right) \psi = 0, \text{ on } \mathcal{I} \setminus \{\alpha_L, \alpha_R\}, \quad (39)$$

$$\psi(\lambda_\alpha+) - \psi(\lambda_\alpha-) = -Q_\alpha \chi_\alpha, \quad (40)$$

$$\psi\left(\frac{l}{2}\right) = \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix} v_-, \quad (41)$$

$$\psi\left(-\frac{l}{2}\right) = - \begin{pmatrix} -B_{10}^\dagger \\ B_{01} \end{pmatrix} v_+. \quad (42)$$

In other words the dual operator takes the form

$$\begin{aligned} \mathfrak{D}^\dagger &= \begin{pmatrix} -D^\dagger & T^\dagger \\ T & D \end{pmatrix} \oplus \left(\bigoplus_{\alpha \in \{L, R\}} \delta_{(s-\lambda_\alpha)} \begin{pmatrix} J_\alpha^\dagger \\ I_\alpha \end{pmatrix} \right) \\ &\oplus \left(\delta_{(s+\frac{l}{2})} \begin{pmatrix} B_{10}^\dagger \\ -B_{01} \end{pmatrix}, \delta_{(s-\frac{l}{2})} \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix} \right), \\ &= \mathcal{D}^\dagger \oplus \delta(s - \lambda_\alpha) Q_\alpha \oplus \left(\delta_{(s+\frac{l}{2})} B_-, \delta_{(s-\frac{l}{2})} B_+ \right). \end{aligned} \quad (43)$$

In terms of \mathfrak{D} and \mathfrak{D}^\dagger the moment map conditions of Eqs. (35) can be written as

$$\text{Vec}(\mathfrak{D}^\dagger \mathfrak{D}) = 0. \quad (44)$$

For a given point of the Taub-NUT space of Figure 2, corresponding to $(t_0, \vec{t}, b_{10}, b_{01})$ satisfying Eqs.(20), we can twist the above operator as follows

$$\begin{aligned} \mathfrak{D}_t^\dagger &= \begin{pmatrix} -D^\dagger - t_3 & T^\dagger - t^\dagger \\ T - t & D + t_3 \end{pmatrix} \oplus \left(\bigoplus_{\alpha \in \{L, R\}} \delta_{(s - \lambda_\alpha)} \begin{pmatrix} J_\alpha^\dagger \\ I_\alpha \end{pmatrix} \right) \\ &\oplus \left(\delta_{(s+\frac{l}{2})} \begin{pmatrix} B_{10}^\dagger & -b_{10}^\dagger \\ -B_{01} & -b_{01} \end{pmatrix} + \delta_{(s-\frac{l}{2})} \begin{pmatrix} -b_{01}^\dagger & B_{01}^\dagger \\ b_{10} & B_{10} \end{pmatrix} \right). \end{aligned} \quad (45)$$

To be exact, whenever adding two operators with one belonging to the instanton bow and another to the Taub-NUT bow data we understand both operators to be tensored with identity and this sum to be acting on the tensor product of the corresponding spaces. For example, $T - t$ stands as a shorthand for $T \otimes 1 - 1 \otimes t$. Unfortunately, in this case using rigorous notation

would make the formula above unreadable. We also allow this shorthand since for a case of a single instanton the vector spaces are one-dimensional and the bow data is abelian so the expression (45) makes perfect sense.

5.2 Connection

From now on we understand ψ to be a section of $E \otimes e \otimes \mathbb{C}^2 \rightarrow \mathcal{I} \setminus \{-\lambda, \lambda\}$, $v_- \in E_{-l/2} \otimes e_{l/2}$ and $v_+ \in E_{l/2} \otimes e_{-l/2}$. We combine v_+ and v_- into a spinor $v = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}$ and denote the data $(\psi(s), \chi_L, \chi_R, v)$ by $\boldsymbol{\psi}$. The twisted operator \mathfrak{D}_t^\dagger to act on the linear Hermitian space formed by such data. For $\boldsymbol{\psi}_1 = (\psi_1(s), \chi_{L1}, \chi_{R1}^\dagger, v_1)$ and $\boldsymbol{\psi}_2 = (\psi_2(s), \chi_{L2}, \chi_{R2}, v_2)$ the natural Hermitian product is given by $(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) = v_1^\dagger v_2 + (\chi_{L1})^\dagger \chi_{L2} + (\chi_{R1})^\dagger \chi_{R2} + \int_{-l/2}^{l/2} \psi_1^\dagger(s) \psi_2(s) ds$. We also define the operator \mathbf{s} acting on $\boldsymbol{\psi}$ as follows

$$\mathbf{s} : (\psi(s), \chi_L, \chi_R, v) \mapsto \left(s\psi(s), -\lambda\chi_L, \lambda\chi_R, \begin{pmatrix} -l/2 & 0 \\ 0 & l/2 \end{pmatrix} v \right). \quad (46)$$

Once we found the orthonormal basis of solutions of $\mathfrak{D}_t^\dagger \boldsymbol{\psi} = 0$ we arrange them as columns of the matrix Ψ , then the orthonormality condition reads $(\Psi, \Psi) = \mathbb{I}$. The instanton connection $\nabla_\mu = \partial_\mu - iA_\mu$ is induced on the kernel of \mathfrak{D}_t^\dagger by the connection $D_\mu = \partial_\mu + isa_\mu$, thus $\nabla_\mu = (\Psi, D_\mu \Psi)$ and the associated $su(2)$ -valued one-form $A = A_0 d\tau + A_j dx^j$ given by

$$A = \left(\Psi, \left(i \frac{\partial}{\partial \tau} - \frac{\mathbf{s}}{V} \right) \Psi \right) d\tau + \left(\Psi, \left(i \frac{\partial}{\partial x_j} - \omega_j \frac{\mathbf{s}}{V} \right) \Psi \right) dx_j \quad (47)$$

6 ADHM Limit

To compare with the ADHM construction we solve $\mathfrak{D}_t^\dagger \Psi = 0$ at the ends of the interval $s = \pm \frac{l}{2}$ to find

$$\psi_{(-\frac{l}{2})} = - \begin{pmatrix} B_{10}^\dagger & -b_{10}^\dagger \\ -B_{01} & -b_{01} \end{pmatrix} \begin{pmatrix} v_+ \\ v_- \end{pmatrix}, \quad \psi_{(\frac{l}{2})} = \begin{pmatrix} -b_{01}^\dagger & B_{01}^\dagger \\ b_{10} & B_{10} \end{pmatrix} \begin{pmatrix} v_+ \\ v_- \end{pmatrix}, \quad (48)$$

The Nahm equations imply that \vec{t} is constant on $[-l/2, l/2]$.

It is illustrative to first consider the case of a single instanton. In this case \vec{T} is constant on each of the three intervals of $\mathcal{I} \setminus \{-\lambda, \lambda\}$. Moreover, the values on the left and on the right intervals are equal, thus for some constant vectors \vec{T}_1 and \vec{T}_2

$$\vec{T}(s) = \begin{cases} \vec{T}_1 & \text{for } -l/2 < s < -\lambda \text{ or } \lambda > s > l/2, \\ \vec{T}_2 & \text{for } -\lambda < s < \lambda. \end{cases} \quad (49)$$

Let $\vec{z}_1 = \vec{t} - \vec{T}_1$ and $\vec{z}_2 = \vec{t} - \vec{T}_2$, then

$$\begin{aligned} & \left[e^{-\kappa_1(\frac{l}{2}-\lambda_2)} \begin{pmatrix} -b_{01}^\dagger & B_{01}^\dagger \\ b_{10} & B_{10} \end{pmatrix} + e^{\kappa_2(\lambda_2-\lambda_1)} e^{\kappa_1(\lambda_1+\frac{l}{2})} \begin{pmatrix} B_{10}^\dagger & -b_{10}^\dagger \\ -B_{01} & -b_{01} \end{pmatrix} \right] \begin{pmatrix} v_+ \\ v_- \end{pmatrix} + \\ & + e^{\vec{\sigma} \cdot \vec{z}_2(\lambda_2-\lambda_1)} \begin{pmatrix} J_L^\dagger \\ I_L \end{pmatrix} \chi_L + \begin{pmatrix} J_R^\dagger \\ I_R \end{pmatrix} \chi_R = 0. \end{aligned} \quad (50)$$

Clearly in the limit of $l \rightarrow 0$ (and since $\lambda < l/2$, we have $\lambda \rightarrow 0$) the above expression reduces to the ADHM linear equation.

For the case of instantons of general charge the exponentials in the above equation become path-ordered exponentials involving the corresponding non-abelian data T . Each of these represents parallel transport along an interval. In the $l \rightarrow 0$ limit, however, all of the intervals in the bow diagram contract to a point and the exponential factors become identities. Therefore the equation for the kernel of \mathfrak{D}_t reduces to the ADHM linear equation for a general case.

7 Proof of Self-duality

The core of this proof is close to the original argument of Nahm [2], but requires some adjustments. We shall need the following relations $(d+ila)b_- = -\frac{1}{2t}(d\kappa + ia)b_-$ and $(d-ila)b_+ = \frac{1}{2t}(d\kappa + ia)b_+$.⁴ We also use the fact that $\mathfrak{D}_t^\dagger \mathfrak{D}_t = 1 \otimes \Delta$, with Δ positive definite (except at some isolated points on the Taub-NUT). Thus it has a well defined inverse $G = (\mathfrak{D}_t^\dagger \mathfrak{D}_t)^{-1}$ (given by the Green's function of Δ) that commutes with the quaternions and the σ -matrices.

⁴ These follow from Eq.(25) and $b_\pm b_\pm^\dagger = t \pm \kappa$. Namely, (25) reads $4itVab_- = 2b_-(b_-db_-^\dagger) - b_-d(b_-^\dagger b_-) = 2(d(b_-b_-^\dagger))b_- - 2(db_-)(b_-^\dagger b_-) - b_-d(b_-^\dagger b_-) = 2(dt - d\kappa)b_- - 4tdb_- - 2dtb_- = -4tdb_- - 2d\kappa b_-$, thus $2t(db_- + ilab_-) = -(d\kappa + ia)b_-$.

As expressed by Eq. (47), the induced connection on the kernel of \mathfrak{D}^\dagger is the instanton connection A_μ , therefore the covariant differential is $dt^\mu \nabla_\mu \equiv dt^\mu (\partial_\mu - iA_\mu) = P dt^\mu D_\mu P = P(d + isa)P = P(d + is\frac{d\tau + \omega}{2V})P$. So the connection is $A_\mu = i(\Psi, D_\mu \Psi)$, then

$$\partial_{[\mu} A_{\nu]} = i(D_{[\mu} \Psi, D_{\nu]} \Psi) - (\Psi, \mathbf{s}\Psi) \partial_{[\mu} a_{\nu]} \quad (51)$$

$$[A_\mu, A_\nu] = (D_{[\mu} \Psi, \Psi)(\Psi, D_{\nu]} \Psi). \quad (52)$$

Then the curvature (field strength) is

$$F_{\mu\nu} = i[\nabla_\mu, \nabla_\nu] = (D_{[\mu} \Psi, (1 - P)D_{\nu]} \Psi) - (\Psi, \mathbf{s}\Psi) \partial_{[\mu} a_{\nu]}, \quad (53)$$

where $1 - P \equiv 1 - \Psi\Psi^\dagger = \mathfrak{D}_t G \mathfrak{D}_t^\dagger$. The second term in the curvature expression is self-dual since da is, while for the first term we have

$$(D_\mu \Psi, (1 - P)D_\nu \Psi) = ([\mathfrak{D}_t^\dagger, D_\mu] \Psi, G[\mathfrak{D}_t^\dagger, D_\nu] \Psi). \quad (54)$$

Since

$$\begin{aligned} \mathfrak{D}_t^\dagger = & \left(\frac{d}{ds} - iT_0 + \Upsilon - \mathfrak{k} \right) \oplus \delta(s - \lambda_\alpha) Q_\alpha \\ & \oplus \left(\delta(s + l/2)(B_-, -b_+) + \delta(s - l/2)(-b_-, B_+) \right). \end{aligned} \quad (55)$$

The commutator $dt^\nu [\mathfrak{D}_t^\dagger, D_\nu]$ is given by

$$\begin{aligned} [\mathfrak{D}_t^\dagger, d + isa] = & (d\mathfrak{k} + ia) \oplus 0 \oplus \left(\delta(s - l/2)(d + ila)b_-, \delta(s + l/2)(d - ila)b_+ \right) \\ = & (d\mathfrak{k} + ia) \left(1 \oplus 0 \oplus \left(-\frac{1}{2t} \delta(s - l/2)b_-, \frac{1}{2t} \delta(s + l/2)b_+ \right) \right). \end{aligned} \quad (56)$$

As the Green's function $G = (\mathfrak{D}^\dagger \mathfrak{D})^{-1}$ is scalar (i.e. commutes with σ -matrices) and $(d\mathfrak{k} - ia) \wedge (d\mathfrak{k} + ia) = (\frac{1}{2}d\mathfrak{x} + ia) \wedge (\frac{1}{2}d\mathfrak{x} - ia)$ is self-dual (see Eq.(33)), the curvature two-form $F = F_{\mu\nu} dx^\mu dx^\nu$ is self-dual as well due to Eqs.(53,54,56).

8 Instantons for the $U(n)$ Gauge Group

Generalizing our construction to instantons in a general gauge group $U(n)$ is fairly straightforward. The corresponding bow diagram is given in Figure 4.

Here the positions of the marked points $\lambda_1, \dots, \lambda_n$ partitioning the interval $[-l/2, l/2]$ is given by the asymptotic of the eigenvalues of the instanton connection around the Taub-NUT circle. All of our previous discussion including the proof of self-duality and the ADHM limit remains valid.

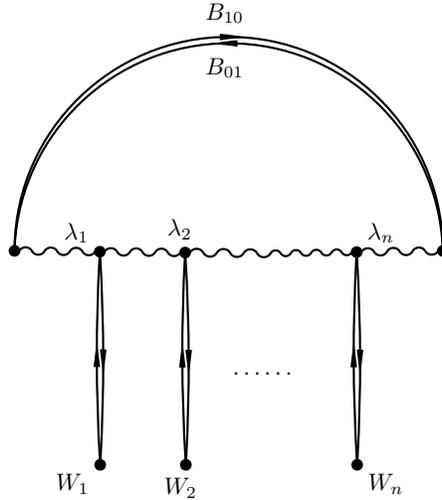


Figure 4: Bow diagram for $U(n)$ Instanton on Taub-NUT.

9 Geometric Meaning of the Nahm Transform for Curved Manifolds

The conventional Nahm transform [26] of some self-dual configuration (or a dimensional reduction of a self-dual configuration) on a flat manifold $M = \mathbb{R}^4/\Lambda$ results in some data on a dual space N of flat connections on M . The kernel of the Nahm transform is the Poincaré bundle $\mathcal{P} \rightarrow M \times N$. Let us denote the two natural projections of the product $M \times N$ on M and N by

p_M and p_N respectively, so that we have the following diagram

$$\begin{array}{ccc}
 & \mathcal{P} & \\
 & \downarrow & \\
 & M \times N & \\
 p_M \swarrow & & \searrow p_N \\
 M & & N.
 \end{array} \tag{57}$$

Then for an instanton bundle $\mathcal{E} \rightarrow M$ its Nahm transform is $p_{N*}(\mathcal{P} \otimes p_M^* \mathcal{E})$. Thus the Poincaré bundle \mathcal{P} plays the role of the kernel of this transform.

For example, for the case of the caloron, flat connections on $\mathbb{R}^3 \times S^1$ have the form $\eta = s dt_0$, where t_0 is the coordinate parameterizing the S^1 factor. The space of such connections forms the dual circle \hat{S}^1 parameterized by s . The Poincaré bundle over the product $\mathbb{R}^3 \times S^1 \times \hat{S}^1$ has natural connection with curvature $\mathcal{F} = ds \wedge dt_0$, and it can be trivalized on either of the two base components, making both pushforward operations p_{N*} and p_{M*} straightforward.

$$\begin{array}{ccc}
 & \mathcal{P} & \mathcal{F} = d\eta = ds \wedge dt_0 \\
 & \downarrow & \\
 & (\mathbb{R}^3 \times S^1) \times \hat{S}^1 & \\
 p_M \swarrow & & \searrow p_N \\
 \mathbb{R}^3 \times S^1 & \eta = s dt_0 & \hat{S}^1.
 \end{array} \tag{58}$$

For a curved manifold M without flat connections, such as the Taub-NUT space, a generalization of the Nahm transform is less straightforward. In order to have a version of the Nahm transform in the diagram below, one has to answer two questions: 1) What is the correct choice of the ‘dual’ manifold N ? and 2) What is the kernel \mathfrak{M} generalizing the Poincaré bundle?

$$\begin{array}{ccc}
 & \mathfrak{M} & \\
 & \downarrow & \\
 & (\text{Taub-NUT}) \times \mathcal{I} & \\
 p_M \swarrow & & \searrow p_N \\
 \text{Taub-NUT} & \eta = s \frac{d\tau + \omega}{2V} & \mathcal{I}.
 \end{array} \tag{59}$$

We propose that for the Taub-NUT space the appropriate choice of N is the space of self-dual Abelian connection on the Taub-NUT (or rather the direct product of this space and \mathbb{R}^3). In order to answer the second question, we have to digress discussing a generalization of instantons to instantons on higher-dimensional spaces.

Instantons on higher dimensional hyperkähler manifolds were defined in [27] in the following manner. Consider the operator $\aleph = I \otimes I + J \otimes J + K \otimes K$ acting on two-forms. Due to the defining quaternionic identities it satisfies

$$\aleph^2 = 2\aleph + 3, \quad (60)$$

and can have eigenvalues 3 or -1 . Note, that on a four-dimensional hyperkähler manifold this operator is related to the Hodge star operation by $*$ = $\frac{1}{2}(\aleph - 1)$, thus the equations

$$\aleph \mathcal{F} = 3\mathcal{F} \quad \text{and} \quad \aleph \mathcal{F} = -\mathcal{F} \quad (61)$$

respectively generalize the self-duality and anti-self-duality conditions to higher dimensions.

Before we proceed, let us observe that the complex structures I, J , and K have a simple action on the vierbein $e^{\hat{\mu}}$ of the Taub-NUT:

$$e^{\hat{0}} = \frac{1}{2\sqrt{V}}(d\tau + \omega), \quad e^{\hat{j}} = \frac{1}{2}\sqrt{V}dx^{\hat{j}}, \quad (62)$$

by multiplying the quaternionic combination $e^{\hat{0}} + Ie^{\hat{1}} + Je^{\hat{2}} + Ke^{\hat{3}}$.

Just as four-dimensional self-duality equations become Bogomolny equations under reduction to three dimensions, we reduce an eight-dimensional self-duality condition $3\mathcal{F} = \aleph \mathcal{F}$ to five dimensions producing the following system of equations on $\mathcal{I} \times \text{Taub-NUT}$:

$$\begin{aligned} 3\mathcal{F}_{\hat{0}s} &= \hat{D}_1\Phi_1 + \hat{D}_2\Phi_2 + \hat{D}_3\Phi_3, \\ 3\mathcal{F}_{\hat{1}s} &= -\hat{D}_0\Phi_1 - \hat{D}_3\Phi_2 + \hat{D}_2\Phi_3, \\ 3\mathcal{F}_{\hat{2}s} &= \hat{D}_3\Phi_1 - \hat{D}_0\Phi_2 - \hat{D}_1\Phi_3, \\ 3\mathcal{F}_{\hat{3}s} &= -\hat{D}_2\Phi_1 + \hat{D}_1\Phi_2 - \hat{D}_0\Phi_3, \\ 2\mathcal{F}_{\hat{\mu}\hat{\nu}} &= \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\mathcal{F}_{\hat{\rho}\hat{\sigma}}, \end{aligned} \quad (63)$$

here Φ_1, Φ_2, Φ_3 are the components of the eight-dimensional connection in the reduced three directions, we used the curvature $\mathcal{F} = \mathcal{F}_{s\hat{\rho}}ds \wedge e^{\hat{\rho}} + \mathcal{F}_{\hat{\mu}\hat{\nu}}e^{\hat{\mu}} \wedge e^{\hat{\nu}}$

vierbein components and $\hat{D}_0 = 2\sqrt{V}D_0$ and $\hat{D}_j = \frac{2}{\sqrt{V}}D_j - 4\omega_j D_0$, which appear in the covariant differential decomposition $D = d\tau D_0 + dx^j D_j = e_{\hat{0}}\hat{D}_{\hat{0}} + e^{\hat{j}}\hat{D}_{\hat{j}}$.

Since Eqs.(63) emerge via a dimensional reduction of higher-dimensional self-duality equations, one might call an object satisfying these equations an *Instapole* or a *Monotone*, but we hope someone will come up with a more poetic name for it.

In our case $\mathcal{F} = d\eta = d\left(s\frac{d\tau+\omega}{2V}\right) = ds \wedge a + sda = \frac{1}{\sqrt{V}}ds \wedge e^0 + sda$ as dictated by the relation (58) between \mathcal{I} and the Taub-NUT. We observe that $\Phi_1 = t_1 = -\frac{1}{2}x_1, \Phi_2 = t_2 = -\frac{1}{2}x_2, \Phi_3 = t_3 = -\frac{1}{2}x_3$ augment \mathcal{F} to produce a solution to the system of Eq. (63). This is exactly the solution defining the object generalizing the Poincare bundle in the diagram (59) that leads to the natural twisting that we used in Eq.(45). It plays the role of the kernel in this generalization of the Nahm transform. It would be very interesting to explore other possible kernels delivered by other Monotone (Instapole) solutions to Eqs.(63).

10 One Instanton Example

Let us now focus on a single $SU(2)$ instanton on the Taub-NUT, i.e. a self-dual curvature configuration with $k_0 = 1$ and $m = 1$. For a single instanton the T 's in the Nahm data are abelian and the Nahm equations are solved by

$$\vec{T}(s) = \begin{cases} \vec{T}_1 & \text{for } -l/2 < s < -\lambda \text{ or } \lambda > s > l/2, \\ \vec{T}_2 & \text{for } -\lambda < s < \lambda. \end{cases} \quad (64)$$

We interpret $\vec{x} = -2\vec{T}_1$ and $\vec{x} = -2\vec{T}_2$ as the locations of the instanton constituents. Let $\vec{z}_1 = \vec{t} - \vec{T}_1, \vec{z}_2 = \vec{t} - \vec{T}_2$ and let $\vec{y} = \vec{T}_2 - \vec{T}_1 = \vec{z}_1 - \vec{z}_2$ measuring the position relative to and the displacement between the constituents. The instanton position along τ is given by T_0 . Since the Taub-NUT metric is invariant with respect to the shifts of τ , we put $T_0 = 0$ without loss of generality. We also gauge away t_0 in favor of the phase of b_{\pm} . Define Q_+ and Q_- by $Q_{\pm}Q_{\pm}^{\dagger} = y \pm \not{y}$. Using the component expressions for the spinors we introduced earlier

$$b_- = \begin{pmatrix} b_{01}^{\dagger} \\ -b_{10} \end{pmatrix}, b_+ = \begin{pmatrix} b_{10}^{\dagger} \\ b_{01} \end{pmatrix}, B_- = \begin{pmatrix} B_{10}^{\dagger} \\ -B_{01} \end{pmatrix}, B_+ = \begin{pmatrix} B_{01}^{\dagger} \\ B_{10} \end{pmatrix}, \quad (65)$$

and since all the components are complex numbers it is straightforward to verify that

$$b_-^\dagger B_- = B_+^\dagger b_+ = e^{i\tau/2} \mathcal{P}, \quad b_+^\dagger B_+ = B_-^\dagger b_- = e^{-i\tau/2} \mathcal{P}, \quad (66)$$

where

$$\mathcal{P} = \sqrt{(T_1 + t)^2 - z_1^2}. \quad (67)$$

The moment maps at $s = \pm l/2$ imply that

$$b_\pm b_\pm^\dagger = |\vec{t}| \pm \mathfrak{k}, \quad B_\pm B_\pm^\dagger = |\vec{T}_1| \pm \mathfrak{T}_1, \quad (68)$$

and moment maps at $s = \pm \lambda$ that $Q_R = Q_+$ and $Q_L = Q_-$.

10.1 Solving the Weyl Equation

On each interval $\mathfrak{D}_t = -\partial_s + \mathfrak{T} - \mathfrak{k} = -\partial_s - \mathfrak{k}$ and $\mathfrak{D}_t^\dagger = \partial_s + \mathfrak{T} - \mathfrak{k} = \partial_s - \mathfrak{k}$, with $\mathfrak{k} = \mathfrak{k}_1$ or \mathfrak{k}_2 in accordance with (64). It follows therefore that $\psi(s)$ has the form

$$\psi(s) = \begin{cases} e^{\mathfrak{k}_1(s+l/2)} \psi_L & \text{for } -l/2 < s < -\lambda, \\ e^{\mathfrak{k}_2 s} \Pi & \text{for } -\lambda < s < \lambda, \\ e^{\mathfrak{k}_1(s-l/2)} \psi_R & \text{for } \lambda < s < l/2. \end{cases} \quad (69)$$

for some constant ψ_L, ψ_R , and Π . As we shall soon verify, the kernel of \mathfrak{D}_t^\dagger is two-dimensional; so from now on we shall understand $\psi(s), \chi_L, \chi_R$ and v to be two-column matrices, so that their first columns deliver one of the solutions and their second columns deliver the second, linearly independent, solution of $\mathfrak{D}_t^\dagger \psi = 0$.

Let $A_L = (B_-, -b_+)$ and $A_R = (-b_-, B_+)$ then the $\mathfrak{D}_t^\dagger \Psi = 0$ conditions read

$$\psi_R - A_R v = 0 \quad \text{at } s = \frac{l}{2} \quad (70)$$

$$e^{-\mathfrak{k}_1(l/2-\lambda)} \psi_R - e^{\mathfrak{k}_2 \lambda} \Pi + Q_R \chi_R = 0 \quad \text{at } s = \lambda, \quad (71)$$

$$e^{-\mathfrak{k}_2 \lambda} \Pi - e^{\mathfrak{k}_1(l/2-\lambda)} \psi_L + Q_L \chi_L = 0 \quad \text{at } s = -\lambda, \quad (72)$$

$$\psi_L + A_L v = 0 \quad \text{at } s = -\frac{l}{2}. \quad (73)$$

It is useful to note the following relations

$$A_L A_L^\dagger = T_1 + t + \mathfrak{X}_1, A_R^\dagger A_L = A_L A_R^\dagger = -e^{i\tau/2} \mathcal{P}, A_L A_R^{-1} = -\frac{e^{i\tau/2}}{\mathcal{P}} (T_1 + t + \mathfrak{X}_1),$$

$$A_R A_R^\dagger = T_1 + t - \mathfrak{X}_1, A_L^\dagger A_R = A_R A_L^\dagger = -e^{-i\tau/2} \mathcal{P}, A_R A_L^{-1} = -\frac{e^{-i\tau/2}}{\mathcal{P}} (T_1 + t - \mathfrak{X}_1),$$

and define μ_+ and μ_- to be such that $\mu_+^2 = A_L A_L^\dagger$ and $\mu_-^2 = A_R A_R^\dagger$ namely

$$\mu_\pm = \sqrt{\frac{T_1 + t + \mathcal{P}}{2}} \pm \sqrt{\frac{T_1 + t - \mathcal{P}}{2}} \frac{\mathfrak{X}_1}{z_1}, \quad (74)$$

then $\mu_+ \mu_- = \mathcal{P}$.

We choose $v = -e^{i\tau/4} A_L^\dagger \frac{\mu_-}{\mathcal{P}} = e^{-i\tau/4} A_R^\dagger \frac{\mu_+}{\mathcal{P}}$, so that now $\psi_L = e^{i\tau/4} \mu_+$, $\psi_R = e^{-i\tau/4} \mu_-$. From the matching conditions Eqs.(71,72) at $s = \pm\lambda$ it follows that

$$\Pi = \frac{1}{2g} \left(e^{-i\tau/4} e^{\lambda \mathfrak{X}_2} (y - \mathfrak{Y}) e^{-(l/2-\lambda)\mathfrak{X}_1} \mu_- + e^{i\tau/4} e^{-\lambda \mathfrak{X}_2} (y + \mathfrak{Y}) e^{(l/2-\lambda)\mathfrak{X}_1} \mu_+ \right), \quad (75)$$

where the function g is given by

$$g = y \cosh 2z_2 \lambda - \frac{\vec{z}_2 \cdot \vec{y}}{z_2} \sinh 2z_2 \lambda = e^{2\mathfrak{X}_2 \lambda} Q_- Q_-^\dagger + Q_+ Q_+^\dagger e^{-2\mathfrak{X}_2 \lambda}, \quad (76)$$

and that

$$\begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} Q_+^\dagger e^{-\lambda \mathfrak{X}_2} \\ Q_-^\dagger e^{\lambda \mathfrak{X}_2} \end{pmatrix} \Upsilon, \quad (77)$$

with

$$\Upsilon = \frac{e^{i\tau/4} e^{\lambda \mathfrak{X}_2} e^{(\frac{l}{2}-\lambda)\mathfrak{X}_1} \mu_+ - e^{-i\tau/4} e^{-\lambda \mathfrak{X}_2} e^{-(\frac{l}{2}-\lambda)\mathfrak{X}_1} \mu_-}{2g}. \quad (78)$$

10.2 Normalization

Let us now check the orthogonality and the normalization of the solution for Ψ of Eqs.(69, 104, 75, 77). To simplify our notation let us introduce $\alpha = \frac{1}{4z_1} \ln \frac{T_1+t+z_1}{T_1+t-z_1}$, so that $\mu_-^2 = T_1 + t - \mathfrak{X}_1 = \mathcal{P} e^{-2\alpha \mathfrak{X}_1}$, and in particular that $\sinh 2\alpha z_1 = z_1/\mathcal{P}$ and $\cosh 2\alpha z_1 = (T_1 + t)/\mathcal{P}$. Introduce $\Delta = \frac{l}{2} - \lambda + \alpha$ and let

$$c_1 = \cosh 2\Delta z_1 = \frac{(T_1 + t) \cosh(l - 2\lambda)z_1 + z_1 \sinh(l - 2\lambda)z_1}{\mathcal{P}}, \quad (79)$$

$$s_1 = \sinh 2\Delta z_1 = \frac{z_1 \cosh(l - 2\lambda)z_1 + (T_1 + t) \sinh(l - 2\lambda)z_1}{\mathcal{P}}, \quad (80)$$

$$c_2 = \cosh 2\lambda z_2, \quad s_2 = \sinh 2\lambda z_2, \quad (81)$$

and $g = y \cosh 2\lambda z_2 + \frac{y^2 + z_2^2 - z_1^2}{2z_2} \sinh 2\lambda z_2$, then we find that $(\Psi, \Psi) = N^2 \mathbb{I}_{2 \times 2}$ with the normalization factor N

$$N^2 = (\Psi, \Psi) = \frac{\mathcal{P}}{g} \left(c_1 c_2 + \frac{y}{z_1} s_1 c_2 + \frac{y}{z_2} c_1 s_2 + \frac{z_1^2 + z_2^2 + y^2}{2z_1 z_2} s_1 s_2 - \cos \frac{\tau}{2} \right). \quad (82)$$

Let us also observe that both Π and Υ appearing in the solution are scalar multiples of unitary matrices, namely

$$\Pi^\dagger \Pi = \frac{\mathcal{P}}{g} \left(y c_1 + \frac{\vec{y} \cdot \vec{z}_1}{z_1} s_1 \right), \quad \Upsilon^\dagger \Upsilon = \frac{\mathcal{P}}{2g^2} \left(c_1 c_2 + s_1 \frac{\vec{z}_2 \cdot \vec{z}_1}{z_1} - \cos \frac{\tau}{2} \right). \quad (83)$$

10.3 One Instanton Connection

We rewrite the Eq.(47) as $A = A^{(0)} d\tau + A^{(3)} - \Phi \frac{1}{2V} (d\tau + \vec{\omega} \cdot d\vec{x})$, where

$$\Phi = (\Psi_N, \mathbf{s}\Psi_N), \quad A^{(0)} = i \left(\Psi_N, \frac{\partial}{\partial \tau} \Psi_N \right), \quad A^{(3)} = i \left(\Psi_N, \frac{\partial}{\partial x_j} \Psi_N \right) dx^j, \quad (84)$$

for an orthonormalized solution Ψ_N satisfying $(\Psi_N, \Psi_N) = \mathbb{I}_{2 \times 2}$. Here we observe that for any $\Psi = N\Psi_N$, with N any nowhere vanishing scalar function, we have

$$\begin{aligned} \left(\Psi_N, \frac{\partial}{\partial x^\mu} \Psi_N \right) &= \frac{1}{2} \left(\left(\Psi_N, \frac{\partial}{\partial x^\mu} \Psi_N \right) - \left(\frac{\partial}{\partial x^\mu} \Psi_N, \Psi_N \right) \right) \\ &= \frac{1}{2N^2} \left((\Psi, \frac{\partial}{\partial x^\mu} \Psi) - \left(\frac{\partial}{\partial x^\mu} \Psi, \Psi \right) \right). \end{aligned} \quad (85)$$

Thus for the solution of Section 10.1 which satisfies $(\Psi, \Psi) = N^2 \mathbb{I}_{2 \times 2}$ we have

$$\Phi = \frac{1}{N^2} (\Psi, \mathbf{s}\Psi), \quad (86)$$

$$A^{(0)} = \frac{i}{N^2} (\Psi, \frac{\partial}{\partial \tau} \Psi) = \frac{i}{2N^2} \left((\Psi, \frac{\partial}{\partial \tau} \Psi) - \left(\frac{\partial}{\partial \tau} \Psi, \Psi \right) \right), \quad (87)$$

$$A^{(3)} = \frac{i}{N^2} (\Psi, d\Psi) = \frac{i}{2N^2} \left(\Psi, (d - \overleftarrow{d}) \Psi \right) \equiv \frac{i}{2N^2} \left((\Psi, d\Psi) - (d\Psi, \Psi) \right), \quad (88)$$

where we introduced the three-dimensional differential $d = dx^j \frac{\partial}{\partial x^j} = dt^j \frac{\partial}{\partial t^j}$.

Given our solution for Ψ of Eqs.(69, 104, 75, 77) one can apply the above formulas, integrating over s making use of some elementary integrals. A relatively straightforward calculation gives

$$N^2 A^{(0)} = \frac{1}{4} (2t - c_1 \mathcal{P}) \frac{\mathbb{X}_1}{z_1^2} + \frac{1}{2\mathcal{P}} \left(\Upsilon_1 - \frac{\vec{T}_1 \cdot \vec{z}_1}{z_1} \frac{\mathbb{X}_1}{z_1} \right) + \frac{i}{2} \frac{s_2}{z_2} \Pi^\dagger (\partial_\tau - \overleftarrow{\partial}_\tau) \Pi + ig \Upsilon^\dagger (\partial_\tau - \overleftarrow{\partial}_\tau) \Upsilon, \quad (89)$$

$$N^2 \Phi = \left(t \left(l + \frac{1}{2t} \right) - \left(\lambda c_1 + \frac{s_1}{2z_1} \right) \mathcal{P} \right) \frac{\mathbb{X}_1}{z_1^2} + \left(\lambda c_2 - \frac{s_2}{2z_2} \right) \Pi^\dagger \frac{\mathbb{X}_2}{z_2^2} \Pi + 2\lambda \Upsilon^\dagger \left((c_2 \vec{y} \cdot \vec{z}_2 - s_2 y z_2) \frac{\mathbb{X}_2}{z_2^2} + \mathbb{Y} - \frac{\vec{y} \cdot \vec{z}_2}{z_2} \frac{\mathbb{X}_2}{z_2} \right) \Upsilon, \quad (90)$$

$$N^2 A^{(3)} = \frac{i}{2} \left\{ \frac{s_1}{\mathcal{P}} \left[\Upsilon_1, d \frac{\mathbb{X}_1}{z_1} \right] + \frac{1}{\mathcal{P}^3} \left(\frac{T_1 + t \vec{z}_1 \cdot d\vec{t}}{z_1} - \frac{\vec{t} \cdot d\vec{t}}{t} \right) [\Upsilon_1, \mathbb{X}_1] + \left(-\frac{1}{2} - \mathcal{P} \left(\frac{l}{2} - \lambda - \frac{s_1}{2z_1} \right) + (c_1 - 1) \frac{T_1}{\mathcal{P}} \right) \frac{[\mathbb{X}_1, d\mathbb{K}]}{z_1^2} + \Pi^\dagger \left(d - \overleftarrow{d} - \left(2\lambda - \frac{s_2}{z_2} \right) \frac{[\mathbb{X}_2, d\mathbb{K}]}{2z_2^2} \right) \Pi + \Upsilon^\dagger \left(2gd - \overleftarrow{d} 2g + 2\lambda \frac{\vec{z}_2 \cdot d\vec{t}}{z_2} [\mathbb{Y}, \mathbb{X}_2] + s_2 \left[\mathbb{Y}, d \frac{\mathbb{X}_2}{z_2} \right] + (c_2 - 1) \frac{y}{z_2^2} [\mathbb{X}_2, d\mathbb{K}] \right) \Upsilon \right\}, \quad (91)$$

where the functions \mathcal{P} and g are defined in Eqs. (67) and (76), the hyperbolic functions c_1, s_1, c_2 and s_2 are in Eqs. (67, 80) and (81), and Π and Υ are given in Eqs. (75) and (78). The normalization factor N^2 is given by Eq. (82).

11 Conclusions

We discussed the topological charges of an instanton configuration on the Taub-NUT space with the maximal symmetry breaking by the Polyakov loop at infinity. These are given by integer monopole charges and an integer instanton number. Solutions with vanishing instanton number correspond to singular monopoles [18]. In their three-dimensional interpretation these have infinite energy, while as configurations on the Taub-NUT space they

are smooth and have finite action. Thus one can regard the Taub-NUT background as a regularization. A simplest solution with zero instanton number was constructed in [21], and its physical properties were explored in [22].

In this manuscript we focussed on the case with vanishing monopole charges. We presented the ADHM-Nahm data for this case. These data is conveniently encoded in a bow diagram, such as in Figure 3 or Figure 4. We used the bow diagram description earlier in [16] to study the moduli spaces of instantons on the Taub-NUT. Here we give the details of the Nahm transform leading to the explicit instanton connection.

As an example illustrating this construction we find a single $SU(2)$ instanton on the Taub-NUT space Eqs.(89, 90, 91).

The bow diagram we presented is not limited to the case of the Taub-NUT background, rather, we chose to limit the scope of this paper to this case to simplify the presentation. In our forthcoming paper [28] we give the bow diagrammatic description of instantons with arbitrary charges on a general ALF spaces of either A_k - or D_k -type.

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Appendix

11.1 Metric Conventions and Moment Maps

The Nahm data on an interval of length l can be organized into a quaternion $t = t_0 e_0 + \vec{t} \cdot \vec{e}$ with the metric and the symplectic forms

$$ds^2 = g(\cdot, \cdot) = l \frac{1}{2} \text{tr} dt dt^\dagger, \quad \omega_j = g(\cdot, e_j \cdot) = -\frac{l}{4} \text{tr} (dt \wedge dt^\dagger e_j - dt^\dagger e_j \wedge dt). \quad (92)$$

With respect to $t \mapsto t + \epsilon$ the moment maps are $\mu_j = -\frac{l}{4} \text{tr} (t^\dagger - t) e_j = l t_j$.

For $q = (b_-, b_+)$ the metric $ds^2 = \frac{1}{2} \text{tr} dq dq^\dagger = db_-^\dagger db_- = db_+^\dagger db_+$. The moment map with respect to $q \mapsto e^{\epsilon e_3}$ is $\mu_j = -\frac{1}{4} \text{tr} q e_3 q^\dagger e_j$.

Coordinates on Taub-NUT are either b_+, b_- or $\vec{t}, 2lt_0 = \tau \sim \tau + 4\pi$. The instanton moduli are $\vec{T}_1, \theta_1 = 2(d_L + d_R)T_0^L$ (or B_+, B_-) and $\vec{T}_2, \theta_2 = 2d_M T_0^M$. The relative coordinates are $\vec{z}_1 = \vec{t} - \vec{T}_1, \vec{z}_2 = \vec{t} - \vec{T}_2$, and the relative position is $\vec{y} = \vec{T}_2 - \vec{T}_1 = \vec{t}_1 - \vec{t}_2$.

We also collect the bifundamental data as

$$b_- = \begin{pmatrix} b_{01}^\dagger \\ -b_{10} \end{pmatrix}, b_+ = \begin{pmatrix} b_{10}^\dagger \\ b_{01} \end{pmatrix}, B_- = \begin{pmatrix} B_{10}^\dagger \\ -B_{01} \end{pmatrix}, B_+ = \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix}, \quad (93)$$

and the fundamental data as

$$Q_- = \begin{pmatrix} J_L^\dagger \\ I_L \end{pmatrix}, Q_+ = \begin{pmatrix} J_R^\dagger \\ I_R \end{pmatrix}. \quad (94)$$

Moment Map

For the Taub-NUT

$$\frac{d}{ds}\aleph + \text{Vec}\left\{\delta\left(s + \frac{l}{2}\right)b_- b_-^\dagger + \delta\left(s - \frac{l}{2}\right)b_+ b_+^\dagger\right\} = 0, \quad (95)$$

and for the instanton Bow Data

$$\left[\frac{d}{ds} - iT_0, \aleph\right] + \text{Vec}\left\{\aleph\aleph + \delta\left(s + \frac{l}{2}\right)B_- B_-^\dagger + \delta\left(s - \frac{l}{2}\right)B_+ B_+^\dagger + \delta(s + \lambda)Q_- Q_-^\dagger + \delta(s - \lambda)Q_+ Q_+^\dagger\right\} = 0. \quad (96)$$

these imply that \vec{T} is constant on each interval and equals \vec{T}_1 for $|s| < \lambda$ and \vec{T}_2 for $|s| > \lambda$ and the conditions at $s = l/2, -l/2, \lambda, -\lambda$ are respectively

$$T_1 + \aleph_1 = B_+ B_+^\dagger, T_1 - \aleph_1 = B_- B_-^\dagger, y + \aleph = Q_+ Q_+^\dagger, y - \aleph = Q_- Q_-^\dagger. \quad (97)$$

Relative coordinates are $\vec{z}_1 = \vec{t} - \vec{T}_1, \vec{z}_2 = \vec{t} - \vec{T}_2, \vec{y} = \vec{T}_2 - \vec{T}_1 = \vec{z}_1 - \vec{z}_2$.

Weyl Equation

$$\left(\frac{d}{ds} - \aleph_{1,2}\right)\psi(s) = 0, \quad (98)$$

$$\psi(\lambda+) - \psi(\lambda-) = -Q_+ \chi_R, \quad \psi(l/2) = (-b_-, B_+)v, \quad (99)$$

$$\psi(-\lambda+) - \psi(-\lambda-) = -Q_- \chi_L, \quad \psi(-l/2) = -(B_-, -b_+)v. \quad (100)$$

Solution of the Weyl Equation

$$v = \frac{1}{P} \begin{pmatrix} -e^{i\frac{\tau}{4}} B_-^\dagger \mu_- \\ e^{-i\frac{\tau}{4}} B_+^\dagger \mu_+ \end{pmatrix}, \quad \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} Q_+^\dagger e^{-\lambda \kappa_2} \\ Q_-^\dagger e^{\lambda \kappa_2} \end{pmatrix} \Upsilon, \quad (101)$$

$$\psi(s) = \begin{cases} e^{-i\frac{\tau}{4}} e^{(s-\frac{l}{2})\kappa_1} \mu_- & \text{for } \lambda < s < l/2, \\ e^{s\kappa_2} \Pi & \text{for } -\lambda < s < \lambda, \\ e^{i\frac{\tau}{4}} e^{(s+\frac{l}{2})\kappa_1} \mu_+ & \text{for } -l/2 < s < -\lambda. \end{cases} \quad (102)$$

Here

$$2g = 2\left(y \cosh 2\lambda z_2 - \frac{\vec{y} \cdot \vec{z}_2}{z_2} \sinh 2\lambda z_2\right), \quad \mathcal{P} = \sqrt{(T_1 + t)^2 - z_1^2}, \quad (103)$$

$$\mu_\pm = \sqrt{\frac{T_1 + t + \mathcal{P}}{2}} \pm \sqrt{\frac{T_1 + t - \mathcal{P}}{2}} \frac{\kappa_1}{z_1}, \quad (104)$$

and

$$\Upsilon = \frac{1}{2g} \left\{ e^{i\frac{\tau}{4}} e^{\lambda \kappa_2} e^{\kappa_1 d} \mu_+ - e^{-i\frac{\tau}{4}} e^{-\lambda \kappa_2} e^{-\kappa_1 d} \mu_- \right\}, \quad (105)$$

$$\Pi = \frac{1}{2g} \left(e^{i\frac{\tau}{4}} e^{-\lambda \kappa_2} (y + \mathfrak{y}) e^{\kappa_1 d} \mu_+ + e^{-i\frac{\tau}{4}} e^{\lambda \kappa_2} (y - \mathfrak{y}) e^{-\kappa_1 d} \mu_- \right). \quad (106)$$

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